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# LARGE DEVIATIONS FOR ONE-DIMENSIONAL SDE WITH DISCONTINUOUS DIFFUSION COEFFICIENT

Large deviation principle is established for a family of solutions to one-dimensional SDE's under the condition that the set of discontinuity points of the diffusion coefficient has zero Lebesgue measure.

## 1. Introduction

In this paper, we establish the *large deviations principle* (LDP) for a family $X^\varepsilon$ of solutions to one-dimensional stochastic differential equations (SDE's) of the form

$$dX_t^\varepsilon = \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dW_t, \quad X_0^\varepsilon = x_0, \quad \varepsilon > 0 \tag{1}$$

under weak assumptions on the diffusion coefficient $\sigma$. LDP for general diffusion processes with continuous coefficients is a well-known result, see [1]. On the other hand, there is a substantial interest in studying the same problem for SDE's with coefficients which allow discontinuities of a certain type, since such SDE's arise naturally, when one is interested in models of diffusive motion in highly inhomogeneous media. This problem was addressed to in [2], [3], and [4], where multidimensional SDE's were considered with coefficients, which have a discontinuity of the jump type along a fixed hyperplane.

Here, we consider the same problem from another side and prove LDP for the family of solutions to (1) under the condition on the diffusion coefficient $\sigma$, which seems to be close to the weakest possible one: in our main result, Theorem 1 below, the main assumption is that the set $\Delta_\sigma$ of discontinuity points of $\sigma$ has zero Lebesgue measure. Clearly, this condition is much weaker than those from [2], [3], or [4], which mean for (1) just that $\sigma$ has a discontinuity of the jump type at a single point. On the other hand, our model is more restrictive than those studied in [2], [3], and [4].

Let us outline the main idea of our approach. Using the *change-of-time* transformation (see, e.g., [7]), one can provide weak solutions to (1) in the form

$$X^\varepsilon = F(\sqrt{\varepsilon}\tilde{W}) \tag{2}$$

with a certain mapping $F : \mathcal{C}([0,\infty)) \to \mathcal{C}([0,\infty))$ and a Wiener process $\tilde{W}$; see Section 3.1 below. It can be shown easily that, if $\sigma$ is continuous, then $F$ is a continuous mapping w.r.t. the standard topology in $\mathcal{C}([0,\infty))$, which corresponds to the uniform convergence on compact sets. In that case, LDP follows directly from LDP for the Wiener process and the so-called *contraction principle* (see, e.g., [5]). In work [6], a pair of *semicontraction principles* was introduced, which allow one to provide separately the corresponding upper and lower bounds for large deviations without the continuity assumptions on the mapping $F$. In the case where the rate functions in these two bounds coincide, this yields LDP; that is, it is possible to give extension of the contraction principle for certain discontinuous mappings. In this paper, we will show that this extension is substantial and allows one to control the large deviation asymptotics for the family of solutions to (1) with a highly discontinuous diffusion coefficient $\sigma$.

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## 2. Main theorem

Let $(S, d)$ be a Polish space, and let $X^\varepsilon, \varepsilon > 0$ be random elements taking values in $S$. Recall some standard definitions (see, e.g., [5].

**Definition 1.** The family $\{X^\varepsilon\}$ satisfies *LDP* with the *rate function* $I : S \to [0, \infty]$, if, for every open set $G$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log P\{X^\varepsilon \in G\} \geq -\inf_{x\in G} I(x), \tag{3}$$

and, for every closed set $F$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log P\{X^\varepsilon \in F\} \leq -\inf_{x\in F} I(x). \tag{4}$$

The rate function $I$ is called *good*, if, for every $a \in [0, \infty)$, the set $\{x : I(x) \leq a\}$ is compact.

If (3) holds, and (4) holds true for every compact set only, then the family $\{X^\varepsilon\}$ is said to satisfy *weak LDP*.

The main result of this paper is given by the following theorem. Let $S = \mathcal{C}([0, \infty))$ with the metric

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \sup_{t\in[0,k]} |f(t) - g(t)| \wedge 1 \right). \tag{5}$$

Let $X^\varepsilon, \varepsilon > 0$ be the family of random elements in $\mathcal{C}([0, \infty))$ corresponding to the weak solutions to (1).

