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ASYMPTOTIC PROPERTIES OF LINEAR REGRESSION PARAMETER ESTIMATOR IN THE CASE OF LONG-RANGE DEPENDENT REGRESSORS AND NOISE

Sufficient conditions of consistency and asymptotic normality of least squares estimator of linear regression model parameter in the case of long-range dependent random regressors and noise are obtained in the paper.

1. Introduction

Estimation of unknown parameters in linear and nonlinear regression models is an important problem of statistics of random processes. Linear regression models with regressors observed with random errors are of a particular interest. The least squares estimator (LSE) is chosen in the paper for parameter estimation as one of the most important and much used regression model parameter estimator.

Among the asymptotic properties of statistical estimators consistency and asymptotic normality are the first two properties that should be considered as they are necessary for further more detailed study of asymptotic behavior of the estimators.

The LSE asymptotic properties of linear regression model parameter with stationary random regressors were considered in the book of A.Ya. Dorogovstev [1] and in [2]. The paper continues this study in the case when random errors in regressors and random noise are long-range dependent and regressors have time dependent trends.

2. Assumptions and results

Consider a regression model

$$X(t)=\sum_{i=1}^{q}\theta_i z_i(t)+\varepsilon(t),\ t\in[0,T], \qquad z_i(t)=a_i(t)+y_i(t),\ i=\overline{1,q}, \tag{2.1}$$

where $\theta^*=(\theta_1,...,\theta_q)\in\mathbb{R}^q$ is a vector of unknown parameters (* means transposition), $a_i:\ [0,\ \infty)\to\mathbb{R}^1,\ i=\overline{1,q}$ are some nonrandom continuous functions and

A1. $y_i(t),\ t\in\mathbb{R}^1,\ i=\overline{1,q}$, are independent mean square continuous measurable stationary Gaussian processes with zero mean and covariance functions (c.f.)

$$B_i(t)=Ey_i(0)y_i(t)=\frac{\sigma_i^2\cos\varkappa_i t}{(1+t^2)^{\frac{\alpha_i}{2}}},$$

where $\sigma_i^2>0,\ \alpha_i\in\left(\frac12,1\right),\ i=\overline{1,q},\ 0\le\varkappa_1<\ldots<\varkappa_q$.

A2. Random noise $\varepsilon(t),\ t\in\mathbb{R}^1$, is a mean square continuous measurable stationary Gaussian process independent of $y_i(t),\ t\in\mathbb{R}^1,\ i=\overline{1,q}$, with zero mean and c.f.

$$B(t)=E\varepsilon(0)\varepsilon(t)=\frac{\sigma^2\cos\varkappa t}{(1+t^2)^{\frac{\alpha}{2}}},\ \sigma^2>0,\ \varkappa>0,\ \varkappa\neq\varkappa_i,\ i=\overline{1,q},\ \alpha\in\left(\frac12,1\right).$$

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Note that c.f. introduced in **A1** and **A2** do not belong to $L_1(\mathbb{R}^1)$, i.e. $y_i(t),\ i=\overline{1,q}$, and $\varepsilon(t),\ t\in\mathbb{R}^1$, are long-range dependent processes.

Definition 2.1. LSE of the unknown parameter θ obtained by the observations

$$\left\{X(t), z_i(t),\ i=\overline{1,q},\ t\in[0,T]\right\}$$

of the form (2.1) is said to be any random vector

$$\widehat{\theta}_T=\widehat{\theta}_T\left(X(t), z_i(t),\ i=\overline{1,q},\ t\in[0,T]\right)$$

having the property

$$Q_T(\widehat{\theta}_T)=\inf_{\tau\in\mathbb{R}^q}Q_T(\tau),\ Q_T(\tau)=\int\limits_0^T\left[X(t)-\sum_{i=1}^q\tau_i z_i(t)\right]^2dt.$$

Introduce the following notation:

$$A^*(t)=(a_1(t),...,a_q(t))\,,\ Y^*(t)=(y_1(t),..,y_q(t)),\ Z(t)=A(t)+Y(t),$$

$$f_1\underset{T\to\infty}{\sim}f_2 \text{ means } \frac{f_1(t)}{f_2(t)}\underset{T\to\infty}{\longrightarrow}1.$$

