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KRYLOV–VERETENNIKOV REPRESENTATION FOR THE m-POINT MOTION OF A DISCRETE-TIME FLOW

We consider a discrete-time stochastic flow which can be regarded as an approximation to a flow of Brownian particles with interaction. For the m-point motion of such discrete-time flow we present a discrete analogue of Krylov-Veretennikov expansion.

1. Introduction

We are interested in Harris stochastic flows of Brownian particles on the line [1]. Let Γ be a continuous real positive definite function on $\mathbb{R}$ such that $\Gamma(0) = 1$ and Γ is Lipschitz outside any neighborhood of zero.

Definition 1.1. The Harris flow $\{x(u,\cdot), u \in \mathbb{R}\}$ with Γ being its local characteristic is a family of Brownian martingales with respect to a joint filtration such that

(i) for every $u_1 \le u_2$ and $t \ge 0$

$$x(u_1,t) \le x(u_2,t)$$

(ii) the joint characteristics are

$$d < x(u_1,\cdot), x(u_2,\cdot) > (t) = \Gamma(x(u_1,t) - x(u_2,t))dt.$$

One of the ways to study functionals of stochastic flows is to analyse its Itô–Wiener expansion. In the case when Γ is smooth enough, the Harris flow can be obtained as a flow of solutions to the following SDE [2]:

$$\begin{aligned} dx(u,t) &= \sum_{k=1}^{\infty} a_k(x(u,t))dw_k(t), \\ x(u,0) &= u, \end{aligned} \tag{1.1}$$

where $\{w_k\}_{k\ge1}$ is a sequence of standard Wiener processes and $a = (a_k)_{k\ge1}$ is a Lipschitz mapping from $\mathbb{R}$ to l_2 such that

$$\sum_{k=1}^{\infty} a_k^2 = 1$$

and

$$\sum_{k=1}^{\infty} a_k(u)a_k(v) = \Gamma(u-v).$$

The Itô–Wiener expansion of the function from value of a solution to SDE was obtained by Krylov and Veretennikov in [8]. The Harris flow could be coalescent [1] and, in this case, there is no stochastic differential equation that generate the flow. One of the ways to study such flow is to construct its approximation using flows with discrete time. Such approach was used in [2]. A flow with discrete time can be defined as a family of random walks with interaction on a line, which is driven by a sequence of independent stationary

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Gaussian processes $\{\xi_n(u), u \in \mathbb{R}\}_{n\geq 1}$. The random walks are defined via the following recurrence equation:

$$x_{n+1}(u) = x_n(u) + \xi_{n+1}(x_n(u)), \qquad (1.2)$$
$$x_0(u) = u,\ u \in \mathbb{R}.$$

We assume that the Gaussian processes $\{\xi_n, n \geq 1\}$ have zero mean and the same continuous covariance function $\Gamma, \Gamma(0) = 1$.

The aim of our paper is to give a construction and an explicit form of the Itô–Wiener expansion for the following random variables

$$\varphi(x_n(u_1), \ldots, x_n(u_m)),$$

where φ is some function $\mathbb{R}^m \to \mathbb{R}$ and $\{x_n(u), u \in \mathbb{R}\}$ is a discrete-time flow (1.2). Such expansion can be regarded as a discrete-time analogue of the Krylov–Veretennikov expansion. Note that for 1-point motion it was obtained in [3]. The article is organized as follows. In the next section we give some known facts related to the Itô–Wiener expansion of Gaussian functionals and the Krylov–Veretennikov representation of a solution to SDE. Also we describe a white noise related to the discrete-time stochastic flow (1.2) and present an Itô–Wiener expansion for $\varphi(x_n(u_1), \ldots, x_n(u_m))$ in terms of multilinear forms from this white noise. In the third section we rewrite the obtained expansion in terms of Gaussian processes $\{\xi_n\}_{n\geq 1}$, which produce the discrete-time flow. An example of a stochastic flow of solutions to SDE and its discrete-time approximation will be given in the last section.

2. An abstract form of a discrete analogue of the Krylov–Veretennikov representation

Let us introduce some basic definitions and notations related to the Itô–Wiener expansion of Gaussian functionals [5, 6].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and H be a separable Hilbert space with its inner product denoted as $(\cdot, \cdot)$ and its norm denoted as $\|\cdot\|$. Let ζ, be a generalized Gaussian random element in H which has zero mean and identity correlation operator, i.e. ζ is a linear map from H to the set of Gaussian random variables such that

$$\forall\ \varphi \in H \quad \mathbb{E}(\zeta, \varphi) = 0,\ \mathbb{E}(\zeta, \varphi)^2 = \|\varphi\|^2.$$

We also call ζ a white noise in H.

Let $H_k, k \geq 1$ be a space of k-linear symmetric Hilbert–Schmidt forms on H and define an inner product in H_k by the rule

$$\forall\ A_k, B_k \in H_k \ \ (A_k, B_k) = \sum_{i_1,\ldots,i_k=1}^{\infty} A_k(e_{i_1}, \ldots, e_{i_k}) B_k(e_{i_1}, \ldots, e_{i_k}),$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis in H. For any $A_k \in H_k$ we define the value of the form A_k at a generalized random element ζ as follows:

$$A_k(\zeta, \ldots, \zeta) := \sum_{i_1,\ldots,i_k=1}^{\infty} A_k(e_{i_1}, \ldots, e_{i_k})(\zeta, e_{i_1}) * \ldots * (\zeta, e_{i_k}),$$

where $*$ denotes the Wick product [6]. Suppose that $\eta \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ and η is measurable with respect to $\sigma(\zeta) = \sigma\{(h, \zeta), h \in H\}$. Then there exists a unique sequence of forms $A_k \in H_k$ ([5]) such that

$$\eta = \sum_{k=0}^{\infty} A_k(\zeta, \ldots, \zeta), \qquad (2.1)$$

where the series converges in square mean. The representation (2.1) is called the Itô–Wiener expansion.

We give an example of a Hilbert space, of a white noise on it and of a form of the Itô–Wiener expansion for some random variable. Let $\{w_t, t \in [0,1]\}$ be a one-dimensional Wiener process, which is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Using this Wiener process we can define a white noise $\dot{w}$ on $L_2([0,1])$ as follows: for $f \in L_2([0,1])$ we put $(\dot{w}, f) = \int_0^1 f(x)dw(x)$. For any Hilbert–Schmidt form A_k on $L_2([0,1])$ there exists a unique function $a_k \in L_2([0,1]^k)$ that is invariant under any permutation of the arguments such that for any $x_1, \ldots, x_k \in L_2([0,1])$

$$A_k(x_1, \ldots, x_k) = \int_0^1 \ldots \int_0^1 a_k(t_1, \ldots, t_k) x_1(t_1) \ldots x_k(t_k) dt_1 \ldots dt_k$$

and the value of the form A_k at $\dot{w}$ has the form

$$A_k(\dot{w}, \ldots, \dot{w}) = \int_0^1 \ldots \int_0^1 a_k(t_1, \ldots, t_k) dw(t_1) \ldots dw(t_k).$$

The Itô–Wiener expansion for a random variable $f(x(t))$, where $\{x(t), t \in [0,1]\}$ is a solution to the following SDE

$$\begin{cases} dx(t) = \sigma(x(t))dw(t) + b(x(t))dt, \\ x(0) = x_0 \end{cases} \tag{2.2}$$

was obtained by Krylov and Veretennikov in [8]. Suppose that $\sigma(\cdot), b(\cdot)$ are Lipschitz functions and

$$\exists\ \mu > 0\ \forall\ x \in \mathbb{R}: \quad |\sigma(x)| \geq \mu.$$

It is known that under such conditions there exists a unique strong solution to the SDE (2.2) and for any bounded Borel measurable function f the random variable $f(x(t))$ is $\sigma(\dot{w})$-measurable. The Krylov–Veretennikov representation can be written in terms of the fundamental solution to a parabolic partial differential equation associated with the SDE (2.2). Denote $a(x) = \frac{1}{2}\sigma^2(x)$ and for fixed $t > 0$ consider

$$\begin{cases} \frac{\partial}{\partial s}u(s,x) + a(x)\frac{\partial^2}{\partial x^2}u(s,x) + b(x)\frac{\partial}{\partial x}u(s,x) = 0,\ 0 < s < t, \\ u(t,x) = \varphi(x),\ \varphi \in C_0^\infty(\mathbb{R}), t \in \mathbb{R}. \end{cases} \tag{2.3}$$