**Theorem 1.** *Let $\sigma$ be measurable and such that, for some positive $c_1, c_2$,*

$$c_1 \leq \sigma^2(x) \leq c_2, \quad x \in \mathbb{R}.$$

*Assume also that the set $\Delta_\sigma$ of discontinuity points of $\sigma$ has zero Lebesgue measure.*

*Then the family $X^\varepsilon, \varepsilon > 0$ satisfies LDP with a good rate function $J$, which equals*

$$J(g) = \frac{1}{2} \int_0^\infty \frac{(\dot{g}(t))^2}{\sigma^2(g(t))} dt \tag{6}$$

*if $g \in \mathcal{C}([0, \infty))$ is an absolutely continuous function with $g(0) = x_0$, $\dot{g} \in L_2([0, \infty))$, and $J(g) = \infty$ otherwise.*

We prove Theorem 1 in Section 4. Before that, we give some auxiliary constructions and statements in Section 3.

## 3. Auxiliary Constructions and Statements

3.1. **Weak solutions to Eqs. (1).** In this section, we provide representation (2) for weak solutions to Eqs. (1), by using the following statement (see Example 4.2 in [7]).

**Proposition 1.** *Let a Borel measurable function $\alpha : \mathbb{R} \to \mathbb{R}$ be bounded and such that $\alpha^2(x) \geq C, x \in \mathbb{R}$ for some positive constant $C$. For some Wiener process $B$, define the process*

$$\phi_t(B) = \int_0^t \alpha^{-2}(x + B_s) ds,$$

*and denote the function inverse to $\phi$ w.r.t. the time variable by $\phi^{-1}$. Put*

$$X_t = B_{\phi_t^{-1}} + x.$$

*Then $X_t$ satisfies SDE*

$$dX_t = \alpha(X_t) dW_t \tag{7}$$

*with the initial condition $X_0 = x$ and some Wiener process $W$.*

In our settings, put $\alpha(x)=\sqrt{\varepsilon}\sigma(x)$. Then the respective process $\phi_t$ has the form

$$\begin{aligned}\phi_t(B)&=\varepsilon^{-1}\int_0^t\sigma^{-2}(x_0+B_s)ds=\int_0^t\sigma^{-2}(x_0+B_s)d\frac{s}{\varepsilon}=|p=\frac{s}{\varepsilon}|\\ &=\int_0^{t/\varepsilon}\sigma^{-2}(x_0+B_{p\varepsilon})dp=\int_0^{t/\varepsilon}\sigma^{-2}(x_0+\sqrt{\varepsilon}\widetilde{W}^\varepsilon_p)dp,\quad \widetilde{W}^\varepsilon_p=\frac{1}{\sqrt{\varepsilon}}B_{p\varepsilon}.\end{aligned}\tag{8}$$

Note that $\widetilde{W}^\varepsilon$ is a Wiener process. Define the transformation $\eta:\mathcal{C}([0,\infty))\to\mathcal{C}([0,\infty))$ as follows:

$$[\eta(f)](t)=\int_0^t\sigma^{-2}(x_0+f(s))ds,\quad t\in[0,\infty),\quad f\in\mathcal{C}([0,\infty)).\tag{9}$$

To simplify the notation, we write $\eta_t(f)=[\eta(f)](t)$ in what follows. In this notation, identity (8) takes the form $\phi_t(B)=\eta_{t/\varepsilon}(\sqrt{\varepsilon}\widetilde{W}^\varepsilon)$. Hence, by Proposition 1, the process

$$X^\varepsilon_t=B_{\phi_t^{-1}}+x_0=\sqrt{\varepsilon}\widetilde{W}^\varepsilon_{\tau^\varepsilon_t}+x_0,\quad \tau^\varepsilon_t:=\eta_t^{-1}(\sqrt{\varepsilon}\widetilde{W}^\varepsilon)$$

for every $\varepsilon>0$ is a weak solution to (1) with this $\varepsilon$. Therefore, formula (2) with the function $F:\mathcal{C}([0,\infty))\to\mathcal{C}([0,\infty))$ defined by

$$[F(f)](t)=f(\eta_t^{-1}(f))+x_0,\quad t\ge 0\tag{10}$$

provides a weak solution to (1). Here, it is irrelevant whether to write $\widetilde{W}^\varepsilon$ or $\widetilde{W}$, because we are interested in the weak solution to (1), only.

It is well known that, for a Wiener process $\tilde{W}$, the respective family $\{\sqrt{\varepsilon}\tilde{W}\}$ satisfies LDP with a good rate function $I$, which equals

$$I(f)=\frac{1}{2}\int_0^\infty(\dot{f}(s))^2ds$$

if $f\in\mathcal{C}([0,\infty))$ is an absolutely continuous function with $f(0)=0$, $\dot{f}\in L_2([0,\infty))$, and $I(f)=\infty$ otherwise. For the Wiener process defined on $[0,T]$ and the respective family of random elements in $\mathcal{C}([0,T])$, a similar statement can be found in Chapter 3, §2 [1]; the proof therein can be extended easily to the case of the Wiener process defined on $[0,\infty)$.