Then

$$\widehat{\theta}_T=\Lambda_T^{-1}T^{-1}\int\limits_0^T Z(t)X(t)dt=\theta+\Lambda_T^{-1}T^{-1}\int\limits_0^T Z(t)\varepsilon(t)dt \tag{2.2}$$

with

$$\Lambda_T=\left(\Lambda_T^{il}\right)_{i,\,l=1}^q,\ \Lambda_T^{il}=T^{-1}\int\limits_0^T z_i(t)z_l(t)dt,\ i,\ l=\overline{1,q}.$$

We will denote by letters k positive constants. Let also

$$d_T^2=\mathrm{diag}\left(d_{iT}^2,\ i=\overline{1,q}\ \right),\ d_{iT}^2=\int\limits_0^T a_i^2(t)dt,\ i=\overline{1,q}\,.$$

Impose some additional conditions on functions $a_i(t),\ t\in[0,+\infty),\ i=\overline{1,q}$.
B1. a_i are bounded: $\sup\limits_{t\in[0,\infty)}|a_i(t)|=k_i<\infty,\ i=\overline{1,q}\,.$

Write

$$J_T=\left(J_T^{il}\right)_{i,\,l=1}^q,\ J_T^{il}=T^{-1}\int\limits_0^T a_i(t)a_l(t)dt.$$

B2. There exists $\lim\limits_{T\to\infty}J_T=J$, where $J=\left(J^{il}\right)_{i=1}^q$ is some positive definite matrix.
It follows from **B2** that

$$\lim_{T\to\infty}T^{-1}d_{iT}^2=\lim_{T\to\infty}J_T^{ii}=J^{ii}>0,\ i=\overline{1,q}.$$

Theorem 2.1. *If conditions* **A1, A2, B1** *and* **B2** *are fulfilled then* $\widehat{\theta}_T\underset{T\to\infty}{\longrightarrow}\theta$ *almost sure (a.s.).*

Introduce matrix measure $\mu_T(dx)$ on $(\mathbb{R}^1,\ \mathcal{B}^1)$ with density matrix $(\mu_T^{j\,l}(x))_{j,\ l=1}^q$,

$$\mu_T^{j\,l}(x) = a_T^j(x)\overline{a_T^l(x)}\left(\int_{\mathbb{R}^1} |a_T^j(x)|^2 dx \int_{\mathbb{R}^1} |a_T^l(x)|^2 dx\right)^{-1/2},$$

$$a_T^j(x) = \int_0^T e^{ixt} a_j(t)dt,\ \ j,l = \overline{1,q}.$$

Note that $d_{jT}^2 = (2\pi)^{-1} \int_{\mathbb{R}^1} |a_T^j(x)|^2 dx,\ j = \overline{1,q}$.

From the condition **A2** it follows that the process $\varepsilon(t),\ t \in \mathbb{R}^1$ has spectral density $f_\varepsilon \in L_2(\mathbb{R}^1)$ which has two discontinuity points $\pm\varkappa$ of the 2-nd type [7].

B3. Family of measures $\mu_T(\cdot)$ converges weakly, as $T \to \infty$, to a measure $\mu(\cdot)$, f_ε is μ-admissible function and $\int_{\mathbb{R}^1} f_\varepsilon(x)\mu(dx)$ is some positive definite matrix.

Remind that μ-admissability of the function f_ε means the fulfillment of relation [8]

$$\int_{-\infty}^{\infty} f_\varepsilon(\lambda)\mu_T(d\lambda) \underset{T\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} f_\varepsilon(\lambda)\mu(d\lambda).$$

Sufficient conditions of μ-admissability of spectral density function f_ε can be found in [8, 9].

Definition 2.2. Matrix measure $\mu(\cdot)$ is said to be spectral measure of regression function $\sum_{i=1}^{q} \theta_i a_i(t)$ [10, 8, 11].

