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EXCHANGEABLE OPTIMAL TRANSPORTATION AND LOG-CONCAVITY

We study the Monge and Kantorovich transportation problems on $\mathbb{R}^\infty$ within the class of exchangeable measures. With the help of the de Finetti decomposition theorem the problem is reduced to an unconstrained optimal transportation problem on a Hilbert space. We find sufficient conditions for convergence of finite-dimensional approximations to the Monge solution. The result holds, in particular, under certain analytical assumptions involving log-concavity of the target measure. As a by-product we obtain the following result: any uniformly log-concave exchangeable sequence of random variables is i.i.d.

1. INTRODUCTION

We consider the Polish linear space $\mathbb{R}^\infty$ equipped with the standard Borel sigma-algebra and two Borel exchangeable probability measures μ (the source measure) and ν (the target measure). A Borel measure is called exchangeable if it is invariant with respect to all permutations of finitely many coordinates, i.e., under every linear operator g of the form

$$g(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, x_{n+1}, \ldots),$$

for every point $x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$, where σ is an element of S_n (the permutation group of n elements). The family of all such transformations g is denoted by $\mathcal{S}_\infty$ and called the infinite permutation group.

We say that a measure π on $X \times Y$, where $X = Y = \mathbb{R}^\infty$, is exchangeable if it is invariant with respect to every mapping

$$(x, y) \mapsto (g(x), g(y)), \; g \in \mathcal{S}_\infty.$$

Finally, a mapping $T \colon \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ is exchangeable if $T \circ g = g \circ T$ for every $g \in \mathcal{S}_\infty$.

Throughout the paper we use the following notation. Let $\mathcal{P}(X)$ denote the space of Borel probability measures on a topological space X, let $\mathcal{P}_{ex}(\mathbb{R}^\infty)$ denote the space of exchangeable probability measures on $\mathbb{R}^\infty$, and let $\mathcal{P}_2(\mathbb{R})$ denote the space of Borel probability measures on $\mathbb{R}$ with finite second moment. We use the notation $W_2(P, Q)$ for the standard quadratic Kantorovich distance between two measures P, Q on a metric space (see [2], [10]).

We are interested in the following transportation problems.

Problem 1.1. Exchangeable Kantorovich problem. Given $\mu, \nu \in \mathcal{P}_{ex}(\mathbb{R}^\infty)$, find the minimum $K(\pi)$ of the functional

$$\pi \mapsto \int (x_1 - y_1)^2 \, d\pi$$

on the set $\mathcal{P}_{ex}(\mu, \nu)$ of exchangeable measures on $\mathbb{R}^\infty \times \mathbb{R}^\infty$ with marginals μ, ν.

Problem 1.2. Exchangeable Monge problem. Given $\mu, \nu \in \mathcal{P}_{ex}(\mathbb{R}^\infty)$, find a Borel exchangeable mapping $T \colon \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ such that the measure

$$\pi = \mu \circ (x, T(x))^{-1}$$

is a solution to the exchangeable Kantorovich problem.

The mapping T is called an exchangeable optimal transportation.

Our motivation for the study of these problems comes from the fact that similar problems on $\mathbb{R}^n$ are equivalent to the standard Monge and Kantorovich problems with the same marginals and the cost function $\sum_{i=1}^{n}(x_i - y_i)^2$ (see [7], [8], [11]). Thus, problems (1.1), (1.2) can be viewed as natural generalizations of the standard Monge–Kantorovich problem to the case of infinite-dimensional exchangeable marginals. Note that two different infinite-dimensional exchangeable marginals on $\mathbb{R}^\infty$ have an infinite Kantorovich distance if one defines it in the standard way (via the minimization of $\int (x - y)_{l_2}^2 d\pi$). In contrast to this, the value of the corresponding minimum of the Kantorovich potential is the squared distance on the space $\mathcal{P}_{ex}(\mathbb{R}^\infty)$. More explanations and results can be found in [7], [12]. See also [9] for similar problems on graphs.

It will be assumed throughout that

$$(1) \qquad\qquad \int x_1^2 d\mu + \int y_1^2 d\nu < \infty.$$

Since the cost function is continuous, the solvability of the Kantorovich problem can be shown by the standard compactness argument.