3.2. **Semicontraction principles.** In this section, we recall some constructions and statements from work [6], which will be used below.

Let the family $X^\varepsilon,\varepsilon>0$ of random elements taking values in $S$ satisfy LDP with a good rate function $I$. Let $S'$ be another Polish space, and let $F:S\to S'$ be some Borel measurable function. Consider the family $Y^\varepsilon=F(X^\varepsilon),\varepsilon>0$. Denote, by $U_F$, the set of continuity points for the function $F$ and assume that the set $\Delta_F=S\backslash U_F$ of discontinuity points for $F$ is negligible in the sense that

$$\limsup_{\varepsilon\to0}\varepsilon\log P(X^\varepsilon\in\Delta_F)=-\infty.\tag{11}$$

For arbitrary $\delta>0$, define

$$\Xi_\delta(y)=\{x\in S:\exists\tilde{x}\in U_F\text{ such that }d(x,\tilde{x})<\delta,d'(F(\tilde{x}),y)<\delta\},$$

$$\Theta_\delta(y)=\{x\in S:\exists\tilde{x}\in U_F\text{ such that }d(x,\tilde{x})<\delta,d'(F(\tilde{x}),y)<\delta,I(\tilde{x})\le I(x)+\delta\},$$

where $d,d'$ are respective metrics in $S,S'$, and put

$$\overline{J}_\delta(y)=\inf_{x\in\Xi_\delta(y)}I(x),\ \overline{J}(y)=\lim_{\delta\to0}\overline{J}_\delta(y),\ \underline{J}_\delta(y)=\inf_{x\in\Theta_\delta(y)}I(x),\ \underline{J}(y)=\lim_{\delta\to0}\underline{J}_\delta(y);$$

here, the limits w.r.t. $\delta$ exist because the families of sets $\{\Xi_\delta(y),\delta>0\},\{\Theta_\delta(y),\delta>0\}$ are monotonous.

Note that the functions $\overline{J}, \underline{J}$ defined above are lower semicontinuous (see Proposition 3 in [6]).

**Proposition 2.** *(Semicontraction principles;* [6]*, Lemma 4).*

(1) *For the family $Y^\varepsilon, \varepsilon > 0$, the lower bound (3) holds true with the rate function $\underline{J}$ for every open set $A \subset S'$.*
(2) *For the family $Y^\varepsilon, \varepsilon > 0$, the upper bound (4) holds true with the rate function $\overline{J}$ for every compact set $B \subset S'$.*

Under the additional assumptions on the functions $\underline{J}$, $\overline{J}$, and $F$, the above statement provides LDP for the family $Y^\varepsilon, \varepsilon > 0$ either in the weak or standard form.

**Proposition 3.** *(*[6]*, Theorem 5).*

(1) *If $\underline{J} = \overline{J} \equiv J$, then the family $Y^\varepsilon, \varepsilon > 0$ satisfies the weak LDP with the rate function $J$.*
(2) *Let, in addition, the function satisfy the following condition:*

$$\text{for every compact set } K \subset S, \text{ the closure of the set } F(U_F \bigcap K) \text{ is compact.} \tag{12}$$

*Then the family $Y^\varepsilon, \varepsilon > 0$ satisfies LDP with the good rate function $J$.*

## 4. Proof of Theorem 1

We put $S = S' = \mathcal{C}([0, \infty))$ with the metric $d = d'$ defined by (5) and apply Proposition 3 with $X^\varepsilon$ instead of $Y^\varepsilon$, $\sqrt{\varepsilon}\tilde{W}$ instead of $X^\varepsilon$, and $F$ defined by (10). To do that, we need to verify condition (11), condition (12), and the identity

$$\underline{J}(g) = \overline{J}(g) = J(g), \quad g \in S \tag{13}$$

with $J$ defined in the statement of Theorem 1; here and below, we write $f, g$, etc. instead of $x, y$, etc.

The main difficulty in the proof is represented by identity (13). We separate the respective proof and show firstly the inequality $\underline{J}(g) \leq J(g)$, and then the inequality $\overline{J}(g) \geq J(g)$. By the definition of the functions $\underline{J}, \overline{J}$, we have $\underline{J} \geq \overline{J}$, which would yield (13).

### 4.1. Auxiliary statements.

In this section, we provide auxiliary statements, which gives one an opportunity to construct the continuity points for the function $F$. Since the set $U_F$ of continuity points is involved substantially in the construction of the functions $\underline{J}, \overline{J}$, such statements will be crucial for the proof of identity (13).