Introduce some notation:

$$\Lambda = \mathrm{diag}\big(\sigma_i^2,\ i = \overline{1,q}\,\big) + J;\ \ \Gamma = \mathrm{diag}\big(J^{ii},\ i = \overline{1,q}\,\big);\ \ b_i = \int_{-\infty}^{\infty} B_i(t)B(t)dt,\ \ i = \overline{1,q};$$

$$\Xi = 2\pi \cdot \Gamma^{\frac{1}{2}} \left(\int_{\mathbb{R}^1} f_\varepsilon(x)\mu(dx)\right) \Gamma^{\frac{1}{2}} + \mathrm{diag}\big(b_i,\ i = \overline{1,q}\,\big).$$

Theorem 2.2. *If conditions* **A1**, **A2** *with* $\alpha \in \left(\frac{3}{4}, 1\right)$, **B1 - B3** *hold, then the distribution of the normed LSE* $T^{\frac{1}{2}}\left(\widehat{\theta}_T - \theta\right)$ *as* $T \to \infty$ *tends to normal distribution* $N\left(0,\ \Lambda^{-1}\,\Xi\,\Lambda^{-1}\right)$.

3. Auxiliary assertions

At first we study asymptotic behavior of Λ_T.

Lemma 3.1. *If conditions* **A1, B1** *and* **B2** *hold, then*

$$\Lambda_T \underset{T\to\infty}{\longrightarrow} \Lambda \text{ a.s.}$$

Proof. For fixed i, l consider general element of the matrix Λ_T:

$$\begin{aligned}\Lambda_T^{il} &= T^{-1}\int_0^T y_i(t)y_l(t)dt + T^{-1}\int_0^T a_i(t)y_l(t)dt + T^{-1}\int_0^T y_i(t)a_l(t)dt + \\ &+ T^{-1}\int_0^T a_i(t)a_l(t)dt = \Delta^{il}(T) + \Delta_i^l(T) + \Delta_l^i(T) + J_T^{il}.\end{aligned} \tag{3.1}$$

Let $i \neq l$. Then for sufficiently large T $(T > T_0)$

$$E\left(\Delta^{il}(T)\right)^2 = T^{-2}\int_0^T\int_0^T B_i(t-s)B_l(t-s)dtds \le$$

$$\le \sigma_i^2\sigma_l^2 T^{-2}\int_0^T\int_0^T \frac{dtds}{(1+(t-s)^2)^{\frac{\alpha_i+\alpha_l}{2}}} \le 2\sigma_i^2\sigma_l^2 T^{-1}\int_{-T}^T \frac{dv}{(1+v^2)^{\frac{\alpha_i+\alpha_l}{2}}}.$$

As subintegral function is integrable $(\alpha_i + \alpha_l > 1)$ then

$$E\left(\Delta^{il}(T)\right)^2 \le 2\sigma_i^2\sigma_l^2 T^{-1}\int_{-\infty}^{\infty} \frac{dv}{(1+v^2)^{\frac{\alpha_i+\alpha_l}{2}}} = 2K_{il}T^{-1}. \tag{3.2}$$

Set $T_n = n^{1+\nu}$, $\nu > 0$. Then $\sum\limits_{n=1}^{\infty} E\left(\Delta^{il}(T_n)\right)^2 < \infty$ and consequently

$$\Delta^{il}(T_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ a.s.}$$

Suppose that $T \in [T_n,\ T_{n+1}]$. Then

$$|\Delta^{il}(T)| \le \sup_{T_n\le T\le T_{n+1}} |\Delta^{il}(T) - \Delta^{il}(T_n)| + |\Delta^{il}(T_n)|.$$

Let us show that

$$\sup_{T_n\le T\le T_{n+1}} |\Delta^{il}(T) - \Delta^{il}(T_n)| \underset{n\to\infty}{\longrightarrow} 0 \text{ a.s.}$$

