

T. I. KOSENKOVA

WEAK CONVERGENCE OF A SERIES SCHEME OF MARKOV CHAINS TO THE SOLUTION OF A LÉVY DRIVEN SDE

Under the assumptions analogous to those of Gnedenko's theorem, the weak convergence of a series scheme of Markov chains to the solution of a Lévy driven SDE is obtained.

1. INTRODUCTION

In this paper, we establish the conditions sufficient for a sequence of stepwise processes defined by a series scheme of Markov chains to converge in \mathbb{D} to a solution of a Lévy driven SDE. This topic is traditional and comes back to the famous Skorokhod invariance principle (see [1]) for the sequence of stepwise processes associated with partial sums of a triangular array of i.i.d. random variables, which satisfy conditions of Gnedenko's theorem (see [2]). In the Markov setting, this result was extended in several directions. In [3], the increments of Markov chains are represented as the images of a fixed measure under some mappings, and the respective limit theorem is formulated in the terms of the latter mappings. This leads to a highly implicit set of conditions on the increments of the Markov chains under consideration.

Another approach, which leads to an explicit and transparent set of conditions, is developed in [4], the so-called “Lévy approximation scheme.” The model studied therein is even more general and contains an additional semi-Markov switching component. The kernel involved into the limit generator in Theorem 1 in [4] is integrable w.r.t. $|u|$ and $|u|^2$ on the sets $\{|u| \leq 1\}$ and $\{|u| \geq 1\}$, respectively. Those properties are clearly stronger than the standard properties of a Lévy measure. This shows that the approach developed in [4] brings some additional moment restrictions in comparison with the conditions of Gnedenko's theorem.

Here, we prove a limit theorem for a sequence of Markov chains under conditions that can be considered as straightforward analogs to those of Gnedenko's theorem, with the the solution of the Lévy driven SDE as the limit process.

The structure of the article is following. In Section 2, the main theorem is introduced. The theorem consists of two statements: about the relative compactness of the associated sequence and about the limit point identification. Section 3 contains the proof of this theorem with two subsections related to the statements of the theorem. The proof of the supplementary statement, the so-called “compact containment condition”, is given in Appendix A.

2. MAIN RESULTS

Let $\{X_k^n, k \leq n\}$, $n \geq 1$ be a sequence of Markov chains in \mathbb{R}^d . Let $\theta_n = \{t_{kn}, 0 \leq k \leq n\}$, $n \geq 1$ be a sequence of partitions of $[0; 1]$ such that there exists a constant $S \geq 1$ such that, for all n ,

$$(1) \quad \frac{1}{nS} \leq \Delta t_{kn} \leq \frac{S}{n},$$

2000 *Mathematics Subject Classification.* Primary 60J10, 60F17, 60B10; Secondary 60J20.

Key words and phrases. Lévy driven SDE, weak convergence of Markov chains.

where $\Delta t_{kn} = t_{kn} - t_{(k-1)n}$. We consider the process

$$X_n(t) = \sum_{k=1}^{n-1} X_k^n \cdot \mathbf{1}_{t \in [t_{(k-1)n}, t_{kn})} + X_n^n \cdot \mathbf{1}_{t \in [t_{(n-1)n}, t_{nn}]}, \quad t \in [0; 1].$$

We introduce a notational convention. By E_k , we denote conditional expectation w.r.t. $\mathfrak{F}_k^n = \sigma\{X_i^n, i \leq k\}$. Let

$$\xi_{kn} = X_k^n - X_{k-1}^n, \quad k \leq n, \quad n \geq 1$$

be an increment of the Markov chain. We assume that this increment has the form

$$\xi_{kn} = \eta_{kn} - \gamma_{kn}(X_{k-1}^n),$$

where η_{kn} and γ_{kn} satisfy the conditions **(G1)**-**(L3)** below. Here, the functions γ_{kn} has the role of a compensating term. Its appearance is related to the fact that, according to Gnedenko's theorem, it is possible to compensate the i.i.d. random variables in such a way that the sums converge weakly. In the Markov setting, instead of the compensating sequence of constants, we consider the compensating sequence of functions.

By

$$\Pi_{kn}(x, dy) = \frac{1}{\Delta t_{kn}} P(\eta_{kn} \in dy | X_{k-1}^n = x),$$

we denote the stochastic kernel that corresponds to η_{kn} . Denote

$$\beta_{kn}(X_{k-1}^n) = E_{k-1} \tau(\eta_{kn}),$$

where

$$\tau(y) = y(1 \wedge \frac{1}{|y|}), \quad y \in \mathbb{R}^d.$$

By $\mathfrak{C}_b^{sep}(\mathbb{R}^d, \mathbb{R})$, we denote the set of bounded continuous functions, which vanish in some neighborhood of zero, i.e.,

$$\mathfrak{C}_b^{sep}(\mathbb{R}^d, \mathbb{R}) = \{f \in C_b(\mathbb{R}^d, \mathbb{R}) : \exists r > 0 \quad f(x) = 0 \quad \text{as } |x| \leq r\}.$$

Let $\Pi(x, dy)$ be a Lévy kernel w.r.t. y , that is $\int_{\mathbb{R}^d} (y^2 \wedge 1) \Pi(x, dy) < \infty$, $x \in \mathbb{R}^d$. Conditions **(G1)**-**(G3')** are straightforward analogs of the conditions of Gnedenko's theorem: **(G1)**

- There exists a function $B : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, such such that, for all $R > 0$,

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \left| \int_{|y| \leq \varepsilon} (\lambda, y)^2 \Pi_{kn}(x, dy) - \frac{1}{\Delta t_{kn}} (\lambda, \beta_{kn}(x))^2 - (B(x)\lambda, \lambda) \right| = 0, \quad \lambda \in \mathbb{R}^d;$$

- for all $f \in \mathfrak{C}_b^{sep}(\mathbb{R}^d, \mathbb{R})$

$$\lim_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \left| \int_{\mathbb{R}^d} f(y) \Pi_{kn}(x, dy) - \int_{\mathbb{R}^d} f(y) \Pi(x, dy) \right| = 0;$$

(G1') for every $R > 0$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{|x| \leq R} \int_{|y| \leq \varepsilon} (\lambda, y)^2 \Pi(x, dy) = 0, \quad \lambda \in \mathbb{R}^d;$$

(G2), there exists $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that, for every $R > 0$,

$$\lim_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \left| \frac{1}{\Delta t_{kn}} (\beta_{kn}(x) - \gamma_{kn}(x)) - a(x) \right| = 0;$$

(G3) for every $R > 0$,

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \int_{|y| \leq \varepsilon} |y|^3 \Pi_{kn}(x, dy) = 0;$$

(G3') for every $R > 0$:

$$\limsup_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \frac{|\gamma_{kn}(x)|^3}{\Delta t_{kn}} = 0.$$

Conditions (L1)-(L3) below are the modification of the linear growth rate conditions.