The paper is organized as follows. In Section 2 we show that the exchangeable Monge problem is equivalent to the classical Monge problem on a convex subset of the Hilbert space l^2. This is shown with the help of the de Finetti-type (ergodic) decomposition for transportation plans. The reduction to l^2 makes possible to apply the standard machinery of transportation theory (duality, convex analysis etc.) to the existence problem.

In Section 3 we pursue a completely different approach, namely, we study the problem whether the optimal transportation is a limit of natural finite-dimensional approximations. We emphasize that this problem is far from being trivial. The affirmative answer is established under quite special assumptions about marginals. Moreover, it implies an unexpected result on the structure of exchangeable measures with additional analytical properties. More precisely, we approximate the marginals by their finite-dimensional projections μ_n, ν_n. We prove that the solutions T_n to the standard Monge problem for μ_n, ν_n converge μ-a.e. to the desired mapping T provided that T_n are uniformly globally Lipschitz:

$$\|T_n(x) - T_n(y)\| \leq K\|x - y\|.$$

This assumption can be verified for certain measures, in particular, in the following model situation: μ is the standard Gaussian measure and ν is uniformly log-concave. Comparing this with the existence result of the Section 2, we obtain the following corollary: every exchangeable uniformly log-concave measure is a countable power of a certain one-dimensional distribution.

2. Reduction to the Hilbert space

Given a Borel probability measure m on $\mathbb{R}$, we denote by m^∞ its countable power (i.i.d. distributions with law m), which is a probability measure on $\mathbb{R}^\infty$.

According to a generalization of the classical de Finetti theorem (see [1], [5]), the exchangeable measures are precisely the mixtures of countable powers.

Theorem 2.1. (the generalized De Finetti theorem). *For every Borel exchangeable measure μ on $\mathbb{R}^\infty$, there exists a Borel probability measure Π on $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ such that*

$$\mu(B) = \int m^\infty(B)\Pi(dm),$$

for every Borel set $B \subset \mathbb{R}^\infty$.

Next, given the de Finetti decomposition of two marginals, we apply the following decomposition theorem, which is a particular case of a result from [12] on ergodic decompositions of optimal transportation plans.

Theorem 2.2. [12]. *Assume we are given the de Finetti decompositions*

$$(2) \qquad \mu = \int_X \mu_x^\infty \, d\sigma_\mu, \ \nu = \int_Y \nu_y^\infty \, d\sigma_\nu$$

of the measures μ, ν, where $X = Y = \mathcal{P}(\mathbb{R})$ and, similarly, the ergodic decomposition of π:

$$(3) \qquad \pi = \int_{\mathcal{P}(\mathbb{R}^2)} \pi_{x,y} d\delta.$$

Then, for δ-almost all (x,y), the measure $\pi_{x,y}$ solves the one-dimensional quadratic Kantorovich problem with marginals μ_x, ν_y:

$$\int (t-s)^2 d\pi_{x,y}(t,s) = W_2^2(\mu_x, \nu_y) = \min_{\theta \in \mathcal{P}(\mu_x, \nu_y)} \int (t-s)^2 d\theta(t,s)$$

and the following representation formula holds:

$$\min_{\pi \in \Pi_{ex}(\mu,\nu)} \int (x_1 - y_1)^2 \, d\pi = \inf_{\delta \in \Pi(\sigma_\mu, \sigma_\nu)} \int W_2^2(\mu^x, \nu^y) \, d\delta.$$

It follows immediately from Theorem 2.2 that the Monge problem can be similarly decomposed in two Monge problems:

1) the Monge problem for the measures σ_μ, σ_ν and the cost function $(\mu, \nu) \mapsto W_2^2(\mu, \nu)$ on $\mathcal{P}(\mathbb{R})$.
2) The one-dimensional Monge problem for the measures μ_x, ν_y and the quadratic cost function.

The following conclusion is straightforward.

Corollary 2.1. *The exchangeable Monge problem admits a solution if and only if problem 1) is solvable and, moreover, problem 2) is solvable for σ_μ-almost all μ^x and σ_ν-almost all ν^y.*

The exchangeable Monge problem is not always solvable. For instance, if μ is a countable power, but ν is not, then there is no optimal transportation of μ onto ν.