Obviously

$$\begin{aligned} \Delta^{il}(T) - \Delta^{il}(T_n) &= T^{-1}\int_0^T y_i(t)y_l(t)dt - T_n^{-1}\int_0^{T_n} y_i(t)y_l(t)dt = \\ &= \left(T^{-1} - T_n^{-1}\right)\int_0^{T_n} y_i(t)y_l(t)dt + T^{-1}\int_{T_n}^T y_i(t)y_l(t)dt = I_1 + I_2, \end{aligned} \tag{3.3}$$

$$|I_1| \le \frac{T_{n+1}-T_n}{T_n}\cdot\left|T_n^{-1}\int_0^{T_n} y_i(t)y_l(t)dt\right| \underset{n\to\infty}{\sim} \frac{1+\nu}{n}\left|\Delta^{il}(T_n)\right|,$$

and $I_1 \underset{n\to\infty}{\longrightarrow} 0$ a.s.. Consider the second summand in (3.3):

$$|I_2| \le T_n^{-1}\int_{T_n}^{T_{n+1}} |y_i(t)|\cdot|y_l(t)|\,dt \le$$

$$\le \frac{1}{2}\left(T_n^{-1}\int_{T_n}^{T_{n+1}} (y_i(t))^2\,dt + T_n^{-1}\int_{T_n}^{T_{n+1}} (y_l(t))^2\,dt\right) = \frac{1}{2}\left(I_3^i(n) + I_3^l(n)\right),$$

$$I_3^i(n) = \frac{T_{n+1}}{T_n}I_4^i(n+1) - I_4^i(n) + \sigma_i^2\frac{T_{n+1}-T_n}{T_n}\ ,\ I_4^i(n) = T_n^{-1}\int_0^{T_n}\left[(y_i(t))^2 - \sigma_i^2\right]dt.$$

Due to Isserlis formula (see, for example [4], p. 30)

$$E\left(I_4^i(n)\right)^2 = T_n^{-2}\int\limits_0^{T_n}\int\limits_0^{T_n}\left(E\left(y_i(t)\right)^2\left(y_i(s)\right)^2-\sigma_i^2\right)dtds =$$

$$= 2T_n^{-2}\int\limits_0^{T_n}\int\limits_0^{T_n} B_i^2(t-s)dtds \le 2K_{ii}T_n^{-1},$$

where $K_{ii}=\sigma_i^4\int\limits_{-\infty}^{\infty}\frac{dv}{(1+v^2)^{\alpha_i}}$. So $\sum\limits_{n=1}^{\infty}E\left(I_4^i(n)\right)^2<\infty$ and $I_4^i \underset{n\to\infty}{\longrightarrow} 0$ a.s.. Thus $I_2 \underset{n\to\infty}{\longrightarrow}$ 0 a.s., and

$$\Delta^{il}(T) \underset{T\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{3.4}$$

Let us prove

$$\Delta_i^l(T) \underset{T\to\infty}{\longrightarrow} 0 \text{ a.s.},\ i,l=1,...,q. \tag{3.5}$$

Evidently $E\Delta_i^l(T)=0$ and

$$E\left(\Delta_i^l(T)\right)^2 \le k_i^2T^{-2}\int\limits_0^T\int\limits_0^T\left|B_l\left(T(t-s)\right)\right|dtds \le$$

$$\le 2k_i^2\sigma_l^2T^{-1}\int\limits_{-T}^{T}\frac{dv}{(1+v^2)^{\frac{\alpha_l}{2}}} \underset{T\to\infty}{\sim} \frac{4k_i^2\sigma_l^2T^{-\alpha_l}}{1-\alpha_l}.$$

Taking $T_n = n^{\frac{1}{\alpha_l}+\nu}$, $\nu>0$, one obtains $\Delta_i^l(T_n) \underset{n\to\infty}{\longrightarrow} 0$ a.s. Further prove of (3.5) is similar to the proof of (3.4).

From (3.4),(3.5) and condition **B2** it follows that for $i\ne l$,

$$\Lambda_T^{il} \underset{T\to\infty}{\longrightarrow} J^{il} \text{ a.s.}$$

Now let $i=l$. Then (3.1) can be rewritten in the form

$$\Lambda_T^{ii} = \Delta^{ii}(T)+2\Delta_i^i(T)+J_T^{ii}.$$

Similarly to the proof of (3.4) one can get

$$\Delta^{ii}(T) \underset{T\to\infty}{\longrightarrow} \sigma_i^2 \text{ a.s.} \tag{3.6}$$

Indeed, $E\Delta^{ii}(T)=B_i(0)=\sigma_i^2$ and

$$E[\Delta^{ii}(T)-\sigma_i^2]^2 = 2T^{-2}\int\limits_0^T\int\limits_0^T B_i^2(t-s)dtds \le 2K_{ii}T^{-1}.$$

If we take $T_n=n^{1+\nu}$, $\nu>0$, then $\sum\limits_{n=1}^{\infty}E[\Delta^{ii}(T_n)-\sigma_i^2]^2<\infty$ and

$$\Delta^{ii}(T_n) \underset{n\to\infty}{\longrightarrow} \sigma_i^2 \text{ a.s.}$$

Further proof of (3.6) is similar to (3.4).