(L1) There exists a family $\{D_r, r > 0\}$ such that

$$\sup_n \max_{k \leq n} \int_{|y| > r(|x|+1)} \Pi_{kn}(x, dy) \leq D_r, \quad x \in \mathbb{R}^d, \quad r > 0,$$

and $D_r \rightarrow 0, \quad r \rightarrow \infty$;

(L1') there exists a constant $D > 0$ such that

$$\sup_n \max_{k \leq n} \int_{|y| > 1} \Pi_{kn}(x, dy) \leq D, \quad x \in \mathbb{R}^d;$$

(L2) there exists a nondecreasing family $\{C_r, r > 0\}$ such that

$$\sup_n \max_{k \leq n} \left[\int_{|y| \leq r(|x|+1)} y^2 \Pi_{kn}(x, dy) \right] \leq C_r (|x| + 1)^2, \quad x \in \mathbb{R}^d;$$

(L3) there exists a constant $K > 0$ such that

$$\sup_n \max_{k \leq n} \frac{1}{\Delta t_{kn}} |\beta_{kn}(x) - \gamma_{kn}(x)| \leq K(|x| + 1), \quad x \in \mathbb{R}^d$$

$$\sup_n \max_{k \leq n} \frac{1}{\Delta t_{kn}} |\gamma_{kn}(x)|^2 \leq K(|x| + 1)^2$$

and

$$|a(x)|^2 + \|B(x)\| \leq K(|x| + 1)^2, \quad x \in \mathbb{R}^d.$$

Theorem 1. (1) Suppose that conditions (L1) – (L3) hold and $\sup_n |X_0^n| < \infty$ a.s. Then $\{X_n\}$ is relatively compact in Skorokhod's space $\mathbb{D}([0; 1])$.

(2) Assume that, in addition, conditions (G1) – (G3') hold. Let X be a limit point of the sequence $\{X_n\}$ in the sense of the weak convergence of finite-dimensional distributions.

Then X is the solution of the martingale problem for the operator

$$Af(x) = \sum_i a_i(x) f'_i(x) + \frac{1}{2} \sum_{i,j} B_{ij}(x) f''_{ij}(x) +$$

$$(2) \quad + \int_{\mathbb{R}^d} (f(x+y) - f(x) - f'(x)\tau(y)) \Pi(x, dy), \quad x \in \mathbb{R}^d, \quad f \in \mathcal{D},$$

where $\mathcal{D} = \mathcal{D}(\mathbb{R}^d, \mathbb{R})$ is the space of test functions.

Remark 1. If, instead of the series scheme of Markov chains $\{X_k^n, k \leq n\}$, $n \geq 1$ we consider a series scheme associated with a triangular array of i.i.d. random variables, and $\Pi(x, dy) \equiv \Pi(dy)$ is a Lévy measure, then conditions (G1) – (G3') are the conditions of Gnedenko's theorem for a triangular array. The form of conditions (L1) – (L3) is a modification of the standard form of the linear growth rate conditions.

Remark 2. Note that (2) can be written in the form

$$(3) \quad \begin{aligned} Af(x) &= \sum_i \tilde{a}_i(x) f'_i(x) + \frac{1}{2} \sum_{i,j} B_{ij}(x) f''_{ij}(x) + \\ &+ \int_{\mathbb{R}^d} (f(x+y) - f(x) - f'(x)y \mathbf{1}_{|y| \leq 1}) \Pi(x, dy), \quad x \in \mathbb{R}^d, \\ &f \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}), \end{aligned}$$

where

$$\tilde{a}(x) = a(x) - \Pi(x, \mathbb{R}^d \setminus B(0, 1)).$$

Let the family $\{\Pi(x, dy)\}$, $x \in \mathbb{R}^d$ be defined by a measure Π_0 and a function $c: \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d$ as follows:

$$\Pi(x, A) = \Pi_0\{\theta : c(x, \theta) \in A\}.$$

Then, combining (3) and Corollary 2 §3 Section 5 [3], we obtain that the limit process is a solution of the SDE

$$(4) \quad \begin{aligned} X(t) &= X(0) + \int_0^t \tilde{a}(X(s)) ds + \int_0^t B(X(s)) dW(s) + \\ &+ \int_0^t \int_{\Theta} c_1(X(s-), u) \tilde{\nu}(du, ds) + \\ &+ \int_0^t \int_{\Theta} c_2(X(s-), u) \nu(du, ds), \quad \text{with} \\ c_1(x, u) &= c(x, u) \mathbf{1}_{|c(x, u)| \leq 1}, \quad c_2(x, u) = c(x, u) \mathbf{1}_{|c(x, u)| > 1}, \end{aligned}$$

where W is a Wiener process, ν is a Poisson point measure with intensity Π_0 , $\tilde{\nu}(du, ds) = \nu(du, ds) - \Pi_0(du)ds$ is the respective compensated Poisson point measure, and W and ν are independent.

Remark 3. The fact that the martingale problem for A is well-posed, together with Theorem 1, implies the weak convergence of the sequence $\{X_n\}$ to the solution of the martingale problem. The problem of the weak uniqueness of a solution of the martingale problem respective to SDE's with jumps was studied in detail. For typical results in this direction, we refer, for instance, to Bass (see [5]) and to Gikhman and Skorokhod (see Theorem 1 §1 section 6 [3]).

3. PROOF OF THEOREM 1

3.1. Relative compactness in \mathbb{D} . To prove the first statement of Theorem 1, it is sufficient, by Theorem 8.6 (see 3 §8 [6]), to show that

$$(5) \quad \limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\tau \in S_0^n} \sup_{u \leq \delta} Eq^2(X_n((\tau + u) \wedge 1), X_n(\tau)) = 0,$$

where $q(x, y) = |x - y| \wedge 1$, and S_0^n is a collection of all discrete \mathfrak{F}_t^n -stopping times bounded by 1. Let $\tau \in S_0^n$ and $\{s_m\}_{m=1}^M$ be the set of all possible values of τ . Then

$$(6) \quad Eq^2(X_n((\tau + u) \wedge 1), X_n(\tau)) = \sum_{m=1}^M E(|X_n((s_m + u) \wedge 1) - X_n(s_m)| \wedge 1)^2 \mathbf{1}_{\tau = s_m}.$$

Let $s_m \in [t_{jn}; t_{(j+1)n})$ and either $s_m + u \in [t_{kn}; t_{(k+1)n})$, $0 \leq j < k$, $k \leq n - 2$, or $(s_m + u) \wedge 1 \in [t_{(n-1)n}; t_{nn}]$. Then the function $\mathbf{1}_{\tau = s_m}$ is \mathfrak{F}_j^n -measurable. Using the above notation, we have, for the one summand on the r.h.s. of (6),

$$(7) \quad E(|X_n((s_m + u) \wedge 1) - X_n(s_m)| \wedge 1)^2 \mathbf{1}_{\tau = s_m} = E\left(\left|\sum_{l=j+1}^k \xi_{ln}\right| \wedge 1\right)^2 \mathbf{1}_{\tau = s_m}.$$