Let $\{T_{t-s}, s < t\}$ be a set of operators that define a solution to (2.3). It is known that $T_t\varphi(x_0) = \mathbb{E}\varphi(x(t))$, where $x(t)$ is a solution to (2.2). Denote $R_t\varphi(x) = \sigma(x)\frac{\partial}{\partial x}T_t\varphi(x)$. Then the Itô–Wiener expansion has the form:

$$\varphi(x(t)) = T_t\varphi(x_0) + \sum_{i=1}^{\infty} \int\limits_{0<t^i<\ldots<t^1<t} \!\!\ldots\!\! \int T_{t^i} R_{t^{i-1}-t^i} \ldots R_{t-t^1}\varphi(x_0) dw_{t^i} \ldots dw_{t^1}. \tag{2.4}$$

Recall that our main object is a discrete-time flow $\{x_n(u), u \in \mathbb{R}\}_{n\geq 1}$, which is defined via the following recurrence equation

$$\begin{aligned} &x_{n+1}(u) = x_n(u) + \xi_{n+1}(x_n(u)), \\ &x_0(u) = u, \end{aligned}$$

where $\{\xi_n(u), u \in \mathbb{R}\}_{n\geq 1}$ is a sequence of independent stationary Gaussian processes with zero mean and a continuous covariance function $\Gamma, \Gamma(0) = 1$. Let us describe a Hilbert space and a white noise $\dot{w}$ on it, such that for any Borel measurable bounded function $\varphi : \mathbb{R}^m \to \mathbb{R}$ and for every $n \geq 1$ a random variable $\varphi(x_n(u_1), \ldots, x_n(u_m))$ is measurable with respect to $\sigma(\dot{w})$. Consider a Hilbert space H^Γ with reproducing kernel $\{\Gamma(u-v), u, v \in \mathbb{R}\}$, [9] i.e.

$$H^\Gamma = \overline{\{\sum_{k=1}^{n} c_k \Gamma(u_k - \cdot), c_k, u_k \in \mathbb{R}, n \geq 1\}}_{\|\cdot\|},$$

where the closure is taken with respect to a norm, which can be introduced as follows:

$$\|\sum_{k=1}^{n} c_k \Gamma(u_k - \cdot)\|^2 = \sum_{k,j=1}^{n} c_k c_j \Gamma(u_k - u_j).$$

Then a white noise $\dot{w}_i$ on H^Γ can be defined by the rule: for $f \in H^\Gamma$ of the form $f = \sum_{k=1}^{n} c_k \Gamma(u_k - \cdot)$ we set

$$(\dot{w}_i, f) = \sum_{k=1}^{n} c_k \xi_i(u_k).$$

One can see that this correspondence produces a white noise on H^Γ. Further we consider a Hilbert space

$$l_2(H^\Gamma) = \{F = (f_1, \ldots, f_k, \ldots), f_k \in H^\Gamma, \sum_{k=1}^{\infty} \|f_k\|^2 < +\infty\}$$

and define a white noise on $l_2(H^\Gamma)$:

$$(\dot{w}, F) = \sum_{i=1}^{\infty} (\dot{w}_i, f_i).$$

In these terms, any element ξ_k from the sequence $\{\xi_n(u), u \in \mathbb{R}\}_{n \geq 1}$ can be obtained by the action of the white noise $\dot{w}$ on the function $F = (\underbrace{0, \ldots, 0}_{k-1}, \Gamma(u - \cdot), 0, \ldots, 0)$:

$$(\dot{w}, F) = \xi_k(u).$$

Consider

$$B(\mathbb{R}^m; \mathbb{R}) = \{f : \mathbb{R}^m \to \mathbb{R} | f \text{ is Borel measurable}, \sup_{\vec{u} \in \mathbb{R}^m} |f(\vec{u})| < +\infty\}$$

with topology induced by the norm $||f||_B = \sup_{\vec{u} \in \mathbb{R}^m} |f(\vec{u})|$. For $\vec{u} \in \mathbb{R}^m$ we write $\xi(\vec{u}) = (\xi(u_1), \ldots, \xi(u_m))$.

For any $\varphi \in B(\mathbb{R}^m; \mathbb{R})$ we have $\varphi(x_n(\vec{u})) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ and $\varphi(x_n(\vec{u}))$ is measurable with respect to the white noise $\dot{w}$ in $l_2(H^\Gamma)$.

We can construct an Itô–Wiener expansion for $\varphi(x_n(\vec{u}))$ in terms of operators Q_k, which are defined via the Itô–Wiener expansion for $\varphi(x_1(\vec{u}))$:

$$\varphi(u_1 + \xi_1(u_1), \ldots, u_m + \xi_1(u_m)) = \sum_{k=0}^{\infty} Q_k \varphi(\vec{u}; \dot{w}_1, \ldots, \dot{w}_1). \tag{2.5}$$

Note that in this case the random variable $\varphi(x_1(\vec{u}))$ is measurable with respect to $\sigma(\xi_1(u_1), \ldots, \xi_1(u_m))$ so $Q_k \varphi$ has the form:

$$Q_k \varphi(\vec{u}; \dot{w}_1, \ldots, \dot{w}_1) = \sum_{i_1, \ldots, i_k = 1}^{m} a_{i_1, \ldots, i_k}(\vec{u}) \xi_1(u_{i_1}) * \ldots * \xi_1(u_{i_k})$$

Lemma 2.1. *(i) For any $\vec{u} \in \mathbb{R}^m$, the mapping*

$$B(\mathbb{R}^m; \mathbb{R}) \ni \varphi \mapsto Q_k \varphi(\vec{u}; \cdot, \ldots, \cdot) \in H_k^\Gamma \tag{2.6}$$

is linear and continuous from $B(\mathbb{R}^m; \mathbb{R})$ to the space of k-linear Hilbert–Schmidt forms on H^Γ.

(ii) For any $\varphi \in B(\mathbb{R}^m;\mathbb{R})$, the mapping

$$\mathbb{R}^m \ni \vec{u} \mapsto Q_k\varphi(\vec{u};\cdot,\dots,\cdot) \in H_k^\Gamma \tag{2.7}$$

is Borel measurable and bounded.

Proof. (i) The linearity of the mapping (2.6) is obvious. To prove its continuity we note that

$$\begin{aligned} &k!\|Q_k\varphi(\vec{u};\cdot,\dots,\cdot)\|_k^2 = \mathbb{E}Q_k\varphi(\vec{u};\dot{w}_1,\dots,\dot{w}_1)^2 \le \\ &\le \mathbb{E}\varphi(u_1+\xi_1(u_1),\dots,u_m+\xi_1(u_m))^2 \le \sup_{\vec{u}\in\mathbb{R}^m}|\varphi(\vec{u})|^2. \end{aligned} \tag{2.8}$$

(ii) First of all, note that the mapping

$$L_2(\Omega,\sigma(\zeta),\mathbb{P}) \ni \eta \mapsto Q_k(\cdot,\dots,\cdot) \in H_k$$

which puts any square integrable measurable with respect to the white noise ζ random variable in correspondence with a Hilbert–Schmidt form of its Itô-Wiener expansion, is continuous and thus is Borel measurable. The continuity of the covariance function Γ implies the existence of a measurable modification of the random process $\{\varphi(u_1+\xi_1(u_1),\dots,u_m+\xi_1(u_m)),\vec{u}\in\mathbb{R}^m\}$. For any $\varkappa \in L_2(\Omega,\sigma(\xi_1),P)$, the function $\{\mathbb{E}\varphi(u_1+\xi_1(u_1),\dots,u_m+\xi_1(u_m))\varkappa,\vec{u}\in\mathbb{R}^m\}$ is Borel measurable, since the function under mathematical expectation is measurable as a function from (u,ω), and its mathematical expectation is finite for all u. In other words, for any linear continuous functional l on $L_2(\Omega,\sigma(\xi_1),P)$ the mapping

$$\mathbb{R}^m \ni \vec{u} \mapsto l(f(u_1+\xi_1(u_1),\dots,u_m+\xi_1(u_m))) \in \mathbb{R}$$

is Borel measurable. Since the Borel $\sigma-$algebra of Hilbert space $L_2(\Omega,\sigma(\xi_1),P)$ is induced by all linear continuous functionals, the mapping $\mathbb{R} \ni \vec{u} \mapsto f(\vec{u}+\xi_1(\vec{u})) \in L_2(\Omega,\sigma(\xi_1),P)$ is Borel measurable. Finally, the mapping (2.7) is Borel measurable as a composition of two measurable mappings. The boundedness follows from the inequality (2.8). The lemma is proved. □

To get the Itô–Wiener expansion for $\varphi(x_n(\vec{u}))$ it is necessary to define such expansion for multilinear form $Q_k\varphi(\vec{u}+\xi_1(\vec{u});\cdot,\dots,\cdot)$. In view of this we extend the domain of operators Q_k to a set of functions with values in some Hilbert space.