For given $x_0 \in \mathbb{R}$ and the function $\sigma$, we define a transformation $\eta : S \to S$ by formula (9), and a transformation $\pi : S \to S$ by

$$[\pi(f)](t) = \eta_t^{-1}(f), \quad f \in S,$$

where the inverse function is taken w.r.t. the time argument; that is, $[\pi(f)](t) = u \iff \eta_u(f) = t$. In what follows, we denote $\pi_t(f) = [\pi(f)](t)$.

**Lemma 1.** *Let a function $f \in S$ be such that*

$$\lambda\{s : f(s) \in \Delta_\sigma\} = 0, \tag{14}$$

*where $\lambda$ is the Lebesgue measure on $\mathbb{R}^+$.*

*Then both the transformations $\eta$ and $\pi$ are continuous at the point $f$.*

*Proof.* Consider an arbitrary sequence $f_n \to f$ in $S$. Then the sequence of functions $f_n, n \geq 1$ converge to $f$ point-wise. Consequently, for every $s$ such that $f(s) \notin \Delta_\sigma$, we have $\sigma^{-2}(f_n(s)) \to \sigma^{-2}(f(s))$, $n \to \infty$. Then, by the dominated convergence theorem, we have, for every $t > 0$,

$$\int_0^t \left|\sigma^{-2}(x_0 + f_n(s)) - \sigma^{-2}(x_0 + f(s))\right| ds \to 0, \quad n \to \infty,$$

which clearly yields $\eta(f_n) \to \eta(f), n \to \infty$ in $S$. This proves the continuity of the transformation $\eta$ at the point $f$.

Now, let us prove the continuity of the transformation $\pi$. Assume the contrary, i.e. let there exist a sequence $f_n \to f$ in $S$ such that, for some $T > 0$,

$$\sup_{t\in[0,T]} |\pi_t(f) - \pi_t(f_n)| \not\to 0.$$

Choosing, if necessary, a subsequence, we can assume without loss of generality that there exists a sequence $t_n \to t \in [0, T]$ such that

$$s_n := \pi_{t_n}(f) \to s, \quad s'_n := \pi_{t_n}(f_n) \to s', \quad \text{and} \quad s \neq s'.$$

Because $\eta$ is continuous at the point $f$ and

$$t_n = \eta_{s_n}(f) = \eta_{s'_n}(f_n),$$

we pass to the limit as $n \to \infty$ and obtain $t = \eta_s(f) = \eta_{s'}(f)$, which yields

$$\int_s^{s'} \sigma^{-2}(x_0 + f(r))dr = 0.$$

Since $s \neq s'$, and $\sigma^{-2}$ is positive, this brings a contradiction, which proves the required statement. □

**Corollary 1.** *Under the conditions of Lemma 1, the transformation $F$ defined by (10) is continuous at the point $f \in S$.*

In the following lemma, we give a sufficient condition for (14) in the terms of the derivative of the function $f$.

**Lemma 2.** *Let a function $f$ be absolutely continuous, and let its derivative $\dot{f}$, which is well defined $\lambda$-a.e., be square integrable and satisfy the condition*

$$\lambda\{s : \dot{f}(s) = 0\} = 0. \tag{15}$$

*Then the measure*

$$\lambda_f(A) = \lambda\{s : f(s) \in A\}, \quad A \in \mathcal{B}(\mathbb{R})$$

*is absolutely continuous w.r.t. the Lebesgue measure. In particular, (14) holds true, because the set $\Delta_\sigma$ has zero Lebesgue measure.*

*Proof.* For a given measurable set $E \subset [0, +\infty)$ and $v \in \mathbb{R}$, denote, by $N(f, E, v)$, the number of the points at the sets $\{x \in E : f(x) = v\}$. The function $f$ belongs locally to the Sobolev class $W^1_p$ with $p = 2$. Hence, we can apply Corollary 2.4.4 in [8] (note that this corollary requires $f \in W^1_{p,loc}$ with some $p$ greater than 1) and get that, for (the continuous modification of) the function $f$, the function $N(f, E, \cdot)$ is measurable; in addition, for every bounded measurable $g$ on $\mathbb{R}$,

$$\int_E g(f(s))|\dot{f}(s)|ds = \int_{\mathbb{R}} g(v)N(f, E, v)dv. \tag{16}$$

Let $A$ be a set of zero Lebesgue measure. Put $g(v) = \mathcal{I}_A(v)$, the indicator function of this set. Then the right-hand side part of (16) equals 0. On the other hand,