Then from condition **B2**, (3.5) and (3.6) it follows that

$$\Lambda_T^{ii} \underset{T\to\infty}{\longrightarrow} \sigma_i^2+J^{ii} \text{ a.s. },$$

and Lemma 1 is proved. □

Corollary 3.1. *If conditions* **A1, A2, B1** *and* **B2** *hold, then for almost all* $\omega \in \Omega$ *there exists such* $T_0 = T_0(\omega)$ *that for any* $T > T_0$ *LSE* $\widehat{\theta}_T$ *given by (2.2) is defined.*

Let us formulate a subcase of homogeneous Hölder-Young-Brascamp-Lieb inequality for $\mathbb{R}^1$ (see [5, 6] for details). Denote by $r(A)$ rank of a matrix A.

Lemma 3.2. *Let* $l_j(x) = x^*\beta_j,\ j = \overline{1,k}$ *be the linear functionals* $l_j\ :\ \mathbb{R}^m \to \mathbb{R}^1$, $\beta_j \in \mathbb{R}^m,\ j = \overline{1,k}$, *M is a matrix with columns* $\beta_j,\ j = \overline{1,k}$.

If functions $f_j \in L_{p_j}(\mathbb{R}^1),\ j = \overline{1,k},\ 1 \le p_j \le \infty,\ z_j = \frac{1}{p_j},\ j = \overline{1,k}$, *such that* $\sum\limits_{j=1}^{k} z_j = m$ *and for arbitrary* $1 \le d \le k$, *and* $\{s_1, ..., s_d\} \subset \{1, ..., k\}$ *the next inequality holds*

$$\sum_{i=1}^{d} z_{s_i} \le r(A),$$

where $A = (\beta_{s_1} ... \beta_{s_d})$, *then*

$$\left| \int\limits_{\mathbb{R}^m} \prod_{j=1}^{k} f_j(l_j(x))dx \right| \le K \prod_{j=1}^{k} \|f_j\|_{p_j},$$

where $K = K(z_1, ..., z_k)$ *is some constant which depends on values* $(z_1, ..., z_k)$ *only (determination of* K *can be found in* [6]*),* $\|\cdot\|_{p_j}$ *is norm in* $L_{p_j}(\mathbb{R}^1)$.

4. Proof of the strong consistency of LSE

Proof. From Lemma 1 and the representation of LSE in the form (2.2) it follows that we need to show that

$$T^{-1} \int\limits_0^T Z(t)\varepsilon(t)dt \underset{T\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{4.1}$$

in order to prove the Theorem 1. For the fixed i

$$T^{-1} \int\limits_0^T z_i(t)\varepsilon(t)dt = T^{-1} \int\limits_0^T a_i(t)\varepsilon(t)dt + T^{-1} \int\limits_0^T y_i(t)\varepsilon(t)dt = I_5(T) + I_6(T). \tag{4.2}$$

The proofs of $I_5(T) \underset{T\to\infty}{\longrightarrow} 0$ a.s. and $I_6(T) \underset{T\to\infty}{\longrightarrow} 0$ a.s. repeat the argument of convergence to zero a.s. of $\Delta_i^l(T)$ and $\Delta^{il}(T)$ from Lemma 1, respectively. □

5. Proof of asymptotic normality of LSE

Proof. Due to Lemma 1 to prove the Theorem 2 it is sufficient to determine the asymptotic distribution of the vector

$$\Psi_T = T^{-\frac{1}{2}} \int\limits_0^T Z(t)\varepsilon(t)dt,$$

as $T^{\frac{1}{2}}\left(\widehat{\theta}_T - \theta\right) = \Lambda_T^{-1}\Psi_T$.