For every $R > 0$, the expression under the sign (\dots) on the r.h.s. of (7) can be estimated by $(|\sum| \wedge 1) \mathbf{1}_{\max_{i \leq n} |X_i^n| \leq R} + 1 \cdot \mathbf{1}_{\max_{i \leq n} |X_i^n| > R}$. Therefore, the r.h.s. of (7) is not greater than

$$(8) \quad 2E \left(\left| \sum_{l=j+1}^k \xi_{ln} \right| \wedge 1 \right)^2 \mathbf{1}_{\tau=s_m} \mathbf{1}_{\max_{i \leq n} |X_i^n| \leq R} + 2P \left(\max_{i \leq n} |X_i^n| > R \right) P(\tau = s_m).$$

By Lemma 1 (see Appendix A), the second term in (8) reads

$$(9) \quad \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left(\max_{i \leq n} |X_i^n| > R \right) = 0.$$

Denote the first term in (8) by S_1 . Fix some $r > 1$ and put

$$(10) \quad \bar{\xi}_{ln} := \xi_{ln} \cdot \mathbf{1}_{|\xi_{ln}| \leq r(|X_{l-1}^n|+1)} \mathbf{1}_{|X_{l-1}^n| \leq R}.$$

Using this notation, S_1 can be estimated as follows:

$$(11) \quad S_1 \leq F \cdot E \left(\sum_{l=j+1}^k \bar{\xi}_{ln} \right)^2 \mathbf{1}_{\tau=s_m} + F \cdot E \left(\left| \sum_{l=j+1}^k \xi_{ln} \mathbf{1}_{|\xi_{ln}| > r(|X_{l-1}^n|+1)} \right| \wedge 1 \right)^2 \mathbf{1}_{\tau=s_m}.$$

Here and below, by F , we denote some positive constant. Denote the first expectation in (11) by I_1 . Because $\mathbf{1}_{\tau=s_m}$ is \mathfrak{F}_j^n -measurable, we obtain

$$(12) \quad I_1 = \sum_{l=j+1}^k E \mathbf{1}_{\tau=s_m} E_j \left(E_{l-1} (\bar{\xi}_{ln})^2 \right) + 2 \sum_{j+1 \leq p < q \leq k} E \mathbf{1}_{\tau=s_m} E_j (|E_{p-1} \bar{\xi}_{pn}| \cdot |E_{q-1} \bar{\xi}_{qn}|).$$

Let us estimate $E_{l-1} (\bar{\xi}_{ln})^2$, $l \leq n$. By (10), it is sufficient to estimate

$$E_{l-1} \xi_{ln}^2 \mathbf{1}_{|\xi_{ln}| \leq r(|X_{l-1}^n|+1)}.$$

Recall that $\xi_{ln} = \eta_{ln} - \gamma_{ln}(X_{l-1}^n)$, and the conditional distributions of η_{ln} w.r.t. \mathfrak{F}_{l-1}^n equal $\Delta t_{ln} \Pi_{ln}(X_{l-1}^n, dy)$. Denote

$$\Gamma_{ln}^r(x) := \{y : |y - \gamma_{ln}(x)| \leq r(|x| + 1)\}.$$

Then

$$(13) \quad E_{l-1} \xi_{ln}^2 \mathbf{1}_{|\xi_{ln}| \leq r(|X_{l-1}^n|+1)} = \int_{\Gamma_{ln}^r(X_{l-1}^n)} (y - \gamma_{ln}(X_{l-1}^n))^2 \Delta t_{ln} \Pi_{ln}(X_{l-1}^n, dy).$$

Using **(L3)**, we obtain that there exists n_0 such that, for all $n \geq n_0$,

$$(14) \quad \max_{l \leq n} |\gamma_{ln}(x)| \leq \frac{1}{2}(|x| + 1).$$

By $\Theta^r(x)$, we denote the ball centered at 0 with radius $r(|x| + 1)$:

$$(15) \quad \Theta^r(x) := \{y : |y| \leq r(|x| + 1)\}.$$

Let $n \geq n_0$. Then, by (14), we have

$$(16) \quad \Gamma_{ln}^r(x) \subseteq \Theta^{r+1}(x). \quad x \in \mathbb{R}^d,$$

Hence,

$$(17) \quad \int_{\Gamma_{l_n}^r(x)} (y - \gamma_{l_n}(x))^2 \Delta t_{l_n} \Pi_{l_n}(x, dy) \leq \int_{\Theta^{r+1}(x)} (y - \gamma_{l_n}(x))^2 \Delta t_{l_n} \Pi_{l_n}(x, dy).$$

Combining this with **(L2)**, **(L3)**, and (1), we obtain

$$(18) \quad E_{l-1} \xi_{l_n}^2 \mathbf{1}_{|\xi_{l_n}| \leq r(|X_{l-1}^n|+1)} \leq \frac{L_0(r)}{n} (|X_{l-1}^n|^2 + 1),$$

where $L_0(r) = F \cdot (C_{r+1} + K)$. Using (18), we get the estimate for $E_{l-1} (\bar{\xi}_{l_n})^2$

$$(19) \quad E_{l-1} (\bar{\xi}_{l_n})^2 \leq \frac{L_0(r)}{n} (R^2 + 1).$$

Further, let us estimate $|E_{p-1} \bar{\xi}_{p_n}|$, $p \leq n$. As before, we will obtain firstly the estimate for

$$(20) \quad \left| E_{p-1} \xi_{p_n} \mathbf{1}_{|\xi_{p_n}| \leq r(|X_{p-1}^n|+1)} \right| = \left| \int_{\Gamma_{p_n}^r(X_{p-1}^n)} (y - \gamma_{p_n}(X_{p-1}^n)) \Delta t_{p_n} \Pi_{p_n}(X_{p-1}^n, dy) \right|.$$

Recall that

$$\beta_{p_n}(x) = \int_{\mathbb{R}^d} y \left(1 \wedge \frac{1}{|y|} \right) \Delta t_{p_n} \Pi_{p_n}(x, dy).$$

Consequently, the integral on the r.h.s. of (20) can be written in the form

$$(21) \quad \begin{aligned} & \beta_{p_n}(x) - \gamma_{p_n}(x) + \gamma_{p_n}(x) \Delta t_{p_n} \Pi_{p_n}(x, \mathbb{R}^d \setminus \Gamma_{p_n}^r(x)) - \\ & - \int_{\Theta^{r+1}(x) \setminus \Gamma_{p_n}^r(x)} y \Delta t_{p_n} \Pi_{p_n}(x, dy) + \\ & + \int_{\Theta^{r+1}(x) \setminus B(0,1)} y \Delta t_{p_n} \Pi_{p_n}(x, dy) - \int_{\mathbb{R}^d \setminus B(0,1)} \frac{y}{|y|} \Delta t_{p_n} \Pi_{p_n}(x, dy). \end{aligned}$$

Here, by $B(0, 1)$, we denote the ball centered at 0 with radius 1. Using (14), we have

$$(22) \quad \Theta^{\frac{1}{2}}(x) \subseteq \Gamma_{p_n}^r(x).$$

Therefore, by **(L1)**,

$$(23) \quad \max_{p \leq n} \Pi_{p_n}(x, \mathbb{R}^d \setminus \Gamma_{p_n}^r(x)) \leq D_{\frac{1}{2}}.$$