$$g(f(s)) = \mathcal{I}_{f(s)\in A} = \mathcal{I}_{f^{-1}(A)}(s).$$

Hence, if we put $E = [0, \infty)$, (16) yields

$$\int_0^\infty \mathcal{I}_{f^{-1}(A)}(s)|\dot{f}(s)|ds = 0.$$

Because $|\dot{f}(s)| > 0$ $\lambda$-a.e., we obtain

$$\lambda_f(A) = \int_0^\infty \mathcal{I}_{f^{-1}(A)}(s)ds = 0,$$

which means that the measure $\lambda_f$ is absolutely continuous. □

Denote, for $f \in S$,

$$\zeta_t(f) = \int_0^t \sigma^2(f(s))ds$$

and write $\zeta_t^{-1}(f)$ for the function inverse to $\zeta_t(f)$ w.r.t. the time variable.

**Lemma 3.** *Let $f, d \in S$ be such that*

$$x_0 + f(t) = g(\zeta_t^{-1}(g)), \quad t \geq 0. \tag{17}$$

*Then*

$$\pi_t(f) = \zeta_t(g), \quad t \geq 0.$$

*Proof.* Since $\pi(f)$ is the inverse transformation for $\eta(f)$ w.r.t. the time variable, the required statement is equivalent to the following:

$$\eta_{\zeta_t(g)}(f) = t, \quad t \geq 0.$$

The latter relation can be obtained straightforwardly:

$$\eta_{\zeta_t(g)}(f) = \int_0^{\zeta_t(g)} \sigma^{-2}(x_0 + f(s))ds = |s = \zeta_u(g), ds = \sigma^2(g(u))du| =$$

$$= \int_0^t \sigma^{-2}(x_0 + f(\zeta_u(g)))\sigma^2(g(u))du = t.$$

In the last identity, we have used (17) with $\zeta_u(g)$ instead of $t$. □

4.2. **Proof of the inequality $\underline{J}(g) \leq J(g)$.** When $J(g) = \infty$, the required statement is trivial. Henceforth, we assume in this section that $J(g) < \infty$. Therefore, $g \in S$ is an absolutely continuous function with $g(0) = x_0$ and $\dot{g} \in L_2([0, \infty))$.

Fix some absolutely continuous function $\chi$ with $\chi(0) = 0$ and $\dot{\chi} \in L_2((0, \infty))$ such that $\dot{\chi} \neq 0$ $\lambda$-a.e. Put $g^a(t) = g(t) + a\chi(t), a > 0$. Then, for every $a \neq b$, the sets $\{\dot{g}^a = 0\}$ and $\{\dot{g}^b = 0\}$ do not have common points at the set $\{\dot{\chi} \neq 0\}$. Therefore, there exists the at most countable set $A_{g,\chi} \subset \mathbb{R}$ such that, for any $a \notin A_{g,\chi}$, the respective function $g^a$ satisfies (15). We fix a sequence $a_n \to 0$ with $a_n \notin A_{g,\chi}$ and write $g_n = g^{a_n} = g + a_n\chi$.

Clearly, every function $g_n$ satisfies (15) by the construction; in addition, we have $\dot{g}_n \to \dot{g}, n \to \infty$ in $L_2((0, \infty))$ and $g_n(t) \to g(t), n \to \infty$ for every $t$.

Let us construct the sequence $f_n \in S$ such that $F(f_n) = g_n$. To do that, we put

$$f_n(t) = g_n(\zeta_t^{-1}(g_n)) - x_0.$$

Then, by Lemma 3 and (10),

$$g_n(t) = x_0 + f_n(\zeta_t(g_n)) = x_0 + f_n(\pi_t(f_n)) = x_0 + f_n(\eta_t^{-1}(f_n)) = [F(f_n)](t), \quad t \geq 0.$$

We have $f_n(t) = g_n(\eta_t(f_n)) - x_0$, and $\eta_\cdot(g_n)$ is a Lipschitz function w.r.t. the time variable, because it is defined as an integral with bounded integrand. Because every $g_n$ is an absolutely continuous function, this means that every $f_n$ is absolutely continuous, as well. Next, one can see easily that the derivative of $f_n$ (recall that this derivative is well defined for $\lambda$-a.a. $t$) satisfies

$$\dot{f}_n(t) = \frac{\dot{g_n}(\eta_t(f_n))}{\sigma^2(g_n(\eta_t(f_n)))} \quad \text{for } \lambda\text{-a.a. } t. \tag{18}$$