Let H be a separable Hilbert space with an orthonormal basis $\{e_j\}_{j=1}^\infty$. Denote by

$$B(\mathbb{R}^m;H) = \{F:\mathbb{R}^m\to H | F \text{ is Borel measurable}, \sup_{\vec{u}\in\mathbb{R}^m}\|F(\vec{u})\| < +\infty\}.$$

Then $F(\vec{u}) = \sum_{j=1}^\infty f_j(\vec{u})e_j$, where $f_j \in B(\mathbb{R}^m;\mathbb{R})$. We define an action of the operators Q_k on function $F \in B(\mathbb{R}^m;H)$ by the rule:

$$Q_kF(\vec{u};\cdot,\dots,\cdot) = \sum_{j=1}^\infty Q_kf_j(\vec{u};\cdot,\dots,\cdot)e_j,$$

where Q_kf_j are defined in (2.5). $Q_kF(\vec{u};\cdot,\dots,\cdot)$ is an H-valued Hilbert–Schmidt form on H^Γ and, for any $x_i \in H^\Gamma$,

$$\sum_{j=1}^\infty Q_kf_j(\vec{u};x_1,\dots,x_k)^2 \le \|x_1\|^2\dots\|x_k\|^2\sum_{j=1}^\infty\|Q_kf_j(\vec{u};\cdot,\dots,\cdot)\|_k^2 \le$$

$$\le \|x_1\|^2\dots\|x_k\|^2\sum_{j=1}^\infty\frac{1}{k!}\mathbb{E}f_j^2(\vec{u}+\xi_1(\vec{u})) \le \text{const}\,\frac{1}{k!}\cdot$$

$$\cdot\mathbb{E}\sum_{j=1}^\infty f_j^2(\vec{u}+\xi_1(\vec{u})) \le \text{const}\sup_{\vec{u}\in\mathbb{R}^m}\|F(\vec{u})\|^2 < +\infty.$$

Therefore the action of the map Q_k on the set $B(\mathbb{R}^m; H)$ is well defined. Denote by (H_k^Γ, H) the Hilbert space of a symmetric k-linear H-valued Hilbert–Schmidt forms on H^Γ. From the last estimations $Q_k : B(\mathbb{R}^m, H) \to B(\mathbb{R}^m, (H_k^\Gamma, H))$. In terms of these operators the Itô–Wiener expansion of the value of a Hilbert-valued function $F \in B(\mathbb{R}^m; H)$ at the point $\vec{u} + \xi(\vec{u})$ has the form:

$$F(\vec{u} + \xi(\vec{u})) = \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} Q_k f_j(\vec{u}; \dot{w}_1, \dots, \dot{w}_1)e_j =$$

$$= \sum_{k=0}^{\infty} Q_k F(\vec{u}; \dot{w}_1, \dots, \dot{w}_1),$$

where

$$Q_k F(\vec{u}; \dot{w}_1, \dots, \dot{w}_1) = \sum_{j=1}^{\infty} Q_k f_j(\vec{u}; \dot{w}_1, \dots, \dot{w}_1)e_j.$$

by definition.

Lemma 2.2. *(i) For any $\vec{u} \in \mathbb{R}^m$ the mapping*

$$B(\mathbb{R}^m; H) \ni \varphi \mapsto Q_k\varphi(\vec{u}; \cdot, \dots, \cdot) \in B(\mathbb{R}^m, (H_k^\Gamma, H))$$

is linear and continuous as a mapping from $B(\mathbb{R}^m; \mathbb{R})$ to the space of k-linear Hilbert–Schmidt forms on H^Γ with values in Hilbert space H.

(ii) For any $\varphi \in B(\mathbb{R}^m; H)$ the mapping

$$\mathbb{R}^m \ni \vec{u} \mapsto Q_k\varphi(\vec{u}; \cdot, \dots, \cdot) \in B(\mathbb{R}^m, (H_k^\Gamma, H))$$

is Borel measurable and bounded.

Proof. The proof is similar to the proof of the Lemma 2.1 and omitted. □

As example, we consider a Hilbert space of the k-linear Hilbert–Schmidt forms on H^Γ with an orthonormal basis $\{E_k^i(\cdot, \dots, \cdot)\}_{i=1}^\infty$. Then for $A_k \in B(\mathbb{R}^m; L_2(\mathbb{R}^k))$

$$A_k(\vec{u}; \cdot, \dots, \cdot) = \sum_{i=1}^{\infty} q_i(\vec{u})E_k^i(\cdot, \dots, \cdot)$$

and

$$A_k(\vec{u} + \xi_1(\vec{u}); \cdot, \dots, \cdot) = \sum_{j=0}^{\infty} Q_j A_k(\vec{u}, \underbrace{\dot{w}_1, \dots, \dot{w}_1}_{j}; \underbrace{\cdot, \dots, \cdot}_{k}),$$

where

$$Q_j A_k(\vec{u}, \dot{w}_1, \dots, \dot{w}_1; \cdot, \dots, \cdot) = \sum_{i=1}^{\infty} Q_j q_i(\vec{u}, \dot{w}_1, \dots, \dot{w}_1)E_k^i(\cdot, \dots, \cdot).$$

by definition.

Note that $Q_j A_k(\vec{u}, \underbrace{\cdot, \dots, \cdot}_{j}; \underbrace{\cdot, \dots, \cdot}_{k})$ is a $j+k-$linear Hilbert–Schmidt form which is symmetric with respect to the first j variables and the last k variables separately. Proceeding recurrently, $Q_{k_j} \dots Q_{k_1} A_{k_0}(\vec{u}, \underbrace{\cdot, \dots, \cdot}_{k_0}, \underbrace{\cdot, \dots, \cdot}_{k_1}, \dots, \underbrace{\cdot, \dots, \cdot}_{k_j})$ is a $k_0 + k_1 + \dots + k_j-$linear Hilbert–Schmidt form which is symmetric with respect to the sets of variables of sizes k_l, $(l = 0, \dots, j)$ separately.

Theorem 2.1. *Let $\{x_n(u), u \in \mathbb{R}\}_{n\geq 1}$ be a discrete-time flow defined by* (1.2)*. Then for any $\varphi \in B(\mathbb{R}^m; \mathbb{R})$ the Itô–Wiener expansion of $\varphi(x_n(\vec{u}))$ has the folllowing form:*

$$\varphi(x_n(\vec{u})) = \sum_{k=0}^{\infty} \sum_{\substack{l_1,\ldots,l_n\geq 0\\ l_1+\ldots+l_n=k}} Q_{l_n}Q_{l_{n-1}}\ldots Q_{l_1}\varphi(\vec{u}; \underbrace{\dot{w}_n,\ldots,\dot{w}_n}_{l_n},\ldots,\underbrace{\dot{w}_1,\ldots,\dot{w}_1}_{l_1}). \tag{2.9}$$

Proof. Let us verify that the iterated action of the operators Q_k in (2.9) is well defined. From the definition of Q_k we have:

$$\varphi(x_n(\vec{u})) = \varphi(x_{n-1}(\vec{u}) + \xi_n(x_{n-1}(\vec{u}))) = \\ = \sum_{k_1=0}^{\infty} Q_{k_1}\varphi(x_{n-1}(\vec{u}); \dot{w}_n,\ldots,\dot{w}_n).$$

The action of the operators Q_k on the function $\{Q_{k_1}\varphi(\vec{u};\cdot,\ldots,\cdot), \vec{u} \in \mathbb{R}^m\}$ with values in a Hilbert space is well-defined in the case when $\|Q_{k_1}\varphi(\vec{u};\cdot,\ldots,\cdot)\|^2_{k_1}$ is a measurable and bounded function in $\vec{u} \in \mathbb{R}^m$. The measurability follows from the statement (ii) of the Lemma 2.1 and the boundedness follows from the inequality:

$$\|Q_{k_1}\varphi(\vec{u};\cdot,\ldots,\cdot)\|^2_{k_1} \leq \frac{1}{k_1!}\mathbb{E}\varphi^2(\vec{u} + \xi_1(\vec{u})) \leq \frac{1}{k_1!}\sup_{\vec{u}\in\mathbb{R}^m}\varphi^2(\vec{u}).$$

Therefore, $Q_{k_2}Q_{k_1}\varphi$ is well defined and

$$Q_{k_1}\varphi(x_{n-2}(\vec{u}) + \xi_{n-1}(x_{n-2}(\vec{u})); \cdot,\ldots,\cdot) = \\ = \sum_{k_2=0}^{\infty} Q_{k_2}Q_{k_1}\varphi(x_{n-2}(\vec{u}); \underbrace{\cdot,\ldots,\cdot}_{k_1}, \underbrace{\dot{w}_{n-1},\ldots,\dot{w}_{n-1}}_{k_2}).$$

Further, the Hilbert space $\mathcal{H} = l_2(L_2(\mathbb{R}))$ can be conceived of as a direct sum of subspaces

$$\mathcal{L}^n = \{F \in \mathcal{H} : F = (\underbrace{0,\ldots,0}_{n-1}, f, 0,\ldots), f \in L_2(\mathbb{R})\}.$$