Thus, we obtain, for the second term of (21),

$$(24) \quad \left| \gamma_{p_n}(x) \Delta t_{p_n} \Pi_{p_n}(x, \mathbb{R}^d \setminus \Gamma_{p_n}^r(x)) \right| \leq \frac{F \cdot D_{\frac{1}{2}}}{n} (|x| + 1).$$

The same argument can be applied to estimate the third term of (21):

$$(25) \quad \left| \int_{\Theta^{r+1}(x) \setminus \Gamma_{p_n}^r(x)} y \Delta t_{p_n} \Pi_{p_n}(x, dy) \right| \leq \frac{F \cdot (r+1) D_{\frac{1}{2}}}{n} (|x| + 1).$$

By **(L1')**, for the fourth and fifth terms of (21), we have

$$(26) \quad \left| \int_{\Theta^{r+1}(x) \setminus B(0,1)} y \Delta t_{p_n} \Pi_{p_n}(x, dy) \right| + \left| \int_{\mathbb{R}^d \setminus B(0,1)} \frac{y}{|y|} \Delta t_{p_n} \Pi_{p_n}(x, dy) \right| \leq \frac{F \cdot (r+1) \cdot D}{n} (|x| + 1).$$

Combining (21), **(L3)** for the first term and estimates (24), (25), and (26) and taking (20) into account, we get

$$(27) \quad \left| E_{p-1} \xi_{p_n} \mathbf{1}_{|\xi_{p_n}| \leq r(|X_{p-1}^n|+1)} \right| \leq \frac{L(r)}{n} (|X_{p-1}^n| + 1),$$

where $L(r) := F \cdot (K + D_{\frac{1}{2}} + (r+1)D_{\frac{1}{2}} + (r+1) \cdot D)$. Thus, we obtain the following estimate for $|E_{p-1}\bar{\xi}_{pn}|$:

$$(28) \quad |E_{p-1}\bar{\xi}_{pn}| \leq \frac{L(r)}{n}(R+1).$$

Combining (19), (28), and the fact that $k-j \leq \delta S \cdot n$ (see (1)), we obtain

$$(29) \quad I_1 \leq P(\tau = s_m)F \cdot (S\delta L_0(r) + S^2\delta^2 L^2(r))(R^2 + 1).$$

Denote the second term of the r.h.s. of (11) by I_2 . Note that $|\sum| \wedge 1$ can be estimated by $\mathbf{1}_{\cup_{i=j+1}^k \{|\xi_{in}| > r(|X_{i-1}^n|+1)\}}$. Moreover, because $\mathbf{1}_{\tau=s_m}$ is \mathfrak{F}_j^n -measurable, we have that I_2 can be estimated in the following way:

$$(30) \quad E\mathbf{1}_{\tau=s_m}E_j\mathbf{1}_{\cup_{i=j+1}^k \{|\xi_{in}| > r(|X_{i-1}^n|+1)\}}.$$

Using the standard fact that $\mathbf{1}_A = 1 - \mathbf{1}_{\bar{A}}$, taking the conditional expectations w.r.t. \mathfrak{F}_l^n , $l = j+1, \dots, k$, and using (23) and (1) for $E_{l-1}\mathbf{1}_{\{|\xi_{in}| > r(|X_{l-1}^n|+1)\}}$, we obtain the following estimate for the expression in (30):

$$(31) \quad \left(1 - \left(1 - \frac{SD_{\frac{1}{2}}}{n}\right)^{k-j}\right)P(\tau = s_m).$$

Because $k-j \leq \delta S \cdot n$, we finally obtain the estimate I_2 in the form

$$(32) \quad I_2 \leq F \cdot S^2 D_{\frac{1}{2}} \delta P(\tau = s_m).$$

Combining (8), (29), and (32) we have finally the following estimate for the r.h.s. of (6):

$$(33) \quad F \cdot \left(P\left(\max_{i \leq n} |X_i^n| > R\right) + (S\delta L_0(r) + S^2\delta^2 L^2(r))(R^2 + 1) + S^2 D_{\frac{1}{2}} \delta\right).$$

Passing to the limit as $\delta \rightarrow 0+$ and using (9), we obtain (5). \square

3.2. Identification of the limit point. To obtain the second statement of Theorem 1, let us firstly note that the family of distributions of $\{X_n\}$ is uniformly continuous in probability; that is, for all $\varepsilon > 0$,

$$(34) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{|t-s| < \delta} P(|X_n(t) - X_n(s)| > \varepsilon) = 0.$$

This property follows from the stronger one (5), proved in Section 3.1. Thus, by Theorem 8.2 (see §8 Section 5, [6]), it is sufficient to show that, for $f \in \mathcal{D}$,

$$(35) \quad \max_{k \leq n} E \left| E_{k-1} \left(\frac{1}{\Delta t_{kn}} (f(X_k^n) - f(X_{k-1}^n)) - Af(X_{k-1}^n) \right) \right| \rightarrow 0, \quad n \rightarrow \infty.$$

The expression under the sign $|\dots|$ in (35) can be written in the form

$$(36) \quad E_{k-1} \left(\frac{1}{\Delta t_{kn}} (f(X_k^n) - f(X_{k-1}^n)) - a^T(X_{k-1}^n) f'(X_{k-1}^n) - \frac{1}{2} \sum_{i,j} B_{ij}(x) f''_{ij}(x) - \int_{\mathbb{R}^d} (f(X_{k-1}^n + y) - f(X_{k-1}^n) - \tau(y)^T f'(X_{k-1}^n)) \Pi(X_{k-1}^n, dy) \right) := g_{kn}(X_{k-1}^n).$$

Recall that $X_k^n = X_{k-1}^n + \eta_{kn} - \gamma_{kn}(X_{k-1}^n)$, and the conditional distribution η_{kn} w.r.t. \mathfrak{F}_{k-1}^n equals $\Delta t_{kn} \Pi_{kn}(X_{k-1}^n, dy)$. Hence, the function g_{kn} on the r.h.s. of (36) has the

form

$$(37) \quad \begin{aligned} g_{kn}(x) = & \int_{\mathbb{R}^d} (f(x+y-\gamma_{kn}(x)) - f(x)) \Pi_{kn}(x, dy) - \\ & - a^T(x) f'(x) - \frac{1}{2} \sum_{i,j} B_{ij}(x) f''_{ij}(x) - \\ & - \int_{\mathbb{R}^d} (f(x+y) - f(x) - \tau(y)^T f'(x)) \Pi(x, dy). \end{aligned}$$

For every $f \in \mathcal{D}$, there exists $\hat{f} \in \mathcal{S}$ such that

$$f(x) = \int_{\mathbb{R}^d} e^{i(\lambda, x)} \hat{f}(\lambda) d\lambda \quad \text{where } \mathcal{S} = \mathcal{S}(\mathbb{R}^d, \mathbb{R}) \text{ is the Schwarz space.}$$