Proof. All these assertions are immediate excepting the "only if" part. We have to show that every exchangeable optimal transportation T induces an optimal transportation mapping $\mathcal{T} \colon \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$. This follows easily from the fact that every exchangeable mapping T is diagonal almost everywhere (i.e., has the form $T(x) = (t(x_1), \ldots, t(x_n), \ldots)$ for some $t \colon \mathbb{R} \mapsto \mathbb{R}$) with respect to any countable power μ_x^∞. This is an immediate consequence of the fact that every exchangeable function f is constant for μ_x^∞-almost all points by the Hewitt–Savage $0-1$ law. Thus, the induced mapping can be defined as follows: $\mathcal{T}(\mu_x) = \mu_x \circ t^{-1}$. The optimality of the latter mapping follows from Theorem 2.2. $\qquad \square$

Since the one-dimensional Monge problem admits a precise solution under appropriate easy-to-check sufficient conditions, the exchangeable Monge problem is reduced to the Monge problem on the metric space

$$(\mathcal{P}_2(\mathbb{R}), W_2(\mathbb{R}))$$

with the cost function W_2^2.

Remarkably, the problem can be further reduced to a problem on a **linear** space. This can be done with the help of the well-known fact that $(\mathcal{P}_2, W_2)$ is isomorphic to a convex subset of $L_2([0,1])$. The distance preserving isomorphism

$$\mathcal{I} \colon \mathcal{P}_2 \mapsto L^2([0,1])$$

has the form

$$\mathcal{I}(\mu) = F_\mu^{-1},$$

where F_μ^{-1} is the inverse distribution function of μ. In case where the distribution function F_μ is not one-to-one, we simply set

$$F_\mu^{-1}(t) = \inf\{s \colon \mu(-\infty, s] > t\}.$$

Thus, the set

$$\mathcal{K} = \mathcal{I}(\mathcal{P}_2(\mathbb{R}))$$

consists of non-decreasing right continuous mappings which belong to $L^2[0,1]$.

Finally, we conclude that the exchangeable Kantorovich and Monge problems are reduced to the same problems on the subset $\mathcal{K}$ of l^2 equipped with the standard l_2-metric.

The existence of optimal transportation mappings on the Hilbert space is known under assumptions given below. It was obtained in [4] and can be constructed with the help of by now standard arguments. Indeed, one can consider the solution (φ, ψ) to the dual Kantorovich problem

(4)
$$\int \varphi d\mu + \int \psi d\nu \to \sup, \ \ \varphi(x) + \psi(y) \le |x - y|_{l^2}^2.$$

It follows from general results on the dual Kantorovich problem that for every solution π to the original Kantorovich problem there exists a solution (φ, ψ) to (4) such that

$$\varphi(x) + \psi(y) \le |x - y|_{l^2}^2$$

and $\varphi(x) + \psi(y) = |x - y|_{l^2}^2$ π-a.e. From these relations we infer that for π-a.e. points (x_0, y_0) one has $y_0 \in \partial \varphi(x_0)$, where $\varphi(x_0)$ is the superdifferential of φ at x_0. To construct the corresponding optimal transportation (and prove uniqueness of solutions to all the associated optimal transportation problems) it is sufficient to ensure that $\partial \varphi(x_0)$ contains a unique element μ-a.e. This was verified in [4] under the assumption of regularity of μ.

Definition 2.1. Assume that we are given a sequence of vectors $\{e_i\}$ such that the closure of $span(\{e_i\})$ contains the topological support of μ. Let us disintegrate μ with respect to e_i:

$$\mu = \int_{X_i^\perp} \mu^x d\mu_i, \quad \mu_i = \mu \circ Pr_i^{-1},$$

where Pr_i is the orthogonal projection onto $X_i^\perp = \{x \colon x \perp e_i\}$ and $\{\mu^x\}$ is the corresponding family of conditional measures.

The measure μ on l^2 is called regular if, for μ_i-almost every x, the conditional measure μ^x is atomless.

Therefore, the following result holds.