In these terms, the Hilbert–Schmidt form $Q_k\varphi(x_{n-1}(\vec{u}) + \xi_n(x_{n-1}(\vec{u})); \cdot,\ldots,\cdot)$ is defined on the subset $\mathcal{L}^{n+1}$ and is measurable with respect to the white noise in $\bigoplus_{k=1}^n \mathcal{L}^k$, so we get

$$Q_{k_1}\varphi(x_{n-1}(\vec{u}) + \xi_n(x_{n-1}(\vec{u})); \dot{w}_{n+1},\ldots,\dot{w}_{n+1}) = \\ = \sum_{k_2=0}^{\infty} Q_{k_2}Q_{k_1}\varphi(x_{n-2}(\vec{u}); \underbrace{\dot{w}_{n+1},\ldots,\dot{w}_{n+1}}_{k_1}, \underbrace{\dot{w}_n,\ldots,\dot{w}_n}_{k_2}).$$

Proceeding recurrently we prove the theorem. □

3. An explicit form of the discrete Krylov-Veretennikov expansion

Note that for any $\varphi \in B(\mathbb{R}^m; \mathbb{R})$ the random variable $\varphi(x_n(u_1),\ldots,x_n(u_m))$ is measurable with respect to $\sigma\{\xi_1,\ldots,\xi_n\} \equiv \sigma\{\xi_1(u), \xi_2(u),\ldots,\xi_n(u), u \in \mathbb{R}\}$. So it is natural to present the Itô–Wiener expansion of this random variable in terms of processes $\{\xi_k\}_{k\geq 1}$. As it was mentioned in the previous section, an action of the operators Q_k can be expressed in terms of Wick product:

$$Q_k\varphi(\vec{u}; \dot{w}_1,\ldots,\dot{w}_1) = \sum_{i_1,\ldots,i_k=1}^{m} a_{i_1,\ldots,i_k}(\vec{u})\ \xi_1(u_{i_1}) * \ldots * \xi_1(u_{i_k}).$$

To obtain the representation for the constants $a_{i_1\ldots i_k}$ we consider the case, when the covariance function Γ of processes $\{\xi_i(u), u \in \mathbb{R}\}$ is such that for any $u_1 < u_2 < \ldots < u_m$ the random vector $\{\xi_i(u_1),\ldots,\xi_i(u_m)\}$ has the density of its distribution. This condition is satisfied, for example, when Γ has the following form: $\Gamma(\cdot) = \int_{\mathbb{R}} \psi(u - \cdot)\psi(u)du,\ \psi \in$

$L_2(\mathbb{R})$, or the spectral measure of Γ has a density with respect to the Lebesgue measure. Indeed, the covariation matrix of the vector $\{\xi_i(u_1), \ldots, \xi_i(u_m)\}$ in the first case is

$$\big(\Gamma(u_i - u_j)\big)_{i,j=1}^{m} = \big((\psi(\cdot - u_i), \psi(\cdot - u_j))_{L_2(\mathbb{R})}\big)_{i,j=1}^{m}$$

and in the second case it is

$$\big(\Gamma(u_j - u_k)\big)_{j,k=1}^{m} = \Big(\int_{\mathbb{R}} e^{i\lambda(u_j - u_k)}\rho(\lambda)d\lambda\Big)_{i,j=1}^{m},$$

where ρ is a spectral density of Γ. In the first case Gram determinant is greater then zero if and only if the system of functions $\{\psi(\cdot - u_1), \psi(\cdot - u_2), \ldots, \psi(\cdot - u_m)\}$ is linearly independent in $L_2(\mathbb{R})$. It is known that for nonzero function $f \in L_2(\mathbb{R})$ and for all distinct $u_1, \ldots, u_k \in \mathbb{R}$ the system $\{f(\cdot - u_1), \ldots, f(\cdot - u_k)\}$ is linearly independent in $L_2(\mathbb{R})$ [10]. The second case is similar to the first one.

Denote the Fourier transform for $\varphi \in L_1(\mathbb{R}^m)$ by

$$\hat{\varphi}(\vec{\alpha}) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \varphi(\vec{x}) e^{i(\vec{\alpha},\vec{x})} d\vec{x}.$$

Lemma 3.1. *Let $p_\Gamma(\cdot, \vec{u})$ be the density of distribution of the random vector $(u_1 + \xi_1(u_1), \ldots, u_m + \xi_1(u_m))$. Then for any $\varphi \in B(\mathbb{R}^m; \mathbb{R}) \bigcap L_1(\mathbb{R}^m)$ and such that $\hat{\varphi} \in L_1(\mathbb{R}^m)$*

$$Q_k\varphi(\vec{u}; \dot{w}_1, \ldots, \dot{w}_1) = \frac{1}{k!} \sum_{i_1,\ldots,i_k}^{m} (-1)^k \int_{\mathbb{R}^m} \frac{\partial^k}{\partial\alpha_{i_1} \ldots \partial\alpha_{i_k}} p_\Gamma(\vec{\alpha}; \vec{u}) \varphi(\vec{\alpha}) d\vec{\alpha}\ \xi_1(u_{i_1}) * \xi_1(u_{i_2}) * \ldots * \xi_1(u_{i_k}). \tag{3.1}$$

Proof. We obtain the proposition of the lemma using well-known expansion for the stochastic exponent [9]: for $\vec{\alpha} \in \mathbb{C}^m$

$$\mathcal{E}(\vec{\alpha}, \xi_1(\vec{u})) \equiv \exp\{\sum_{i=1}^{m} \alpha_i \xi_1(u_i) - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j \Gamma(u_i - u_j)\} =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{m} \alpha_{i_1} \ldots \alpha_{i_k}\ \xi(u_{i_1}) * \ldots * \xi(u_{i_k}),$$

where the series converges in the square mean.

Using inverse Fourier transform, we have:

$$\varphi(\vec{u} + \xi_1(\vec{u})) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-i(\vec{u}+\xi_1(\vec{u}),\vec{\alpha})} \hat{\varphi}(\vec{\alpha}) d\vec{\alpha} =$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-i(\vec{u},\vec{\alpha}) - \frac{1}{2}\sum_{l,j=1}^{m} \alpha_l \alpha_j \Gamma(u_l - u_j)} \mathcal{E}(-i\vec{\alpha}, \xi_1(\vec{u})) \hat{\varphi}(\vec{\alpha}) d\vec{\alpha} =$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \overline{\hat{p}_\Gamma(\vec{\alpha}, \vec{u})} \hat{\varphi}(\vec{\alpha}) \mathcal{E}(-i\vec{\alpha}, \xi_1(\vec{u})) d\vec{\alpha}.$$

For any $\vec{\alpha} \in \mathbb{R}^m$ $\mathcal{E}(-i\vec{\alpha}, \xi_1(\vec{u})) \in L_2(\Omega, \sigma(\xi), \mathbb{P})$ and

$$||\mathcal{E}(-i\vec{\alpha}, \xi_1(\vec{u}))||_{L_2} = \exp\{\frac{1}{2} \sum_{l,m=1}^{m} \alpha_l \alpha_j \Gamma(u_l - u_j)\}.$$

Since $\hat{\varphi} \in L_1(\mathbb{R}^m)$, we get:

$$\int_{\mathbb{R}^m} ||\mathcal{E}(-i\vec{\alpha}, \xi_1(\vec{u}))||_{L_2} e^{-\frac{1}{2}\sum_{l,j=1}^{m} \alpha_l \alpha_j \Gamma(u_l - u_j)} |\hat{\varphi}(\vec{\alpha})| d\vec{\alpha} = \int_{\mathbb{R}^m} |\hat{\varphi}(\vec{\alpha})| d\vec{\alpha} < +\infty,$$

so, the Bochner integral $\int_{\mathbb{R}^m} \exp\{-i(\vec{u},\vec{\alpha})-\frac{1}{2}\sum_{l,j=1}^m \alpha_l\alpha_j\Gamma(u_l-u_j)\}\mathcal{E}(-i\vec{\alpha},\xi_1(\vec{u}))\hat{\varphi}(\vec{\alpha})d\vec{\alpha}$ is well-defined [11]. Denote by P_k an operator which for any $\eta \in L_2(\Omega,\sigma(\xi),\mathbb{P})$ assigns k-th term of Itô-Wiener expansion of η. Since Bochner integral commutes with continuous linear operators [11],

$$\int_{\mathbb{R}^m} \overline{\hat{p}_\Gamma(\vec{\alpha},\vec{u})}\hat{\varphi}(\vec{\alpha})\mathcal{E}(-i\vec{\alpha},\xi_1(\vec{u}))d\vec{\alpha} = \sum_{k=0}^{\infty} P_k\Big(\int_{\mathbb{R}^m} \overline{\hat{p}_\Gamma(\vec{\alpha},\vec{u})}\hat{\varphi}(\vec{\alpha})\mathcal{E}(-i\vec{\alpha},\xi_1(\vec{u}))d\vec{\alpha}\Big) =$$

$$= \sum_{k=0}^{\infty}\int_{\mathbb{R}^m} \overline{\hat{p}_\Gamma(\vec{\alpha},\vec{u})}\hat{\varphi}(\vec{\alpha})P_k\Big(\mathcal{E}(-i\vec{\alpha},\xi_1(\vec{u}))\Big)d\vec{\alpha}.$$