Therefore, the first term in (37) can be written as

$$(38) \quad \int_{\mathbb{R}^d} \hat{f}(\lambda) e^{i(\lambda, x)} \int_{\mathbb{R}^d} [e^{i(\lambda, y - \gamma_{kn}(x))} - 1] \Pi_{kn}(x, dy) d\lambda.$$

The term under the sign [...] in (38) can be represented as follows:

$$(39) \quad \begin{aligned} e^{i(\lambda, y - \gamma_{kn}(x))} - 1 = & e^{-i(\lambda, \gamma_{kn}(x))} [e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y))] - \\ & - e^{-i(\lambda, \gamma_{kn}(x))} [e^{i(\lambda, \gamma_{kn}(x))} - 1 - i(\lambda, \tau(y))]. \end{aligned}$$

In view of the definition of β_{kn} , we get

$$\int_{\mathbb{R}^d} i(\lambda, \tau(y)) \Pi_{kn}(x, dy) = \frac{i(\lambda, \beta_{kn}(x))}{\Delta t_{kn}}.$$

Using (38) and (39), we obtain, for the function g_{kn} , the expression

$$(40) \quad \begin{aligned} g_{kn}(x) = & \int_{\mathbb{R}^d} \hat{f}(\lambda) e^{i(\lambda, x)} \cdot \left(e^{-i(\lambda, \gamma_{kn}(x))} \int_{\mathbb{R}^d} [e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y))] \Pi_{kn}(x, dy) - \right. \\ & - \int_{\mathbb{R}^d} [e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y))] \Pi(x, dy) - \\ & \left. - i(\lambda, a(x)) + \frac{1}{2} \lambda^T B(x) \lambda - \right. \\ & \left. - e^{-i(\lambda, \gamma_{kn}(x))} \frac{1}{\Delta t_{kn}} [e^{i(\lambda, \gamma_{kn}(x))} - 1 - i(\lambda, \beta_{kn}(x))] \right) d\lambda. \end{aligned}$$

We note that, for all $R > 0$,

$$(41) \quad \max_{k \leq n} E |g_{kn}(X_{k-1}^n)| \leq \max_{k \leq n} \sup_{|x| \leq R} |g_{kn}(x)| + \max_{k \leq n} \int_{|x| > R} |g_{kn}(x)| P_{X_{k-1}^n}(dx).$$

To estimate the first and second summands in (41), we need the following statements, which will be proved below.

Proposition 1. *If conditions (G1) - (G3') and (L1) - (L3) hold, then, for all $R > 0$ and $f \in \mathcal{D}$,*

$$(42) \quad \max_{k \leq n} \sup_{|x| \leq R} |g_{kn}(x)| \rightarrow 0, \quad n \rightarrow \infty.$$

Proposition 2. *Let $f \in \mathcal{D}$ and g_{kn} be defined by (37). If conditions (L1) - (L3) are fulfilled, and $\sup_n |X_0^n| < \infty$ a.s., then*

$$(43) \quad \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{k \leq n} E \mathbf{1}_{\{X_{k-1}^n > R\}} |g_{kn}(X_{k-1}^n)| = 0.$$

Taking (41) into account, these two statements are sufficient to obtain (35) and, thus, the second statement of Theorem 1.

Proof of Proposition 1.

For $0 < \varepsilon < 1$ and $\delta > 0$, consider the function

$$(44) \quad f_\delta^\varepsilon(x) = \begin{cases} 1, & x \in \overline{B}(0, \varepsilon); \\ 1 - \frac{|x| - \varepsilon}{\delta}, & x \in \overline{B}(0, \varepsilon + \delta) \setminus \overline{B}(0, \varepsilon); \\ 0, & \mathbb{R}^d \setminus \overline{B}(0, \varepsilon + \delta). \end{cases}$$

Note that, for every $x \in \mathbb{R}^d$,

$$(45) \quad f_\delta^\varepsilon(x) \rightarrow \mathbf{1}_{\overline{B}(0, \varepsilon)}(x) \quad \text{as } \delta \rightarrow 0.$$

The function under the sign (\dots) in (40) can be represented in the form

$$(46) \quad \begin{aligned} I_{kn}(x, \lambda) := & \left(e^{-i(\lambda, \gamma_{kn}(x))} \int_{\mathbb{R}^d} (1 - f_\delta^\varepsilon(y)) \left[e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y)) \right] \Pi_{kn}(x, dy) - \right. \\ & \left. - \int_{\mathbb{R}^d} (1 - f_\delta^\varepsilon(y)) \left[e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y)) \right] \Pi(x, dy) \right) + \\ & + \left(e^{-i(\lambda, \gamma_{kn}(x))} \cdot \frac{i(\lambda, \gamma_{kn}(x) - \beta_{kn}(x))}{\Delta t_{kn}} - i(\lambda, a(x)) \right) + \\ & + \left(e^{-i(\lambda, \gamma_{kn}(x))} \int_{\mathbb{R}^d} f_\delta^\varepsilon(y) \left[e^{i(\lambda, y)} - 1 - i(\lambda, y) \right] \Pi_{kn}(x, dy) - \right. \\ & \left. - e^{-i(\lambda, \gamma_{kn}(x))} \frac{1}{\Delta t_{kn}} \left[e^{i(\lambda, \gamma_{kn}(x))} - 1 - i(\lambda, \gamma_{kn}(x)) \right] + \frac{1}{2} (B(x)\lambda, \lambda) \right) - \\ & - \left(\int_{\mathbb{R}^d} f_\delta^\varepsilon(y) \left[e^{i(\lambda, y)} - 1 - i(\lambda, y) \right] \Pi(x, dy) \right) =: \\ & =: J_{kn}(x, \lambda) + S_{kn}(x, \lambda) + U_{kn}^{\varepsilon, \delta}(x, \lambda) - U^{\varepsilon, \delta}(x, \lambda). \end{aligned}$$

Let us prove that for every $\lambda \in \mathbb{R}^d$

$$(47) \quad \max_{k \leq n} \sup_{|x| \leq R} |I_{kn}(x, \lambda)| \rightarrow 0, \quad n \rightarrow \infty.$$

By **(L3)** and (1), we have

$$(48) \quad \limsup_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \left| 1 - e^{-i(\lambda, \gamma_{kn}(x))} \right| = 0.$$

Note that $J_{kn}(x, \lambda)$ can be presented as follows:

$$(49) \quad \begin{aligned} & e^{-i(\lambda, \gamma_{kn}(x))} \int_{\mathbb{R}^d} (1 - f_\delta^\varepsilon(y)) \left(e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y)) \right) [\Pi_{kn}(x, dy) - \Pi(x, dy)] + \\ & + \left(e^{-i(\lambda, \gamma_{kn}(x))} - 1 \right) \int_{\mathbb{R}^d} (1 - f_\delta^\varepsilon(y)) \left[e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y)) \right] \Pi(x, dy). \end{aligned}$$

Using **(G1)** for the first term of (49) and (48) for the second one, we obtain

$$(50) \quad \max_{k \leq n} \sup_{|x| \leq R} |J_{kn}(x, \lambda)| \rightarrow 0, \quad n \rightarrow \infty, \quad \lambda \in \mathbb{R}^d.$$