Theorem 2.3. [4]. *Let μ, ν be two Borel probability measures on*

$$(\mathcal{P}(\mathbb{R}), W_2^2(\mathbb{R})) \sim (\mathcal{K}, \|\cdot\|_{l^2}) \subset l^2.$$

Assume that

$$\int |x|_{l_2}^2 d\mu + \int |y|_{l_2}^2 d\nu < \infty$$

and the source measure μ is regular in the sense of Definition 2.1. Then there exists unique solutions π, (φ, ψ) to the primal and the dual Kantorovich problems and a unique solution to the Monge problem that has the form

$$T(x) = x - \partial\varphi(x).$$

3. Finite-dimensional approximations and log-concavity

In this section we pursue a completely different approach to the existence for the Monge problem. We construct the optimal mapping as a limit of finite-dimensional approximations. In should be emphasized that it is usually hard to capture the decomposition structure given by the de Finetti theorem if the exchangeable measure is given as a limit of finite-dimensional approximations. This is the reason why the main result of this section looks completely unrelated to the de Finetti decomposition and abstract sufficient conditions obtained in the previous section.

The projection $P_n \colon \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$ onto the first n coordinates will be denoted by P_n:

$$P_n(x) = (x_1, x_2, \cdots, x_n, 0, 0, \cdots).$$

Let us consider the projections $\mu_n = \mu \circ P_n^{-1}$, $\nu_n = \nu \circ P_n^{-1}$ of the marginals.

Clearly, the measures μ_n, ν_n are exchangeable as well (considered as measures on $\mathbb{R}^n$, i.e., invariant with respect to all permutations of the first n coordinates). Let π_n be the solution to the corresponding finite-dimensional exchangeable Monge–Kantorovich problem

$$\int (x_1 - y_1)^2 \, dm \to \inf$$

where the infimum is taken among all $2n$-dimensional exchangeable measures with marginals μ_n, ν_n. Equivalently, one can solve the standard Monge–Kantorovich problem with the cost function $\sum_{i=1}^n (x_i - y_i)^2$ instead.

Let

$$T_n(x) = \nabla\Phi_n(x)$$

be the corresponding optimal transportation mapping.

Assumption A. There exists $K > 0$ such that the potentials Φ_n satisfy the inequality

$$\Phi_n(a) - \Phi_n(b) - \langle \nabla\Phi_n(b), a - b \rangle \le K\|a - b\|^2$$

for all n, $a, b \in \mathbb{R}^n$.

Equivalently, the dual potentials Ψ_n satisfy the inequality

$$\Psi_n(a) - \Psi_n(b) - \langle \nabla\Psi_n(b), a - b \rangle \ge \frac{\|a - b\|^2}{K}.$$

Remark 3.1. Clearly, assumption **A** is equivalent to the requirement that every optimal mapping $\mathbb{R}^n \ni x \mapsto \nabla\Phi_n(x)$ is K-Lipschitz:

$$|\nabla\Phi_n(x) - \nabla\Phi_n(y)| \le K|x - y|$$

on $\mathbb{R}^n$.

Theorem 3.1. *Under assumptions **A** and (1), there exists a solution π to problem (1.2).*

Proof. Since the marginals of $\{\pi_n\}$ constitute tight sequences, the sequence $\{\pi_n\}$ of measures on $\mathbb{R}^\infty \times \mathbb{R}^\infty$ is tight. Hence one can extract a weakly convergent subsequence (denoted for brevity again by $\{\pi_n\}$) $\pi_n \to \pi$. Clearly, π is exchangeable and its marginals are μ and ν. Let us show that π is a solution to problem (1.1). Indeed, assuming the contrary, we obtain that there exists another exchangeable measure $\tilde{\pi}$ such that

$$\int (x_1 - y_1)^2 d\tilde{\pi} < \int (x_1 - y_1)^2 d\pi.$$

It follows from the weak convergence and (1) that

$$(5) \qquad \int (x_1 - y_1)^2 d\pi = \lim_n \int (x_1 - y_1)^2 d\pi_n.$$

Hence $\int (x_1 - y_1)^2 d\tilde{\pi} < \int (x_1 - y_1)^2 d\pi_N$ for some N. But this contradicts the optimality of π_N, because the projection of $\tilde{\pi}$ onto $\mathbb{R}^N \times \mathbb{R}^N$ satisfies the corresponding constraints and gives a better value to the Kantorovich functional.