Finally, we have:

$$\varphi(\vec{u}+\xi_1(\vec{u})) = \sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\int_{\mathbb{R}^m}\sum_{i_1,\ldots,i_k=1}^{m}\alpha_{i_1}\ldots\alpha_{i_k}\overline{\hat{p}_\Gamma(\vec{\alpha},\vec{u})}\hat{\varphi}(\vec{\alpha})d\vec{\alpha}\ \xi_1(u_{i_1})*\ldots*\xi_1(u_{i_k}) =$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{R}^m}\sum_{i_1,\ldots,i_k=1}^{m}(-1)^k\frac{\partial^k}{\partial\alpha_{i_1}\ldots\partial\alpha_{i_k}}p_\Gamma(\vec{\alpha};\vec{u})\varphi(\vec{\alpha})d\vec{\alpha}\ \xi_1(u_{i_1})*\xi_1(u_{i_2})*\ldots*\xi_1(u_{i_k}).$$

The lemma is proved. □

Note, that in the case when $\Gamma(\cdot)=\int_{\mathbb{R}}\psi(u-\cdot)\psi(u)du$ for some $\psi\in L_2(\mathbb{R})$, $\psi(u)=\psi(-u)$, $\|\psi\|=1$, then for any $i\geq 1$ there exists a Wiener process $\widetilde{w}_i$ on an extended probability space such that $\xi_i(u)=\int_{\mathbb{R}}\psi(u-v)d\widetilde{w}_i(v)$ and $\{\widetilde{w}\}_{i\geq 1}$ are independent. In this case the Wick product can be rewritten in terms of multiple Ito integrals [9]:

$$\xi_1(u_{i_1})*\xi_1(u_{i_2})*\ldots*\xi_1(u_{i_k}) = \int_{\mathbb{R}^k}\psi(u_{i_1}-v_1)\star\ldots\star\psi(u_{i_k}-v_k)\ d\widetilde{w}_1(v_1)\ldots d\widetilde{w}_1(v_k),$$

where

$$\psi(u_{i_1}-v_1)\star\ldots\star\psi(u_{i_k}-v_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\prod_{j=1}^{k}\psi(u_{i_j}-v_{\sigma(j)}),$$

and S_k is the symmetric group of all permutations of $\{1,\ldots,k\}$. So we can rewrite (3.1) in the following way:

$$Q_k\varphi(\vec{u};\widetilde{w}_1,\ldots,\widetilde{w}_1) = \frac{1}{k!}\int_{\mathbb{R}^m}\sum_{i_1,\ldots,i_k=1}^{m}(-1)^k\frac{\partial^k}{\partial\alpha_{i_1}\ldots\partial\alpha_{i_k}}p_\Gamma(\vec{\alpha};\vec{u})\varphi(\vec{\alpha})d\vec{\alpha}\cdot$$

$$\cdot\int_{\mathbb{R}^k}\psi(u_{i_1}-v_1)\star\ldots\star\psi(u_{i_k}-v_k)\ d\widetilde{w}_1(v_1)\ldots d\widetilde{w}_1(v_k) =$$

$$= \frac{1}{k!}\sum_{i_1,\ldots,i_k=1}^{m}(-1)^k\int_{\mathbb{R}^k}\int_{\mathbb{R}^m}\frac{\partial^k}{\partial\alpha_{i_1}\ldots\partial\alpha_{i_k}}p_\Gamma(\vec{\alpha};\vec{u})\varphi(\vec{\alpha})d\vec{\alpha}\ \psi(u_{i_1}-v_1)\star\ldots\star$$

$$\star\psi(u_{i_k}-v_k)\ d\widetilde{w}_1(v_1)\ldots d\widetilde{w}_1(v_k).$$

To write the Itô–Wiener expansion for $\varphi(x_n(\vec{u}))$ we give some notations. For $k\geq 0$ and $m\geq 1$ we denote by

$$J(k,m)=\{r=(i_1,\ldots,i_k), i_j\in\{1,\ldots,m\}\}$$

a set of multi-indexes. For a vector $\vec{u}\in\mathbb{R}^m$ and some index $r\in J(k;m)$ we write

$$\vec{u}_r=(u_{i_1},\ldots,u_{i_k})\in\mathbb{R}^k$$

and

$$D^r f(\vec{x}) = \frac{\partial^k}{\partial x_{i_1}\ldots\partial x_{i_k}}f(\vec{x}).$$

Also we denote $\xi^{\circledast k}(\vec{u}_r) = \xi(u_{i_1}) * \ldots * \xi(u_{i_k})$. Denote by $S(\mathbb{R}^n)$ the Schwartz space, i.e. $S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{\vec{u}\in\mathbb{R}^n} |\vec{u}^\alpha D^\beta f(\vec{u})| \le +\infty,\ \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n),\ \alpha_i, \beta_i \in \mathbb{N}_0, i = 1, \ldots, n\}$

Theorem 3.1. *Suppose that the covariance function Γ has the following form $\Gamma(\cdot) = \int_{\mathbb{R}} \psi(u-\cdot)\psi(u)du$, where $\psi \in S(\mathbb{R})$ and $p_\Gamma(\cdot, \vec{u})$ is the density of distribution of the vector $\{u_1 + \xi_1(u_1), \ldots, u_m + \xi_1(u_m)\}$, $u_1 < \ldots < u_m$. Then for any $\varphi \in S(\mathbb{R}^m)$*

$$\varphi(x_n(\vec{u})) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{k_1! \ldots k_n!} \sum_{r_n \in J(k_n,m)} \cdots \sum_{r_1\in J(k_1,m)}$$

$$(-1)^{k_1+\ldots+k_n} \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} D^{r_n} p_\Gamma(\vec{\alpha}^{(n)}, \vec{\alpha}^{(n-1)}) D^{r_{n-1}} p_\Gamma(\vec{\alpha}^{(n-1)}, \vec{\alpha}^{(n-2)}) \ldots$$

$$D^{r_1} p_\Gamma(\vec{\alpha}^{(1)}, \vec{u}) \varphi(\vec{\alpha}^{(n)}) \xi_n^{\circledast k_n}(\vec{\alpha}_{r_n}^{(n-1)}) \xi_{n-1}^{\circledast k_{n-1}}(\vec{\alpha}_{r_{n-1}}^{(n-2)}) \cdot \ldots \cdot \xi_1^{\circledast k_1}(\vec{u}_{r_1}) d\vec{\alpha}^{(n)} \ldots d\vec{\alpha}^{(1)},$$

where the series converges in the square mean.

Proof. The proposition of the theorem is obtained by the iteration of the expansion formula (3.1) for $\varphi(x_1(\vec{u}))$. Denote by

$$I_{i_1,\ldots,i_k}(\varphi)(\vec{u}) = \frac{1}{k!} \int_{\mathbb{R}^m} \varphi(\vec{y}) \frac{\partial^k}{\partial y_{i_1} \ldots \partial y_{i_k}} p_\Gamma(\vec{y}, \vec{u}) d\vec{y},$$

where $\vec{u} \in \mathbb{R}^m$ such that $u_j \neq u_i,\ j \neq i$. Using inverse Fourier transform, we get another representation for $I_{i_1,\ldots,i_k}(\varphi)(\vec{u})$:

$$I_{i_1,\ldots,i_k}(\varphi)(\vec{u}) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} (-i)^k \hat{\varphi}(\vec{y}) y_{i_1} \ldots y_{i_k} \exp\{i(\vec{u}, \vec{y}) - \frac{1}{2} \sum_{l,j=1}^{m} \Gamma(u_l - u_j) y_l y_j\} d\vec{y}.$$