Indeed, using the formula for the derivative of the inverse function, one can write

$$\begin{aligned}\dot{g_n}(t) &= \dot{f}_n(\eta_t^{-1}(f_n))\frac{d}{dt}[\eta_t^{-1}(f_n)] = \dot{f}_n(\eta_t^{-1}(f_n))\left(\left(\frac{d}{dt}\eta_t(f_n)\right)|_{t=\eta_t^{-1}(f_n)}\right)^{-1}\\ &= \dot{f}_n(\eta_t^{-1}(f_n))\sigma^2(f_n(\eta_t^{-1}(f_n)) = \dot{f}_n(\eta_t^{-1}(f_n))\sigma^2(g_n(t)).\end{aligned}$$

This means that, for $\lambda$-a.a. $t$,

$$\dot{f}(\eta_t^{-1}(f)) = \frac{\dot{g}(t)}{\sigma^2(g(t))}.$$

This is equivalent to (18), because both the function $\eta.(f_n) : [0,\infty) \to [0,\infty)$ and its inverse one $\pi.(f_n) = \zeta.(g_n)$ are Lipschitz and, therefore, transform the Lebesgue measure into an absolutely continuous one.

It follows from (18) that every $f_n$ satisfies (15). Then $f_n$ satisfies (14) by Lemma 2, and $f_n$ is a continuity point for the mapping $F$ by Lemma 1.

On the other hand, $f_n(0) = 0$ by construction, because $g_n(0) = x_0$. Using (18) once again, we obtain

$$\begin{aligned}I(f_n) &= \frac{1}{2}\int_0^\infty (\dot{f}_n(t))^2 dt = \frac{1}{2}\int_0^\infty \frac{(\dot{g_n}(\eta_t(f_n)))^2}{\sigma^4(g_n(\eta_t(f_n)))}dt\\ &= \left|s = \eta_t(f_n), ds = \sigma^{-2}(x_0 + f_n(t))dt\right|\\ &= \frac{1}{2}\int_0^\infty \frac{(\dot{g_n}(s))^2}{\sigma^2(g_n(s))}ds = J(y_n).\end{aligned}$$

In what follows, we study the limit behavior of the integrals $I(f_n) = J(g_n)$. We begin with the following simple auxiliary statement.

**Lemma 4.** *Let $h_n \to h$ in $L_2((0,\infty))$. For a uniformly bounded sequence of functions $\{\theta_n(t)\}$ such that $\theta_n(t) \to \theta(t), t \in A$ for a given measurable set $A$,*

$$\int_A h_n^2(t)\theta_n(t)dt \to \int_A h^2(t)\theta(t)dt, \quad n \to \infty.$$

*Proof* follows directly from the dominated convergence theorem.

**Lemma 5.** $\lim_{n\to\infty} J(g_n) = J(g)$.

*Proof.* Divide the integration interval $(0,\infty)$ into three sets

$$A = \{t : y(t) \notin \Delta_\sigma\}, \quad B = \{t : y(t) \in \Delta_\sigma, \dot{y}(t) = 0\}, \quad C = \{t : y(t) \in \Delta_\sigma, \dot{y}(t) \neq 0\}.$$

Put $\theta_n(t) = \sigma^{-2}(g_n(t))$, $\theta(t) = \sigma^{-2}(g(t))$, $h_n(t) = \dot{g}_n(t)$, and $h(t) = \dot{g}(t)$ in the previous lemma. Then

$$\int_A \frac{(\dot{g}_n(t))^2}{\sigma^2(g_n(t))}dt \to \int_A \frac{(\dot{g}(t))^2}{\sigma^2(g(t))}dt, \quad n \to \infty.$$

Next, because the derivative $\dot{g}(t)$ equals zero on the set $B$, the equality

$$\int_B \frac{(\dot{g}(t))^2}{\sigma^2(g(t))}dt = 0$$

holds. Moreover, because $\sigma^{-2}$ is bounded and $a_n \to 0$, we have

$$\int_B \frac{(\dot{g}_n(t))^2}{\sigma^2(g_n(t))}dt = a_n^2 \int_B \frac{(\dot{\chi}(t))^2}{\sigma^2(g_n(t))}dt \to 0 = \int_B \frac{(\dot{g}(t))^2}{\sigma^2(g(t))}dt.$$

Finally, the Lebesgue measure of the set $C$ equals zero by Lemma 2. Hence, the respective integrals are negligible. $\square$

Let us finalize the proof. We note that $g_n \to g$ in $S$, and every $f_n$ is a continuity point for $F$. Therefore, for any $\delta > 0$, there exists $N$ such that $f_n \in \Theta_\delta(g), n \geq N$. To see that, one can take the continuity point $\tilde{f}$ in the definition of $\Theta_\delta(g)$ equal to $f_n$. Therefore,

$$\underline{J}_\delta(g) = \inf_{f \in \Theta_\delta(g)} I(f) \leq \lim_{n\to\infty} I(f_n) = J(g).$$

Because $\delta > 0$ is arbitrary, this provides the required inequality $\underline{J}(g) \leq J(g)$.