By the change of variables

$$\int \partial_{x_i} \Phi_n^2 \, d\mu = \int \partial_{x_i} \Phi_n^2 \, d\mu_n = \int y_i^2 d\nu_n = \int y_i^2 d\nu < \infty$$

for every $i \le n$. Let us pass to a subsequence of the sequence $\{\partial_{x_i} \Phi_n\}$ (denoted again by $\{\partial_{x_i} \Phi_n\}$). Applying the diagonal method, one can assume without loss of generality that

$$\partial_{x_i} \Phi_n \to T_i$$

weakly in $L^2(\mu)$ for every i. We show that $T = (T_1, T_2, \ldots, T_n, \ldots)$ is the desired mapping. By standard measure-theoretical arguments it is sufficient to show that $\partial_{x_i} \Phi_n \to T_i$ in measure.

Consider the quantity

$$D_n = \int \Big(\Phi_n(x) + \Psi_n(y) - \sum_{i=1}^n x_i y_i \Big) d\pi,$$

where Ψ_n is the Legendre transform of Φ_n (the dual potential). Since the integrand is nonnegative, one has $D \ge 0$. Since

$$\int \Phi_n d\pi = \int \Phi_n d\mu = \int \Phi_n d\mu_n = \int \Phi_n d\pi_n,$$

$$\int \Psi_n d\pi = \int \Psi_n d\nu = \int \Psi_n d\nu_n = \int \Psi_n d\pi_n,$$

and $\Phi_n + \Psi_n = \sum_{i=1}^n x_i y_i$ π_n-almost everywhere, we have

$$D_n = \int \sum_{i=1}^n x_i y_i (d\pi_n - d\pi).$$

Exchangeability of π, π_n implies that all the pairs (x_i, y_i) are equally distributed. Hence

$$D_n = n \int x_1 y_1 (d\pi_n - d\pi).$$

We obtain, in particular, that

$$(6) \qquad \lim_{n \to \infty} \frac{D_n}{n} = 0.$$

This follows easily from the weak convergence $\pi_n \to \pi$ and (5).

On the other hand,

$$D_n = \lim_m D_{n,m},$$

where

$$D_{n,m} = \int (\Phi_n(x) + \Psi_n(y) - \sum_{i=1}^{n} x_i y_i) d\pi_m = \int (\Phi_n(x) + \Psi_n(\nabla \Phi_m) - \sum_{i=1}^{n} x_i \partial_{x_i} \Phi_m) d\pi_m.$$

Indeed, by the same arguments as above we show that

$$\int (\Phi_n(x) + \Psi_n(y)) d\pi_m = \int \Phi_n d\mu + \int \Psi_n d\nu$$

whenever $m \geq n$ and

$$\int x_i y_i d\pi_m \to \int x_i y_i d\pi$$

for every $i \leq n$.

Taking into account the identity

$$\Phi_n(x) = -\Psi_n(\nabla \Phi_n) + \sum_{i=1}^{n} x_i \partial_{x_i} \Phi_n$$

we obtain

$$D_{n,m} = \int \Psi_n(\nabla \Phi_m(x)) - \Psi_n(\nabla \Phi_n(x)) - \sum_{i=1}^{n} x_i (\partial_{x_i} \Phi_m - \partial_{x_i} \Phi_n) d\mu.$$

Assumption **A** implies that

$$D_{n,m} \geq \frac{1}{K} \int |Pr_n \nabla \Phi_m - \nabla \Phi_n|^2 d\mu.$$

Passing to the limit as $m \to \infty$ and applying the $L^2(\mu)$-weak convergence $\partial_{x_i} \Phi_m \to T_i$ we obtain by the well-known properties of the L^2-weak convergence that

$$K D_n \geq \int |Pr_n T - \nabla \Phi_n|^2 d\mu.$$

Since $Pr_n T$ and $\nabla \Phi_n$ commute with permutations of the first n coordinates, we have

$$\frac{K D_n}{n} \geq \int (T_1 - \partial_{x_1} \Phi_n)^2 d\mu.$$

Then (6) implies that $\partial_{x_1} \Phi_n \to T_1$ in measure. By exchangeability the same holds for every x_i: $\lim_n \partial_{x_i} \Phi_n = T_i$. The proof is complete. □

As an interesting byproduct we obtain a characterization of the uniformly log-concave exchangeable measures.