From this formula, the function $I_{i_1,\ldots,i_k}(\varphi)(\cdot)$ is well defined for all $\vec{u} \in \mathbb{R}^m$. We verify that for $\varphi \in S(\mathbb{R}^m)$ the function $I_{i_1,\ldots,i_k}(\varphi)(\cdot) \in S(\mathbb{R}^m)$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$

$$\int_{\mathbb{R}^m} \hat{\varphi}(\vec{y}) y_{i_1} \ldots y_{i_k} \frac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1} \ldots \partial u_m^{\alpha_m}} \exp\{i(\vec{u}, \vec{y}) - \frac{1}{2} \sum_{l,j=1}^{m} \Gamma(u_l - u_j) y_l y_j\} d\vec{y} =$$

$$= \int_{\mathbb{R}^m} \hat{\varphi}(\vec{y}) y_{i_1} \ldots y_{i_k} e^{i(\vec{u},\vec{y}) - \frac{1}{2}\sum_{l,j=1}^m \Gamma(u_l - u_j) y_l y_j} \mathcal{P}(\vec{y}, \vec{u}) d\vec{y},$$

where $\mathcal{P}$ is a polynomial from the variables $\{y_1, \ldots, y_m, \Gamma(u_i - u_j), \ldots,\ \Gamma^{(|\alpha|)}(u_i - u_j), i, j = 1, 2, \ldots, m\}$. Since $\varphi \in S(\mathbb{R}^m)$, the function $\hat{\varphi}$ also belongs to $S(\mathbb{R}^m)$, and the last integral locally in $\vec{u}$ uniformly converges and so $I_{i_1,\ldots,i_k}(\varphi)(\cdot)$ is differentiable. Each term of the polynomial $\mathcal{P}$ has the form $C\vec{y}^\gamma \prod_{k=1}^{K_1} \Gamma(u_{i_k} - u_{j_k}) \ldots \prod_{k=1}^{K_{|\alpha|}} \Gamma^{(|\alpha|)}(u_{i_k} - u_{j_k})$, where γ is some multi-index. Since $\varphi \in S(\mathbb{R}^m)$ $\{\hat{\varphi}(\vec{y})\vec{y}^\gamma,\ \vec{y} \in \mathbb{R}^m\} \in S(\mathbb{R}^m)$, so it is sufficient to prove that for any $g \in S(\mathbb{R}^m)$

$$\sup_{\vec{u}\in\mathbb{R}^m} |\vec{u}^\beta \prod_{k=1}^{K_1} \Gamma(u_{i_k} - u_{j_k}) \ldots \prod_{k=1}^{K_{|\alpha|}} \Gamma^{(|\alpha|)}(u_{i_k} - u_{j_k}) \cdot$$

$$\cdot \int_{\mathbb{R}^m} \hat{g}(\vec{y}) \exp\{i(\vec{u}, \vec{y}) - \sum_{l,j=1}^{m} \Gamma(u_l - u_j) y_l y_j\} d\vec{y}|$$

is finite.

Since for every $k \in \mathbb{N}_0$, $\Gamma^{(k)}$ is bounded, we can estimate the last expression by

$$C \sup_{\vec{u}\in\mathbb{R}^m} |\vec{u}^\beta \int_{\mathbb{R}^m} g(\vec{y}) p_\Gamma(\vec{y}, \vec{u}) d\vec{y}| = \sup_{\vec{u}\in\mathbb{R}^m} |\vec{u}^\beta \mathbb{E} g(\vec{u} + \xi_1(\vec{u}))|.$$

Since $g \in S(\mathbb{R}^m)$, $\sup_{\vec{u}\in\mathbb{R}^m} |\vec{u}^\beta g(\vec{u})| < +\infty$ we obtain that $I_{i_1,\dots,i_k}(\varphi)(\cdot) \in S(\mathbb{R}^m)$.

By (3.1)

$$\varphi(x_n(\vec{u})) = \varphi\Big(x_{n-1}(\vec{u}) + \xi_n\big((x_{n-1}(\vec{u}))\big)\Big) = \sum_{k_n=0}^{\infty} Q_{k_n}\varphi(x_{n-1}(\vec{u}); \dot{w}_n, \dots, \dot{w}_n) =$$

$$= \sum_{k_n=0}^{\infty} \frac{(-1)^{k_n}}{k_n!} \sum_{i_1,\dots,i_{k_n}}^{m} I_{i_1,\dots,i_{k_n}}(\varphi)(x_{n-1}(\vec{u}))\cdot\xi_n(v_1)*\xi_n(v_2)*\dots*\xi_n(v_{k_n}))\big|_{v_j=x_{n-1}(u_{i_j})}.$$

Under assumption $\Gamma(\cdot) = \int_{\mathbb{R}} \psi(u-\cdot)\psi(u)du$, the Wick product has the form:

$$\xi_j(u_{i_1}) * \xi_j(u_{i_2}) * \dots * \xi_j(u_{i_k}) = \int_{\mathbb{R}^k} \psi(u_{i_1} - v_1) \star \dots \star \psi(u_{i_k} - v_k) d\widetilde{w}_j(v_1) \dots d\widetilde{w}_j(v_k),$$

where $\{\widetilde{w}_j\}_{j\geq 1}$ are independent Brownian motions on $\mathbb{R}$.

Denote $K_{i_1,\dots,i_k}(\vec{u},\vec{v}) = \psi(u_{i_1} - v_1) \star \dots \star \psi(u_{i_k} - v_k)$, and in this terms the previous expansion has the form

$$\varphi(x_n(\vec{u})) = \sum_{k_n=0}^{\infty} \frac{(-1)^{k_n}}{k_n!} \sum_{i_1,\dots,i_{k_n}}^{m} \int_{\mathbb{R}^k} I_{i_1,\dots,i_{k_n}}(\varphi)(x_{n-1}(\vec{u}))\cdot$$
$$\cdot K_{i_1,\dots,i_{k_n}}(x_{n-1}(\vec{u}),\vec{v}) d\widetilde{w}_n(v_1) \dots d\widetilde{w}_n(v_k)$$

Since $\psi \in S(\mathbb{R})$, for fixed $\vec{v} \in \mathbb{R}^m$ the function $K_{i_1,\dots,i_k}(\cdot,\vec{v}) \in C^\infty(\mathbb{R}^m)$ is bounded. Also $I_{i_1,\dots,i_k}(\varphi) \in S(\mathbb{R}^m)$, so $I_{i_1,\dots,i_k}(\varphi)(\cdot)K_{i_1,\dots,i_k}(\cdot,\vec{v}) \in S(\mathbb{R}^m)$ and we can apply the formula (3.1):

$$\varphi(x_n(\vec{u})) = \sum_{k_n=0}^{\infty} \frac{(-1)^{k_n}}{k_n!} \sum_{i_1,\dots,i_{k_n}}^{m} \int_{\mathbb{R}^{k_n}} \sum_{k_{n-1}=0}^{\infty} \frac{(-1)^{k_{n-1}}}{k_{n-1}!}\cdot$$
$$\cdot \sum_{j_1,\dots,j_{k_{n-1}}=1}^{m} \int_{\mathbb{R}^{k_{n-1}}} I_{j_1,\dots,j_{k_{n-1}}}\big(I_{i_1,\dots,i_{k_n}}(\varphi)(\cdot)K_{i_1,\dots,i_{k_n}}(\cdot,\vec{v}^{(1)})\big)(x_{n-2}(\vec{u}))\cdot$$
$$\cdot K_{j_1,\dots,j_{k_{n-1}}}(x_{n-2}(\vec{u}),\vec{v}^{(2)}) d\widetilde{w}_{n-1}(\vec{v}^{(2)}) d\widetilde{w}_n(\vec{v}^{(1)}).$$

Denote by

$$A_{k_{n-1}}(\vec{u},\vec{v}^{(1)},\dot{\widetilde{w}}_{n-1},\dots,\dot{\widetilde{w}}_{n-1}) = \frac{(-1)^{k_{n-1}}}{k_{n-1}!} \sum_{j_1,\dots,j_{k_{n-1}}=1}^{m}$$

$$\int_{\mathbb{R}^{k_{n-1}}} I_{j_1,\dots,j_{k_{n-1}}}\big(I_{i_1,\dots,i_{k_n}}(\varphi)(\cdot)K_{i_1,\dots,i_{k_n}}(\cdot,\vec{v}^{(1)})\big)(\vec{u})K_{j_1,\dots,j_{k_{n-1}}}(\vec{u},\vec{v}^{(2)}) d\widetilde{w}_{n-1}(\vec{v}^{(2)}).$$

We verify that

$$\int_{\mathbb{R}^{k_n}} \sum_{k_{n-1}=0}^{\infty} A_{k_{n-1}}(x_{n-2}(\vec{u}),\vec{v}^{(1)},\dot{\widetilde{w}}_{n-1},\dots,\dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)}) =$$

$$= \sum_{k_{n-1}=0}^{\infty} \int_{\mathbb{R}^{k_n}} A_{k_{n-1}}(x_{n-2}(\vec{u}),\vec{v}^{(1)},\dot{\widetilde{w}}_{n-1},\dots,\dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)}).$$