4.3. **Proof of the inequality $\overline{J}(g) \geq J(g)$.** We consider separately two cases: $J(g) < \infty$ and $J(g) = \infty$. In the first case, the required inequality is provided by the following lemma.

**Lemma 6.** *Let $J(g) < \infty$. Then, for every $\gamma > 0$, there exists $\delta > 0$ such that*

$$\inf_{f \in \Xi_\delta(g)} I(f) > J(g) - \gamma.$$

*Proof.* Assume the contrary; that is, let there exist some $\gamma > 0$ and the sequences of functions $f_n, \tilde{f}_n, n \geq 1$ such that $\tilde{f}_n \in U_F$, $d(f_n, \tilde{f}_n) < 1/n, d(g, F(\tilde{f}_n)) < 1/n$, and

$$I(f_n) \leq J(g) - \gamma.$$

Since the rate functional $I$ is "good", the sequence $f_n, n \geq 1$ belongs to the compact set $\{f : I(f) \leq a\}$ with $a = J(g) - \gamma$. Passing, if necessary, to a subsequence, we can assume that $f_n \to f$, and, therefore, $\tilde{f}_n \to f$. In addition, we have $F(\tilde{f}_n) \to g$.

Denote $\tilde{g}_n = F(\tilde{f}_n)$ and $g_n(t) = \tilde{g}_n(t) + (f_n - \tilde{f}_n)(\zeta_t(\tilde{g}_n))$. Then we have

$$\tilde{f}_n(t) = \tilde{g}_n(\zeta_t^{-1}(\tilde{g}_n)), \quad f_n(t) = g_n(\zeta_t^{-1}(\tilde{g}_n)).$$

Clearly, $g_n \to g$ and $\tilde{g}_n \to g$ in $S$. Performing calculations similar to those made in the previous section, we can represent the rate function $I(f_n)$ in the form

$$\begin{aligned} I(f_n) &= \frac{1}{2}\int_0^\infty (\dot{f}_n(t))^2 dt = \frac{1}{2}\int_0^\infty (\dot{g}_n(\zeta_t^{-1}(\tilde{g}_n)))^2 \cdot \frac{1}{\sigma^4(\tilde{g}_n(\zeta_t^{-1}(\tilde{g}_n)))} dt \\ &= \frac{1}{2}\int_0^\infty \frac{(\dot{g}_n(s))^2}{\sigma^2(\tilde{g}_n(s))} ds. \end{aligned} \tag{19}$$

Consider the set $A = \{s : \dot{g}(s) \neq 0\}$ and denote

$$h_n(s) = \frac{\dot{g}_n(s)}{\sigma(\tilde{g}_n(s))}\mathcal{I}_A(s), \quad h(s) = \frac{\dot{g}(s)}{\sigma(g(s))}\mathcal{I}_A(s).$$

Then

$$J(g) = \frac{1}{2}\|h\|_{L_2}^2, \quad I(f_n) \geq \frac{1}{2}\|h_n\|_{L_2}^2.$$

We will show that there exists

$$h_n \to h, \quad n \to \infty \quad \text{weakly in } L_2((0,\infty)). \tag{20}$$

Because (20) yields

$$\|h\|_{L_2} \leq \liminf_{n\to\infty} \|h_n\|_{L_2},$$

this will give the contradiction with the assumption made above, which will complete the proof of the lemma.

First, we note that

$$\dot{g}_n \to \dot{g}, \quad n \to \infty \quad \text{weakly in } L_2((0,\infty)). \tag{21}$$

Indeed, because

$$\int_0^\infty \frac{(\dot{g}_n(s))^2}{\sigma^2(\tilde{g}_n(s))} ds \leq 2I(f_n) \leq 2J(g) - 2\gamma$$

and $\sigma$ is bounded, the sequence $\dot{g}_n, n \geq 1$ is bounded in $L_2((0,\infty))$. Therefore, every its subsequence contains a weakly convergent subsequence, say $\dot{g}_{n_k}, k \geq 1$ (see, e.g., [9], Chapter IV §3). Denote the respective limit by $q$. Because $g_n \to g$ in $S$, one has, for every $t$,

$$g(t) = \lim_{n\to\infty} \Big[g_n(0) + (\dot{g}_n, \mathcal{I}_{[0,t]})_{L_2}\Big] = g(0) + \lim_{k\to\infty} (\dot{g}_{n_k}, \mathcal{I}_{[0,t]})_{L_2} = g(0) + \int_0^t q(s)\,ds.$$

This means that $g$ is absolutely continuous and $\dot{g} = q \in L_2((0,\infty))$. Because every subsequence of $\dot{g}_n, n \geq 1$ contains a subsequence weakly convergent to $\dot{g}$, we have (21).