We recall that a probability measure μ on $\mathbb{R}^n$ is called log-concave if it has the form $e^{-V} \cdot \mathcal{H}^k|_L$, where L is an affine subspace of dimension $k \in \{0, 1, \cdots, n\}$ with the corresponding k-dimensional Lebesgue measure $\mathcal{H}^k$ and V is a convex function.

In what follows we consider uniformly log-concave measures. Roughly speaking, these are measures with potentials V satisfying the inequality

$$V(x) - V(y) - \langle \nabla V(y), x - y \rangle \geq \frac{K}{2} |x - y|^2, \quad K > 0,$$

which is equivalent to $D^2 V \geq K \cdot \mathrm{Id}$ in the smooth (finite-dimensional) case.

More precisely, we say that a probability measure μ is K-uniformly log-concave ($K > 0$) if, for any $\varepsilon > 0$, the measure $\hat{\mu} = \frac{1}{Z} e^{\frac{K-\varepsilon}{2}|x|^2} \cdot \mu$ is log-concave with a suitable renormalization factor Z. According to a result of Borell [3], the projections of log-concave measures are log-concave (this is in fact a corollary of the Brunn–Minkowski inequality). It can be easily checked that the uniform log-concavity is preserved by projections as well. We can extend this notion to the infinite-dimensional case. Namely,

we call a probability Borel measure μ on a locally convex space X log-concave (K-uniformly log-concave with $K > 0$) if its images $\mu \circ l^{-1}$, $l \in X^*$ under linear continuous functionals are all log-concave (K-uniformly log-concave with $K > 0$).

Another classical result we apply below is the famous Cafarelli contraction theorem. Here is its version from [6].

Theorem 3.2. (Caffarelli's contraction theorem). *Let $\nabla\Phi$ be the optimal transportation of a probability measure $\mu = e^{-V}dx$ to $\nu = e^{-W}dx$. Assume that for some positive numbers c, C one has $D^2V \leq C \cdot \mathrm{Id}$, $D^2W \geq c \cdot \mathrm{Id}$. Then $\nabla\Phi$ is Lipschitz with $\|\nabla\Phi\|_{Lip} \leq \sqrt{\frac{C}{c}}$.*

Remark 3.2. Clearly, Theorem 3.2 provides a tool for verification of Assumption **A**. The authors do not know other tools to establish **A** with a comparable level of generality.

Remark 3.3. As we already mentioned, the lower bound for the potentials of measures is preserved under projections. It is interesting that the upper bound for the potential

$$(7) \qquad\qquad D^2W \leq K$$

is preserved under projections as well, i.e., all projections of $\nu = e^{-W}dx$ which satisfy (7) have again the same property. For smooth potentials this can be checked by direct computations.

Theorem 3.3. *Every exchangeable uniformly log-concave measure ν is a countable power of a one-dimensional uniformly log-concave measure.*

Proof. Theorem 3.1 implies existence of an exchangeable transportation mapping T of the standard Gaussian measure $\gamma = \gamma^\infty$, $\gamma(dx) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ to ν. Indeed, Assumption **A** follows from the Caffarelli contraction theorem and the fact that the finite-dimensional projections of ν are uniformly log-concave. The result now follows from Corollary 2.1. $\square$

Remark 3.4. The assumption of the uniform log-concavity in Theorem 3.3 is important and cannot be replaced by the weaker assumption of log-concavity. There exist log-concave exchangeable measures that are not product measures. For example, let $m = e^{-V(x)}dx$ be a one-dimensional log-concave probability measure. The measure

$$\tilde{\nu} = \prod_{i=1}^{\infty} e^{-V(x_i+t)}dx_i \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$$

is log-concave on $\mathbb{R}^\infty \times \mathbb{R}$. Its projection onto the space of x-coordinates is log-concave by the result of C. Borell and exchangeable, but is not a product measure.

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