Note that $A_{k_{n-1}}(\vec{u};\vec{v}^{(1)};\cdot,\dots,\cdot)$ is $k_{n-1}-$linear Hilbert–Schmidt form. For fixed N

$$\int_{\mathbb{R}^{k_n}} \sum_{k_{n-1}=0}^{N} A_{k_{n-1}}(x_{n-2}(\vec{u}),\vec{v}^{(1)},\dot{\widetilde{w}}_{n-1},\dots,\dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)}) =$$

$$= \sum_{k_{n-1}=0}^{N} \int_{\mathbb{R}^{k_n}} A_{k_{n-1}}(x_{n-2}(\vec{u}), \vec{v}^{(1)}, \dot{\widetilde{w}}_{n-1}, \ldots, \dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)}).$$

The series $\sum_{k_{n-1}=0}^{\infty} A_{k_{n-1}}(x_{n-2}(\vec{u}), \vec{v}, \dot{\widetilde{w}}_{n-1}, \ldots, \dot{\widetilde{w}}_{n-1})$ converges in square mean to $I_{i_1,\ldots,i_{k_n}}(\varphi)(x_{n-1}(\vec{u}))K_{i_1,\ldots,i_{k_n}}(x_{n-1}(\vec{u}), \vec{v})$. Since the Wiener processes $\widetilde{w}_{n-1}$ and $\widetilde{w}_n$ are independent, the random values $\{\int_{\mathbb{R}^{k_n}} A_j(\vec{u}, \vec{v}^{(1)}, \dot{\widetilde{w}}_{n-1}, \ldots, \dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)})\}_{j\geq 0}$ are orthogonal in $L_2(\Omega)$.

The series $\sum_{k_{n-1}=0}^{\infty} \int_{\mathbb{R}^{k_n}} A_{k_{n-1}}(x_{n-2}(\vec{u}), \vec{v}^{(1)}, \dot{\widetilde{w}}_{n-1}, \ldots, \dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)})$ converges in the square mean since

$$\sum_{j\geq 0} \mathbb{E}\Big(\int_{\mathbb{R}^{k_n}} A_j(x_{n-2}(\vec{u}), \vec{v}^{(1)}, \dot{\widetilde{w}}_{n-1}, \ldots, \dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)})\Big)^2 \leq \mathbb{E}\varphi(x_n(\vec{u}))^2 < +\infty.$$

Therefore we have

$$\int_{\mathbb{R}^{k_n}} \sum_{k_{n-1}=0}^{\infty} A_{k_{n-1}}(\vec{u}, \vec{v}^{(1)}, \dot{\widetilde{w}}_{n-1}, \ldots, \dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)}) =$$

$$= \sum_{k_{n-1}=0}^{\infty} \int_{\mathbb{R}^{k_n}} A_{k_{n-1}}(\vec{u}, \vec{v}^{(1)}, \dot{\widetilde{w}}_{n-1}, \ldots, \dot{\widetilde{w}}_{n-1}) d\widetilde{w}_n(\vec{v}^{(1)}).$$

Finally,

$$\begin{aligned} \varphi(x_n(\vec{u})) = &\sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{\infty} \sum_{i_1,\ldots,i_{k_n}=1}^{m} \sum_{j_1,\ldots,j_{k_{n-1}}=1}^{m} \frac{(-1)^{k_n}}{k_n!} \frac{(-1)^{k_{n-1}}}{k_{n-1}!} \int_{\mathbb{R}^{k_n}} \int_{\mathbb{R}^{k_{n-1}}} \\ &I_{j_1,\ldots,j_{k_{n-1}}}\big(I_{i_1,\ldots,i_{k_n}}(\varphi)(\cdot) K_{i_1,\ldots,i_{k_n}}(\cdot, \vec{v}^{(1)})\big)(x_{n-2}(\vec{u}))\cdot \\ &\cdot K_{j_1,\ldots,j_{k_{n-1}}}(x_{n-2}(\vec{u}), \vec{v}^{(2)}) d\widetilde{w}_{n-1}(\vec{v}^{(2)}) d\widetilde{w}_n(\vec{v}^{(1)}). \end{aligned} \tag{3.2}$$

By Fubini's theorem for stochastic integrals ([12], Theorem IV.65) we can change the order of integration in the expression

$$\int_{\mathbb{R}^{k_n}} \int_{\mathbb{R}^m} I_{i_1,\ldots,i_{k_n}}(\varphi)(\vec{y}) K_{i_1,\ldots,i_{k_n}}(\vec{y}, \vec{v}^{(1)}) \frac{\partial^{k_{n-1}}}{\partial y_{j_1} \ldots \partial y_{j_{k_{n-1}}}} p_\Gamma(\vec{y}, \vec{u}) d\vec{y} d\widetilde{w}_n(\vec{v}^{(1)}) =$$

$$= (-1)^{k_{n-1}} \int_{\mathbb{R}^{k_n}} \int_{\mathbb{R}^m} \frac{\partial^{k_{n-1}}}{\partial y_{j_1} \ldots \partial y_{j_{k_{n-1}}}} \Big(I_{i_1,\ldots,i_{k_n}}(\varphi)(\vec{y}) K_{i_1,\ldots,i_{k_n}}(\vec{y}, \vec{v}^{(1)}) \Big) \cdot$$

$$\cdot\, p_\Gamma(\vec{y}, \vec{u}) d\vec{y} d\widetilde{w}_n(\vec{v}^{(1)})$$

since $\frac{\partial^{k_{n-1}}}{\partial y_{j_1} \ldots \partial y_{j_{k_{n-1}}}} I_{i_1,\ldots,i_{k_n}}(\varphi)(\vec{y}) K_{i_1,\ldots,i_{k_n}}(\vec{y}, \cdot) \in S(\mathbb{R}^{k_n})$ and

$$\int_{\mathbb{R}^{k_n}} \int_{\mathbb{R}^m} \Big(\frac{\partial^{k_{n-1}}}{\partial y_{j_1} \ldots \partial y_{j_{k_{n-1}}}} I_{i_1,\ldots,i_{k_n}}(\varphi)(\vec{y}) K_{i_1,\ldots,i_{k_n}}(\vec{y}, \vec{v}^{(1)}) \Big)^2 p_\Gamma(\vec{y}, \vec{u}) d\vec{y} d\vec{v}^{(1)} < +\infty.$$

So we can rewrite (3.2) terms of the Wick product:

$$\varphi(x_n(\vec{u})) = \sum_{k_n=0}^{\infty} \sum_{r_n \in J(k_n, m)}^{m} \sum_{k_{n-1}=0}^{\infty} \sum_{r_{n-1} \in J(k_{n-1}, m)} \frac{(-1)^{k_n}}{k_n!} \frac{(-1)^{k_{n-1}}}{k_{n-1}!} \int_{\mathbb{R}^m} \cdot$$

$$\cdot \int_{\mathbb{R}^m} D^{r_n} p_\Gamma(\vec{\alpha}^{(n)}; \vec{\alpha}^{(n-1)}) \xi_n^{\circledast k_n}(\vec{\alpha}_{r_n}^{(n-1)}) D^{r_{n-1}} p_\Gamma(\vec{\alpha}^{(n-1)}; x_{n-2}(\vec{u})) d\vec{\alpha}^{(n-1)}$$

$$\xi_{n-1}^{\circledast k_{n-1}}(\vec{v})\Big|_{\vec{v}=x_{n-2}(\vec{u}_{r_n})} \varphi(\vec{\alpha}^{(n)}) d\vec{\alpha}^{(n)}.$$

Proceeding recurrently we prove the theorem. □

4. Example

In this section, we consider an example of a stochastic flow of solutions to SDE and a discrete-time flow which approximates it. For such flows we compare the first terms of their Krylov–Veretennikov representations.

Let us consider the following SDE with the space-time white noise W [7]:

$$x(u,t) = u + \int_{\mathbb{R}} \int_0^t \psi(x(u,s) - v) W(dv, ds), \tag{4.1}$$

where $\psi \in S(\mathbb{R})$, $\int_{\mathbb{R}} \psi^2(u)du = 1$, $\psi(u) = \psi(-u)$ and W is a Wiener sheet on $\mathbb{R}\times[0,+\infty)$.