Let $\lambda^* = (\lambda_1, ..., \lambda_q) \in \mathbb{R}^q$ be an arbitrary fixed vector, $\mathcal{F}$ – σ-algebra generated by $\left\{Y(t),\ t \in \mathbb{R}^1\right\}$. Conditional distribution relatively to $\mathcal{F}$ of random variable $\lambda^*\Psi_T$ is Gaussian with expected value $E\left\{\lambda^*\Psi_T|\mathcal{F}\right\} = 0$ and variance

$$E\left\{(\lambda^*\Psi_T)^2|\mathcal{F}\right\} = \lambda^* \left(T^{-1} \int\limits_0^T \int\limits_0^T Z(t)Z^*(s)B(t-s)dtds\right)\lambda = \lambda^*\Xi_T\lambda,$$

where equalities are valid a.s. [1]. Then for characteristic function of the vector Ψ_T we have

$$\varphi_T(\lambda) = Ee^{i\lambda^*\Psi_T} = E\left\{E\left(e^{i\lambda^*\Psi_T}|\mathcal{F}\right)\right\} = Ee^{-\frac{1}{2}\lambda^*\Xi_T\lambda}.$$

The diagonal element of Ξ_T is

(5.1)
$$\Xi_T^{ii} = T^{-1}\int_0^T\int_0^T B(t-s)a_i(t)a_i(s)dtds + 2T^{-1}\int_0^T\int_0^T B(t-s)a_i(s)y_i(t)dtds + \\ + T^{-1}\int_0^T\int_0^T B(t-s)y_i(t)y_i(s)dtds = I_7 + I_8 + I_9.$$

For the first term it is easy to get, using conditions **B2** and **B3**, that

(5.2)
$$I_7 \underset{T\to\infty}{\sim} J^{ii}d_{iT}^{-2}\int_0^T\int_0^T B(t-s)a_i(t)a_i(s)dtds = \\ = 2\pi J^{ii}\int_{\mathbb{R}^1} f_\varepsilon(x)\mu_T^{i\,i}(dx) \underset{T\to\infty}{\longrightarrow} 2\pi J^{ii}\int_{\mathbb{R}^1} f_\varepsilon(x)\mu^{i\,i}(dx).$$

Furthermore,

$$EI_8^2 = 4T^{-2}\int_0^T\int_0^T\int_0^T\int_0^T B(t-s)B(u-v)B_i(t-u)a_i(s)a_i(v)dtdsdudv \leq \\ \leq 4k_i^2\sigma_i^2\sigma^4T^{-2}\int_0^T\int_0^T\int_0^T\int_0^T \frac{dtdsdudv}{(1+(t-s)^2)^{\frac{\alpha}{2}}(1+(u-v)^2)^{\frac{\alpha}{2}}(1+(t-u)^2)^{\frac{\alpha_i}{2}}}.$$

Using the change of variables $u_1 = t-s,\ u_2 = u-v,\ u_3 = t-u,\ u_4 = v$, with unit Jacobian, integral can be estimated as follows:

(5.3)
$$EI_8^2 \leq 32k_i^2\sigma_i^2\sigma^4T^{-1}\int_{-T}^T \frac{du_1}{(1+u_1^2)^{\frac{\alpha}{2}}}\int_{-T}^T \frac{du_2}{(1+u_2^2)^{\frac{\alpha}{2}}}\int_{-T}^T \frac{du_3}{(1+u_3^2)^{\frac{\alpha_i}{2}}} \underset{T\to\infty}{\sim} \\ \underset{T\to\infty}{\sim} \frac{256k_i^2\sigma_i^2\sigma^4T^{2-2\alpha-\alpha_i}}{(1-\alpha)^2(1-\alpha_i)},$$

where $2-2\alpha-\alpha_i < 0$ as $\alpha \in \left(\frac{3}{4}, 1\right)$ and $\alpha_i \in \left(\frac{1}{2}, 1\right)$.