Note that, for the integrand in (49), the following inequality holds:

$$\left| (1 - f_\delta^\varepsilon(y)) \left[e^{i(\lambda, y)} - 1 - i(\lambda, \tau(y)) \right] \right| \leq \mathbf{1}_{|y| > \varepsilon} \cdot F \cdot (1 \vee |\lambda|), \quad y \in \mathbb{R}^d.$$

Here and below, F stands for some positive constant. By **(L1)**,

$$\max_{k \leq n} \Pi_{kn}(x, \{|y| > \varepsilon\}) \leq D \frac{\varepsilon}{2R}, \quad |x| \leq R.$$

Hence, using **(G1)**, we obtain

$$(51) \quad \max_{k \leq n} \sup_{|x| \leq R} |J_{kn}(x, \lambda)| \leq F \cdot (1 \vee |\lambda|) D \frac{\varepsilon}{2R}.$$

Further, applying **(G2)** and (48) to the function $S_{kn}(x, \lambda)$, we obtain

$$(52) \quad \max_{k \leq n} \sup_{|x| \leq R} |S_{kn}(x, \lambda)| \rightarrow 0, \quad n \rightarrow \infty, \quad \lambda \in \mathbb{R}^d.$$

In addition, **(L3)** implies the inequality

$$(53) \quad \max_{k \leq n} \sup_{|x| \leq R} |S_{kn}(x, \lambda)| \leq F \cdot (R + 1) |\lambda|.$$

Recall the function $U_{kn}^{\varepsilon, \delta}(x, \lambda)$:

$$(54) \quad \begin{aligned} U_{kn}^{\varepsilon, \delta}(x, \lambda) &= e^{-i(\lambda, \gamma_{kn}(x))} \int_{\mathbb{R}^d} f_{\delta}^{\varepsilon}(y) \left[e^{i(\lambda, y)} - 1 - i(\lambda, y) \right] \Pi_{kn}(x, dy) - \\ &- e^{-i(\lambda, \gamma_{kn}(x))} \frac{1}{\Delta t_{kn}} \left[e^{i(\lambda, \gamma_{kn}(x))} - 1 - i(\lambda, \gamma_{kn}(x)) \right] + \frac{1}{2} (B(x)\lambda, \lambda). \end{aligned}$$

Adding and subtracting the expressions $\frac{(\lambda, y)^2}{2}$ and $\frac{(\lambda, \gamma_{kn}(x))^2}{2}$ from the terms under the sign [...], respectively, we obtain the following form of $U_{kn}^{\varepsilon}(x, \lambda)$:

$$(55) \quad \begin{aligned} e^{-i(\lambda, \gamma_{kn}(x))} \cdot \left(- \int_{\mathbb{R}^d} f_{\delta}^{\varepsilon}(y) \frac{(\lambda, y)^2}{2} \Pi_{kn}(x, dy) + \frac{(\lambda, \beta_{kn}(x))^2}{2\Delta t_{kn}} + \frac{1}{2} (B(x)\lambda, \lambda) \right) + \\ + \Gamma_{kn}^{\varepsilon, \delta}(x, \lambda). \end{aligned}$$

Here,

$$(56) \quad \begin{aligned} \Gamma_{kn}^{\varepsilon, \delta}(x, \lambda) &:= \left(1 - e^{-i(\lambda, \gamma_{kn}(x))} \right) \frac{1}{2} (B(x)\lambda, \lambda) + \\ &+ e^{-i(\lambda, \gamma_{kn}(x))} \left(\int_{\mathbb{R}^d} \varpi_{\varepsilon}^{\delta}(y, \lambda) \Pi_{kn}(x, dy) - \right. \\ &\left. - \frac{\omega_{kn}(x, \lambda)}{\Delta t_{kn}} + \frac{(\lambda, \gamma_{kn}(x))^2 - (\lambda, \beta_{kn}(x))^2}{2\Delta t_{kn}} \right), \\ \varpi_{\varepsilon}^{\delta}(y, \lambda) &:= f_{\delta}^{\varepsilon}(y) \left[e^{i(\lambda, y)} - 1 - i(\lambda, y) + \frac{(\lambda, y)^2}{2} \right] \quad \text{and} \\ \omega_{kn}(x, \lambda) &:= e^{i(\lambda, \gamma_{kn}(x))} - 1 - i(\lambda, \gamma_{kn}(x)) + \frac{(\lambda, \gamma_{kn}(x))^2}{2}. \end{aligned}$$

In addition, for some $F > 0$ and for all $\delta > 0$ and $\varepsilon > 0$

$$(57) \quad \begin{aligned} |\varpi_{\varepsilon}^{\delta}(y, \lambda)| &\leq F \cdot |\lambda|^3 |y|^3 \mathbf{1}_{|y| \leq \varepsilon + \delta}, \\ |\omega_{kn}(x, \lambda)| &\leq F \cdot |\lambda|^3 |\gamma_{kn}(x)|^3, \quad y \in \mathbb{R}^d, \quad |x| \leq R. \end{aligned}$$

Furthermore, by **(G2)**, there exists N such that, for all $n \geq N$,

$$(58) \quad (\lambda, \gamma_{kn}(x))^2 \leq (\lambda, \beta_{kn}(x))^2 + (\Delta t_{kn} \cdot (\lambda, a(x)))^2 + \Delta t_{kn} |\Upsilon_{kn}(x)| |\lambda|^2,$$

where

$$(59) \quad \max_{k \leq n} \sup_{|x| \leq R} \Upsilon_{kn}(x) \rightarrow 0, \quad n \rightarrow \infty.$$

Combining (56) with (48), (57), conditions **(G3)**, **(G3')**, and (58), we obtain

$$(60) \quad \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \lim_{\delta \rightarrow 0} \left| \Gamma_{kn}^{\varepsilon, \delta}(x, \lambda) \right| = 0, \quad \lambda \in \mathbb{R}^d.$$

Passing to the limit as $\delta \rightarrow 0+$ in the first term of (55) and using (45), by **(G1)** and (60), we obtain

$$(61) \quad \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \max_{k \leq n} \sup_{|x| \leq R} \left| U_{kn}^{\varepsilon, \delta}(x, \lambda) \right| = 0, \quad \lambda \in \mathbb{R}^d.$$

In addition, using (54), we obtain that there exists positive F such that, for all $k \leq n$, and $n \geq 1$,

$$(62) \quad \left| U_{kn}^{\varepsilon, \delta}(x, \lambda) \right| \leq F \cdot \left(\int_{|y| \leq \varepsilon + \delta} |\lambda|^2 |y|^2 \Pi_{kn}(x, dy) + \frac{1}{\Delta t_{kn}} |\lambda|^2 |\gamma_{kn}(x)|^2 + \|B(x)\| |\lambda|^2 \right), \quad |x| \leq R.$$

Hence, by **(L2)** and **(L3)**, we have

$$(63) \quad \max_{k \leq n} \sup_{|x| \leq R} \left| U_{kn}^{\varepsilon, \delta}(x, \lambda) \right| \leq F \cdot (R + 1)^2 |\lambda|^2.$$

Here, again, F is a positive constant.