We build an approximation in the form of a discrete-time flow using the following sequence of series of stationary Gaussian processes:

$$\{\xi_k^n(u) = \int_{\mathbb{R}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(u-v)W(dv,ds), u \in \mathbb{R}, k = 1, \ldots, n\}_{n\geq 1}$$

via the recurrence equation:

$$\begin{aligned} & x_{k+1}^n(u) = x_k^n(u) + \xi_{k+1}^n(x_k^n(u)), \\ & x_0^n(u) = u. \end{aligned} \tag{4.2}$$

We define a sequence of processes $\{x_n(u,t), u \in \mathbb{R}, t \in [0,1]\}_{n\geq 1}$ as a sequence of polygonal lines on time interval $[0,1]$ with edges at the points $\left(x_k^n(u), \frac{k}{n}\right)$. According to the Theorem 4 in [2] there is an estimation on the rate of convergence of the approximation scheme (4.2):

$$\mathbb{E} \sup_{u\in[0,1]} |x_n^n(u) - x(u,1)| \leq \frac{c}{\sqrt{n}} \tag{4.3}$$

So, for any continuous bounded function $\varphi : \mathbb{R}^m \to \mathbb{R}$

$$\varphi\Big(x_n(\vec{u},1)\Big) \xrightarrow{L_2} \varphi(x(\vec{u},1)).$$

Therefore, the expansion terms of $\varphi(x_n^n(\vec{u}))$ converge.

We give an informal calculation of the explicit form of the first term of the Itô–Wiener expansion for $\varphi(x(\vec{u},t))$ and derive a representation for $\varphi(x_{[kt]}^n(\vec{u}))$. Consider the following Cauchy problem:

$$\begin{aligned} & \frac{\partial}{\partial s}U(s,\vec{u}) + \frac{1}{2}\sum_{i,j=1}^m \Gamma(u_i - u_j)\frac{\partial^2}{\partial u_i \partial u_j}U(s,\vec{u}) = 0, s < t, \vec{u} \in \mathbb{R}^m, \\ & U(t,\vec{u}) = f(\vec{u}), \end{aligned}$$

where $\Gamma(\cdot) = \int_{\mathbb{R}} \psi(u - \cdot)\psi(u)du$.

Denote by $\{T_t\}_{t\geq 0}$ the set of operators that define a solution to this boundary value problem, i.e. $T_{t-s}f(\vec{u}) = U(s,\vec{u})$ and define $R_{t-s}^i f(\vec{u},v) = \psi(u_i - v)\frac{\partial}{\partial u_i}T_{t-s}f(\vec{u})$. By Itô formula applied to the function $\{U(t,\vec{u}), t \geq, \vec{u} \in \mathbb{R}\}$ we have:

$$f(x(\vec{u},t)) = T_t f(\vec{u}) + \sum_{i=1}^m \int_{\mathbb{R}} \int_0^t R_{t-s}^i f(x(\vec{u},s),v)W(dv,ds). \tag{4.4}$$

Note, that from this expression it is follows that $T_t f(\vec{u}) = \mathbb{E}f(x(\vec{u},t))$. Applying Itô formula to the function $R_{t-s}^i f(x(\vec{u},t),v)$ and using (4.4) we get the first term of Krylov–Veretennikov expansion:

$$A_1 f := \sum_{i=1}^m \int_{\mathbb{R}} \int_0^t T_s \psi(u_i - v)\frac{\partial}{\partial u_i}T_{t-s}f(\vec{u})W(dv,ds). \tag{4.5}$$

We can use the results of the previous section to write the first term of expansion for $f(x^n_{[kt]}(\vec{u}))$. Suppose that $f \in S(\mathbb{R}^m)$. Let us denote by $\{K_{\frac{k}{n}}\}_{k=1}^n$ the semigroup of operators which are defined by the rule:

$$K_{\frac{1}{n}} f(\vec{u}) = \mathbb{E} f(x^n_1(\vec{u})) = \int_{\mathbb{R}^m} f(\vec{x}) p_n(\vec{x}, \vec{u}) d\vec{x},$$

where $p_n(\cdot, \vec{u})$ is the density of distribution of the random vector $(u_1 + \xi^n_1(u_1), \ldots, u_m + \xi^n_1(u_m))$. Then

$$K_{\frac{k}{n}} := K^k_{\frac{1}{n}} f(\vec{u}) = \mathbb{E} f(x^n_k(\vec{u})).$$

Denote by $S^i_n f(\vec{u}) = \int_{\mathbb{R}^m} f(\vec{x}) \frac{\partial}{\partial x_i} p_n(\vec{x}, \vec{u}) d\vec{x}$. The first term of the Itô–Wiener expansion of $\varphi(x^n_k(\vec{u}))$ in terms of operators $\{Q_k\}_{k \geq 0}$, which were defined in Section 2, has the form:

$$\sum_{j=0}^{k-1} Q^j_0 Q_1 Q^{k-j-1}_0 \varphi(\vec{u}; \dot{w}_j),$$

where

$$\{w_k(v) = \int_{\mathbb{R}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(v - y) W(dy, ds), v \in \mathbb{R}\}$$

is a Wiener process.

Since $Q^k_0 f = K_{\frac{k}{n}} f$ and from the Lemma 3.1 for $f \in S(\mathbb{R}^m)$:

$$Q_1 f(\vec{u}) = -\sum_{i=1}^{m} \int_{\mathbb{R}} \int_0^{\frac{1}{n}} S^i_n f(\vec{u}) \psi(u_i - v) W(dv, ds),$$

by the Theorem 3.1 the first term of Itô–Wiener expansion of $f(x^n_k(\vec{u}))$ has the form:

$$A_{1,n} f := -\sum_{j=0}^{k-1} \sum_{i=1}^{n} \int_{\mathbb{R}} \int_{\frac{j}{n}}^{\frac{j+1}{n}} K_{\frac{j}{n}} (\psi(u_j - v) S^i_n K_{\frac{k-j-1}{n}} f(\vec{u})) W(dv, ds).$$

Note that the operator $K_{\frac{j}{n}}$ can be exchanged with the integral with respect to Brownian sheet if ([12], Theorem IV.65)

$$\int_{\mathbb{R}} \int_{\mathbb{R}^m} \left(\psi(y_j - v) S^i_n K_{\frac{k-j-1}{n}} f(\vec{y})\right)^2 p^{(j)}_n(\vec{y}, \vec{u}) d\vec{y} dv < +\infty,$$

where $p^{(j)}_n(\cdot, \vec{u})$ is the density of distribution of the random vector $x^n_j(\vec{u})$. By Fubini's theorem the last integral is equal to

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}} \psi^2(y_j - v) \left(S^i_n K_{\frac{k-j-1}{n}} f(\vec{y})\right)^2 p^{(j)}_n(\vec{y}, \vec{u}) dv d\vec{y} =$$

$$= \int_{\mathbb{R}^m} \left(S^i_n K_{\frac{k-j-1}{n}} f(\vec{y})\right)^2 p^{(j)}_n(\vec{y}, \vec{u}) d\vec{y}.$$

As is was proved in the previous section, for $f \in S(\mathbb{R}^m)$ $K_{\frac{k-j-1}{n}} f \in S(\mathbb{R}^m)$, and so $S^i_n K_{\frac{k-j-1}{n}} f \in S(\mathbb{R}^m)$, therefore the last integral is finite. Integrating by parts we obtain for every $j = 0, \ldots, k-1$:

$$S^i_n K_{\frac{k-j-1}{n}} f(\cdot) = -\int_{\mathbb{R}^m} p_n(\vec{x}, \cdot) \frac{\partial}{\partial x_i} K_{\frac{k-j-1}{n}} f(\vec{x}) d\vec{x} =$$

$$= -K_{\frac{1}{n}} \frac{\partial}{\partial x_i} K_{\frac{k-j-1}{n}} f(\cdot).$$

Therefore, the first term has the form:

$$A_{1,n}f := \sum_{i=1}^{m}\sum_{j=0}^{k-1}\int_{\mathbb{R}}\int_{\frac{j}{n}}^{\frac{j+1}{n}} K_{\frac{j}{n}}(\psi(u_i - v)K_{\frac{1}{n}}\frac{\partial}{\partial u_i}K_{\frac{k-j-1}{n}}f(\vec{u}))W(dv, ds). \tag{4.6}$$

As one can see, this formula is similar to the first term of Krylov–Veretennikov representation for $f(x(u,t))$ (4.5).

From the estimation (4.3) we get the rate of convergence for the first term of expansion:

Proposition. *Let* $\varphi \in \mathrm{Lip}(\mathbb{R}^m)$, *then*

$$\mathbb{E}|A_1\varphi(\vec{u}) - A_{1,n}\varphi(\vec{u})| \le \frac{c_1}{\sqrt{n}}.$$

Proof. The proposition follows from the estimations:

$$\begin{aligned}\mathbb{E}(A_1\varphi(\vec{u}) - A_{1,n}\varphi(\vec{u}))^2 &\le \mathbb{E}(\varphi(x(\vec{u},1)) - \varphi(x_n^n(\vec{u}))^2 \le \\ &\le L^2\mathbb{E}\|x(\vec{u},1) - x_n^n(\vec{u})\|^2 \le \\ &\le mL^2\mathbb{E}\sup_{u\in[0,1]}|x(u,1) - x_n^n(u)|^2 \le \frac{mL^2c}{n}.\end{aligned}$$

□

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