Now consider the behavior of the last term of (5.1). Under the conditions **A1, A2** $B_i(t)B(t) \in L_1(\mathbb{R}^1)$, and by Lebesgue dominated convergence theorem

(5.4)
$$EI_9 = T^{-1}\int_0^T\int_0^T B(t-s)B_i(t-s)dtds = \int_{-T}^T\left(1-\frac{|t|}{T}\right)B_i(t)B(t)dt \underset{T\to\infty}{\longrightarrow} b_i.$$

Using the normality of the processes $y_i(t),\ t\in\mathbb{R}^1,\ i=\overline{1,q}$, we obtain

$$E\left(I_9 - EI_9\right)^2 = 2T^{-2}\int_0^T\int_0^T\int_0^T\int_0^T B(t-s)B(u-v)B_i(t-u)B_i(s-v)dtdsdudv = \\ = 2\sigma_i^4\sigma^4T^{-2}\int_{\mathbb{R}^4} f_1(t-s)f_2(u-v)f_3(t-u)f_4(s-v)f_5(t)f_6(s)f_7(u)f_8(v)dtdsdudv,$$

where $f_1(t) = f_2(t) = \dfrac{\cos \varkappa t}{(1+t^2)^{\frac{\alpha}{2}}}\chi_T(|t|),\ t \in \mathbb{R}^1,\ f_3(t) = f_4(t) = \dfrac{\cos \varkappa_i t}{(1+t^2)^{\frac{\alpha_i}{2}}}\chi_T(|t|),\ t \in \mathbb{R}^1,\ f_j(t) = \chi_T(t),\ t \in \mathbb{R}^1,\ j = 5,6,7,8,\ \chi_T(t)$ is indicator of the set $[0,T]$. We will use for the last integral Hölder-Young-Brascamp-Lieb inequality (Lemma 2). In this case

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we put $p_1 = p_2 = 1,\ p_3 = p_4 = 2,\ p_j = 4,\ j = 5,6,7,8$, then one can show that assumptions of Lemma 2 are fulfilled and so

$$\begin{aligned} E\left(I_9 - EI_9\right)^2 \le & 2K\sigma_i^4\sigma^4T^{-1}\left(\int_{-T}^{T}\left|\frac{\cos \varkappa t}{(1+t^2)^{\frac{\alpha}{2}}}\right|dt\right)^2 \int_{-T}^{T}\frac{\cos^2 \varkappa_i t}{(1+t^2)^{\alpha_i}}dt \le \\ & \le 8Kc_i\sigma_i^4\sigma^4T^{-1}\left(\int_0^T \frac{dt}{(1+t^2)^{\frac{\alpha}{2}}}\right)^2 \underset{T\to\infty}{\sim} \frac{8Kc_i\sigma_i^4\sigma^4T^{1-2\alpha}}{(1-\alpha)^2}, \end{aligned} \tag{5.5}$$

where $K = K\left(1,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$, $c_i = \int_{-\infty}^{\infty} \frac{\cos^2 \varkappa_i t}{(1+t^2)^{\alpha_i}}dt < \infty$ and $1 - 2\alpha < 0$.

From (5.2)–(5.5) it follows that

$$\Xi_T^{ii} \underset{T\to\infty}{\xrightarrow{\mathbf{P}}} \Xi^{ii}. \tag{5.6}$$

Similarly one can show that for elements Ξ_T^{il} with $i \neq l$

$$\Xi_T^{il} \underset{T\to\infty}{\xrightarrow{\mathbf{P}}} \Xi^{il}. \tag{5.7}$$

Thus, from (5.6) and (5.7) it follows that $\Xi_T \underset{T\to\infty}{\xrightarrow{\mathbf{P}}} \Xi$. Note that $\lambda^*\Xi_T\lambda \ge 0$. Then by Lebesgue dominated convergence theorem we get

$$\lim_{T\to\infty}\varphi_T(\lambda) = \lim_{T\to\infty} Ee^{-\frac{1}{2}\lambda^*\Xi_T\lambda} = e^{-\frac{1}{2}\lambda^*\Xi\lambda},$$

and Theorem 2 is proved. □

6. Example

Consider a regression model

$$X(t) = \theta_1\left[\cos \omega t + y_1(t)\right] + \theta_2\left[\sin \omega t + y_2(t)\right] + \varepsilon(t), \tag{6.1}$$

where $y_1(t),\ y_2(t),\ t \in \mathbb{R}^1$, satisfy **A1**, $\varepsilon(t),\ t \in \mathbb{R}^1$, satisfies **A2**, $\omega > 0$ is some known number, $\omega \neq \varkappa$. Vector parameter $\theta = (\theta_1,\ \theta_2)$ belongs to a bounded open set, $\theta_1^2 + \theta_2^2 > 0$.