Finally, using

$$\left| f_{\delta}^{\varepsilon}(y) \left[e^{i(\lambda, y)} - 1 - i(\lambda, y) \right] \right| \leq \mathbf{1}_{|y| \leq \varepsilon + \delta} |\lambda|^2 |y|^2,$$

(G1'), and (45), we obtain

$$(64) \quad \lim_{\varepsilon \rightarrow 0} \sup_{|x| \leq R} \lim_{\delta \rightarrow 0} \left| U^{\varepsilon, \delta}(x, \lambda) \right| = 0, \quad \lambda \in \mathbb{R}^d.$$

It follows from **(L2)** and **(G1)** that there exists some $F > 0$ such that

$$(65) \quad \sup_{|x| \leq R} \left| U^{\varepsilon, \delta}(x, \lambda) \right| \leq F(R + 1)^2 |\lambda|^2.$$

Combining (50), (52), (61), and (64), we obtain the needed convergence (47). In addition, from (51), (53), (63), and (65), we have

$$\sup_n \max_{k \leq n} \sup_{|x| \leq R} |I_{kn}(x, \lambda)| \leq P(\lambda),$$

where $P(\lambda)$ is some polynomial w.r.t. λ . Because of $\hat{f} \in \mathcal{S}$, the function $\hat{f} \cdot P(\lambda) \in \mathcal{S}$ is integrable on \mathbb{R}^d . Then, using representation (40), convergence (47), and the dominated convergence theorem, we obtain the statement of the proposition. \square

Proof of Proposition 2.

We will show that, for some n_0 ,

$$(66) \quad \sup_{n \geq n_0} \max_{k \leq n} \sup_{x \in \mathbb{R}^d} |g_{kn}(x)| < \infty;$$

this, together with Lemma 1 below, would provide the required statement. For a given $f \in \mathcal{D}$, take R_f such that

$$(67) \quad f(x) = 0, \quad f'(x) = 0, \quad f''(x) = 0, \quad |x| > R_f.$$

Combining (67) and (37), we obtain

$$(68) \quad g_{kn}(x) = \int_{\mathbb{R}^d} f(x + y - \gamma_{kn}(x)) \Pi_{kn}(x, dy) - \int_{\mathbb{R}^d} f(x + y) \Pi(x, dy), \quad |x| > R_f.$$

If $|x+y| > R_f$, then the term $f(x+y)$ under the second integral vanishes. If $|x+y| \leq R_f$ and $|y| \leq \frac{1}{2}(|x|+1)$, then

$$|x| \leq 2R_f + 1.$$

This means that, for $|x| > 2R_f + 1$, the second integral in (68) equals

$$\int_{|y| \geq \frac{1}{2}(|x|+1)} f(x+y)\Pi(x, dy).$$

From **(G1)** and **(L1)**, we obtain

$$(69) \quad \left| \int_{\mathbb{R}^d} f(x+y)\Pi(x, dy) \right| \leq \sup_{x \in \mathbb{R}^d} |f(x)| \cdot D_{\frac{1}{2}}.$$

A similar argument can be applied to the first integral in (68). Using **(L3)**, we obtain that there exists n_0 such that $|\gamma_{kn}(x)| \leq \frac{1}{4}(|x|+1)$ for $n \geq n_0$. Then, for $|x| > 4R_f + 3$, the first integral in (68) equals

$$\int_{|y| \geq \frac{1}{2}(|x|+1)} f(x+y-\gamma_{kn}(x))\Pi_{kn}(x, dy).$$

Using **(L1)**, we obtain

$$(70) \quad \left| \int_{\mathbb{R}^d} f(x+y-\gamma_{kn}(x))\Pi_{kn}(x, dy) \right| \leq \sup_{x \in \mathbb{R}^d} |f(x)| \cdot D_{\frac{1}{2}}.$$

□

APPENDIX A. COMPACT CONTAINMENT CONDITION

Lemma 1. *Let $\sup_n |X_0^n| < \infty$ a.s., and let conditions **(L1)** – **(L3)** be fulfilled. Then*

$$\lim_{R \rightarrow \infty} \liminf_{n \rightarrow \infty} P \left(\max_{k \leq n} |X_k^n| \leq R \right) = 1.$$

The proof.

The proof consists of three steps.

Step I. Fix any $r > 1$. For all n , denote, by τ_n^r , a Markov time w.r.t. $\{\mathfrak{F}_k^n, k \leq n\}$. We have

$$(71) \quad \tau_n^r = \inf \{k \leq n : |\xi_{1n}| \leq r(|X_0^n| + 1), \dots, |\xi_{k-1n}| \leq r(|X_{k-2}^n| + 1), \\ |\xi_{kn}| > r(|X_{k-1}^n| + 1)\},$$

if the set is nonempty, and $\tau_n^r = \infty$ otherwise. Put $\tilde{\xi}_{kn} = \xi_{kn} \cdot \mathbf{1}_{\{\tau_n^r > k\}}$. We denote

$$\tilde{X}_k^n = \sum_{k=1}^n \tilde{\xi}_{kn}.$$

Note that

$$(72) \quad P(\max_{k \leq n} |X_k^n| \leq R) \geq P \left(\{\max_{k \leq n} |X_k^n| \leq R\} \cap \{\tau_n^r = \infty\} \right).$$

Clearly, when $\tau_n^r = \infty$, we have $X_k^n = \tilde{X}_k^n$. Consequently, the probability on the r.h.s. of (72) is equal to

$$(73) \quad P \left(\{\max_{k \leq n} |\tilde{X}_k^n| \leq R\} \cap \{\tau_n^r = \infty\} \right).$$

Combining (73) and the fact that $P(A \cap B) \geq P(A) + P(B) - 1$, we obtain the following estimate for $P(\max_{k \leq n} |X_k^n| \leq R)$:

$$(74) \quad P(\max_{k \leq n} |X_k^n| \leq R) \geq P \left(\max_{k \leq n} |\tilde{X}_k^n| \leq R \right) + P(\tau_n^r = \infty) - 1.$$

We claim that

$$(75) \quad \lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} P(\tau_n^r = \infty) = 1.$$

Indeed, $\{\tau_n^r = \infty\} = \bigcap_{k \leq n} \{|\xi_{kn}| \leq r(|X_{k-1}^n| + 1)\}$. Then, to obtain the lower bound for $P(\tau_n^r = \infty)$, it is sufficient to obtain the upper bound for $E_{k-1} \mathbf{1}_{|\xi_{kn}| > r(|X_{k-1}^n| + 1)}$. By **(L3)** and (1), we obtain $|\gamma_{kn}(x)| \leq c(|x| + 1)$, $n \geq 1$, where $c = \sqrt{KS}$. Hence, using **(L1)** and (1), we have

$$(76) \quad E_{k-1} \mathbf{1}_{|\xi_{kn}| > r(|X_{k-1}^n| + 1)} \leq \frac{SD_{r+c}}{n}.$$

Consequently, we obtain

$$(77) \quad P(\tau_n^r = \infty) \geq \left(1 - \frac{SD_{r+c}}{n}\right)^n.$$

Thus, (75) holds.