Now we can prove (20). Take an arbitrary $u \in L_2((0,\infty))$ and write

$$(h_n, u)_{L_2} = (\dot{g}_n, v_n)_{L_2}, \quad (h, u)_{L_2} = (\dot{g}, v)_{L_2}$$

with

$$v_n(s) = \frac{u(s)}{\sigma(\tilde{g}_n(s))}\mathcal{I}_A(s), \quad v(s) = \frac{u(s)}{\sigma(g(s))}\mathcal{I}_A(s).$$

By Lemma 2, one has $g(s) \notin \Delta_\sigma$ for $\lambda$-a.a. $s \in A$. Therefore,

$$\sigma(\tilde{g}_n(s)) \to \sigma(g(s)),$$

because $\tilde{g}_n \to g$ in $S$. Since $\sigma$ is separated from zero, this provides by the dominated convergence theorem that $v_n$ converges to $v$ (strongly) in $L_2((0,\infty))$. Then, by (21), we obtain

$$(h_n, u)_{L_2} = (\dot{g}_n, v_n)_{L_2} \to (\dot{g}, v)_{L_2} = (h, u)_{L_2},$$

which completes the proof of (20). □

In the case $J(g) = \infty$, the required inequality is provided by the following lemma.

**Lemma 7.** *Let $J(g) = \infty$. Then, for every $N > 0$, there exists $\delta > 0$ such that*

$$\inf_{f\in\Xi_\delta(g)} I(f) > N.$$

*Proof.* Let $g(0) \neq x_0$ and $\delta < \Big(|g(0) - x_0| \wedge 1\Big)/2$. Then, for every functions $f, \tilde{f}$ such that $d(f, \tilde{f}) < \delta, d(g, F(\tilde{f})) < \delta$, we have $f(0) \neq 0$ because

$$[F(\tilde{f})](0) = \tilde{f}(0) + x_0$$

(see (10)). This means that, for such $\delta$,

$$\inf_{f\in\Xi_\delta(g)} I(f) = \infty.$$

Now, let $g(0) = x_0$. Like the proof of the previous lemma, we assume the contrary. In other words, we suppose that, for some $N$, there exist the sequences $f_n, \tilde{f}_n, n \geq 1$ such that

$$\tilde{f}_n \in U_F, \quad d(f_n, \tilde{f}_n) < 1/n, \quad d(g, F(\tilde{f}_n)) < 1/n, \quad I(f_n) \leq N.$$

Define the functions $g_n, \tilde{g}_n$ in the same way as in the previous proof. Then representation (19) holds true. Like the proof of the previous lemma, this representation and the assumptions $I(f_n) \leq N$ and $g_n \to g$ in $S$ imply that $g$ is absolutely continuous, and (21) holds true. Because $g(0) = x_0$, $g$ is absolutely continuous with

$$\int_0^\infty (\dot{g}(s))^2\,ds \leq \liminf_{n\to\infty} \int_0^\infty (\dot{g}_n(s))^2\,ds < \infty,$$

and $\sigma$ is separated from zero, we have $J(g) < \infty$. This contradiction provides the required statement. □

4.4. **Completion of the proof of Theorem 1.** Using the Fubini theorem, one can show easily that, for every $\varepsilon > 0$, the expectation of the random variable

$$\lambda\{s : \sqrt{\varepsilon}\tilde{W}_s \in \Delta_\sigma\}$$

equals zero. Because this random variable is non-negative, this means that

$$\lambda\{s : \sqrt{\varepsilon}\tilde{W}_s \in \Delta_\sigma\} = 0$$

with probability one. Hence, by Lemma 1,

$$P(\sqrt{\varepsilon}\tilde{W} \in \Delta_F) = 0,$$

which proves (11). Condition (12) can be verified easily using the Arzelà–Ascoli theorem and the fact that the time change transformation involved into definition (10) of the mapping $F$ is Lipschitz.

We have verified (11), (12), and identity (13). Hence, we can apply Proposition 3, which provides the required statement. □

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