It is easily seen that

$$d_{1T}^2 = \frac{T}{2} + \frac{1}{4\omega}\sin 2\omega T,\ d_{2T}^2 = \frac{T}{2} - \frac{1}{4\omega}\sin 2\omega T;$$

$$J_T = \begin{pmatrix} \frac{1}{2} + \frac{1}{4\omega}T^{-1}\sin 2\omega T & \frac{1}{4\omega}T^{-1}(1-\cos 2\omega T) \\ \frac{1}{4\omega}T^{-1}(1-\cos 2\omega T) & \frac{1}{2} - \frac{1}{4\omega}T^{-1}\sin 2\omega T \end{pmatrix},$$

and fulfillment of conditions **B1, B2** is obvious with

$$\lim_{T\to\infty} J_T = J = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}E.$$

It is known (see, for example [12]) that the regression function from equation (6.1) has spectral density

$$\mu(dx) = \begin{pmatrix} \delta_\omega(dx) & i\rho_\omega(dx) \\ -i\rho_\omega(dx) & \delta_\omega(dx) \end{pmatrix},$$

where measure δ_ω and signed measure ρ_ω are concentrated at the points $\pm\omega$ and

$$\delta_\omega\left(\{\pm\omega\}\right) = \frac{1}{2}, \rho_\omega\left(\{\pm\omega\}\right) = \pm\frac{1}{2}.$$

It means that

$$\int\limits_{\mathbb{R}^1} f_\varepsilon(x)\mu(dx) = f_\varepsilon(\omega)J.$$

Moreover, as proved in [9], spectral density f_ε is μ-admissible. Thus, condition **B3** is also fulfilled.

Now we can use Theorem 2 and claim that the normed LSE $T^{\frac{1}{2}}\left(\widehat{\theta}_T - \theta\right)$ is asymptotically normal with zero mean and covariance matrix

$$\Lambda^{-1}\Xi\Lambda^{-1} = \begin{pmatrix} \dfrac{2\pi f_\varepsilon(\omega) + 4b_1}{(2\sigma_1^2+1)^2} & 0 \\ 0 & \dfrac{2\pi f_\varepsilon(\omega) + 4b_2}{(2\sigma_2^2+1)^2} \end{pmatrix}, \tag{6.2}$$

where $b_i = \int\limits_{-\infty}^{\infty} B_i(t)B(t)dt = 2\pi\int\limits_{-\infty}^{\infty} f_i(\lambda)f_\varepsilon(\lambda)d\lambda,\ f_i(\lambda)$ is spectral density of random processes $y_i(t),\ t\in\mathbb{R}^1,\ i=1,2.$

Note that the value $f_\varepsilon(\omega)$ can be found by formula (see [7])

$$f_\varepsilon(\omega) = \frac{2^{-\frac{1+\alpha}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)}\left[(\omega+\varkappa)^{\frac{\alpha-1}{2}}K_{\frac{\alpha-1}{2}}(\omega+\varkappa) + |\omega-\varkappa|^{\frac{\alpha-1}{2}}K_{\frac{\alpha-1}{2}}(|\omega-\varkappa|)\right],$$

where K_ν is modified Bessel function of the 3d type and order ν, and the value of $b_i,\ i=1,2$, can be represented using the formula 4 on page 390 of [13] in the form

$$b_i = \frac{\sqrt{\pi}\,2^{\frac{1-\alpha-\alpha_i}{2}}}{\Gamma(\frac{\alpha+\alpha_i}{2})}\left[(\varkappa+\varkappa_i)^{\frac{\alpha+\alpha_i-1}{2}}K_{\frac{\alpha+\alpha_i-1}{2}}(\varkappa+\varkappa_i) + \right.$$
$$\left. + |\varkappa-\varkappa_i|^{\frac{\alpha+\alpha_i-1}{2}}K_{\frac{\alpha+\alpha_i-1}{2}}(|\varkappa-\varkappa_i|)\right].$$

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