Let us take (74) into account. To obtain the statement of the lemma, it is sufficient to show that

$$(78) \quad \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left(\max_{k \leq n} |\tilde{X}_k^n| > R\right) = 0.$$

Step II. The proof of (78).

Denote

$$\psi_{k-1n} = E_{k-1} \tilde{\xi}_{kn}, \quad M_k^n = \tilde{X}_k^n - \sum_{i=1}^k \psi_{i-1n},$$

and notice that M_k^n is a martingale w.r.t. \mathfrak{F}_k^n . Using Doob's maximum inequality for the martingale part of \tilde{X}_k^n and Chebyshev's inequality for the predictable part of \tilde{X}_k^n , we obtain

$$P\left(\max_{k \leq n} |\tilde{X}_k^n| > R\right) \leq \frac{F}{R^2} \left(E(M_n^n)^2 + E\left(\sum_{k=1}^n |\psi_{k-1n}|\right)^2 \right).$$

Here and below, F is some positive constant. Note that

$$(M_n^n)^2 = \left(\tilde{X}_n^n - \sum_{k=1}^n \psi_{(k-1)n}\right)^2 \leq 2\left(\tilde{X}_n^n\right)^2 + 2\left(\sum_{k=1}^n |\psi_{(k-1)n}|\right)^2.$$

Using the inequality $(a_1 + \dots + a_n)^2 \leq n \cdot (a_1^2 + \dots + a_n^2)$, we get

$$(79) \quad P\left(\max_{k \leq n} |\tilde{X}_k^n| > R\right) \leq \frac{F}{R^2} \left(E\left(\tilde{X}_n^n\right)^2 + n \cdot \sum_{k=1}^n E\psi_{k-1n}^2 \right).$$

We will show in **Step III** below that there exists n_0 such that, for $n \geq n_0$, the following estimates hold for ψ_{k-1n} and $E\left(\tilde{X}_n^n\right)^2$:

$$(80) \quad E\left(\tilde{X}_k^n\right)^2 \leq L_2(r) \left(1 + \frac{L_1(r)}{n}\right)^k, \quad k \leq n,$$

$$(81) \quad E\psi_{k-1n}^2 \leq \frac{L_3(r)}{n^2} \left(1 + \frac{L_1(r)}{n}\right)^k, \quad k \leq n,$$

where $L_i(r)$, $i = 1, 2, 3$ are positive constants.

Using (79), (80), and (81), we obtain (78).

Step III. The proof of inequalities (80) and (81).

In the above notation,

$$\psi_{k-1n} = E_{k-1} \tilde{\xi}_{kn} = \mathbf{1}_{\{\tau_n^r > k-1\}} E_{k-1} \xi_{kn} \mathbf{1}_{|\xi_{kn}| \leq r(|X_{k-1}^n|+1)}.$$

Using (27), we get that there exists n_0 such that for $n \geq n_0$

$$(82) \quad |\psi_{k-1n}| \leq \mathbf{1}_{\{\tau_n^r > k-1\}} \frac{L(r)}{n} (|X_{k-1}^n| + 1),$$

where $L(r) := K + D_{\frac{1}{2}} + (r+1)D_{\frac{1}{2}} + (r+1) \cdot D$. Because $|X_{k-1}^n \mathbf{1}_{\{\tau_n^r > k-1\}}| \leq |\tilde{X}_{k-1}^n|$, we obtain

$$(83) \quad |\psi_{k-1n}| \leq \frac{L(r)}{n} (|\tilde{X}_{k-1}^n| + 1).$$

To prove (80), we note that

$$E|\tilde{X}_k^n - \tilde{X}_{k-1}^n|^2 = E|\tilde{X}_k^n|^2 - E|\tilde{X}_{k-1}^n|^2 - 2E|\psi_{k-1n}| \cdot |\tilde{X}_{k-1}^n|.$$

Using (83), we have

$$(84) \quad E|\psi_{k-1n}| \cdot |\tilde{X}_{k-1}^n| \leq 2 \frac{L(r)}{n} (E|\tilde{X}_{k-1}^n|^2 + 1).$$

On the other hand,

$$\begin{aligned} E|\tilde{X}_k^n - \tilde{X}_{k-1}^n|^2 &= EE_{k-1}(\tilde{\xi}_{kn})^2 = \\ &= E\mathbf{1}_{\{\tau_n^r > k-1\}} E_{k-1} \xi_{kn}^2 \mathbf{1}_{|\xi_{kn}| \leq r(|X_{k-1}^n|+1)}. \end{aligned}$$

Using (18), we get

$$(85) \quad E|\tilde{X}_k^n - \tilde{X}_{k-1}^n|^2 \leq \frac{1}{n} [L_0(r)E|\tilde{X}_{k-1}^n|^2 + L_0(r)],$$

where $L_0(r) = C_{r+1} + K$. Combining this with (84), we have

$$(86) \quad E|\tilde{X}_k^n|^2 \leq \left(1 + \frac{L_1(r)}{n}\right) E|\tilde{X}_{k-1}^n|^2 + \frac{L_1(r)}{n}.$$

Iterating the estimate in (86), we get

$$(87) \quad E|\tilde{X}_k^n|^2 \leq L_2(r) \left(1 + \frac{L_1(r)}{n}\right)^k, \quad k \leq n,$$

where $L_i(r), i = 1, 2$ are positive constants. Inequality (81) follows from (83) and (87). \square

Acknowledgements

I would like to express my deep gratitude to Dr. A.M. Kulik, my research supervisor, for his helpful and constructive recommendations on this paper. I would also like to thank Dr. A.Yu. Pilipenko for his useful advices and critique of this work.

REFERENCES

1. I. I. Gikhman, A. V. Skorokhod, *Introduction to the Theory of Stochastic Processes*, Dover, New York, 1996.
2. B. V. Gnedenko, A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Cambridge, MA, 1954.
3. I. I. Gikhman, A. V. Skorokhod, *Stochastic Differential Equations and Their Applications*, Springer, Berlin, 1972.
4. V. S. Koroliuk, N. Limnios, I. V. Samoilenko, *Lévy approximation of impulsive recurrent process with semi-Markov switching*, Theory of Probability and Mathematical Statistics (2009), no. 80, 85–92.
5. R. F. Bass, *Uniqueness in law for pure jump Markov processes*, Probab. Th. Rel. Fields (1988), no. 79, 271–287.

6. S. N. Ethier, T. G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley, New York, 1986.

CHAIR OF PROBABILITY THEORY, STATISTICS AND ACTUARIAL MATHEMATICS, FACULTY OF MECHANICS AND MATHEMATICS, TARAS SHEVCHENKO NATIONAL UNIVERSITY OF KYIV, 6, GLUSHKOV AVE., KYIV 03127, UKRAINE

E-mail address: tanya.kosenkova@gmail.com