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LYAPUNOV-TYPE INEQUALITIES FOR QUASILINEAR SYSTEMS WITH ANTIPERIODIC BOUNDARY CONDITIONS

НЕРІВНОСТІ ТИПУ ЛЯПУНОВА ДЛЯ КВАЗІЛІНІЙНИХ СИСТЕМ З АНТИПЕРІОДИЧНИМИ ГРАНИЧНИМИ УМОВАМИ

We establish some new Lyapunov-type inequalities for one-dimensional p-Laplacian systems with antiperiodic boundary conditions. The lower bounds of eigenvalues are presented.

Встановлено деякі нові нерівності типу Ляпунова для одновимірних p-лапласових систем з антиперіодичними граничними умовами. Наведено нижні межі для власних значень.

1. Introduction. The Lyapunov inequality and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations. A classical result of Lyapunov [12] states that if $u(t)$ is a nontrivial solution of the differential system

$$\begin{aligned} &u''(t) + r(t)u(t) = 0, \quad t \in (a,b), \\ &u(a) = 0 = u(b), \end{aligned}$$

where $r(t)$ is a continuous and nonnegative function defined in $[a,b]$, then

$$\int_a^b r(t)dt > \frac{4}{b-a},$$

and the constant 4 cannot be replaced by a larger number.

Since the appearance of Lyapunov's fundamental paper, various proofs and generalizations or improvements have appeared in the literature. For authors, who contributed to the Lyapunov-type inequalities, we refer to [1, 3 – 11, 14 – 16, 20 – 23] and the references quoted therein, especially to the survey papers [1, 3, 22]. In recent years, the Lyapunov inequality has been extended in many directions.

For example, in 2004, Pinasco [18] has generalized the classical Lyapunov inequality for the half-linear differential equation

$$\begin{aligned} &(|u'(t)|^{p-2}u'(t))' + r(t)|u(t)|^{p-2}u(t) = 0, \quad t \in (a,b), \\ &u(a) = u(b) = 0. \end{aligned} \tag{1.1}$$

He obtained Lyapunov inequality for system (1.1) as follows:

$$\int_a^b r(t)dt \geq \frac{2^p}{(b-a)^{p/q}},$$

where $p > 1$, $1/p + 1/q = 1$ and $r \in C([a,b],(0,+\infty))$.

In 2006, Pinasco [17] studied the following Dirichlet–Neumann problem:

$$(|u'(t)|^{p-2}u'(t))' + r(t)|u(t)|^{p-2}u(t) = 0, \quad t \in (a,b),$$

$$u'(a) = u(b) = 0.$$

He gave the following Lyapunov-type inequality:

$$\int_a^b r(t)dt > \frac{1}{(b-a)^{p-1}}.$$

Although there is an extensive literature on the Lyapunov-type inequalities for various classes of differential equations, there is not much done for the linear Hamiltonian systems.

In 2006, Napoli and Pinasco [13] have interested in the problem of finding the Lyapunov-type inequality for the following quasilinear systems:

$$\begin{aligned} &-(|u_1'(t)|^{p_1-2}u_1'(t))' = r_1(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}u_1(t),\\ &-(|u_2'(t)|^{p_2-2}u_2'(t))' = r_2(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2-2}u_2(t) \end{aligned} \tag{1.2}$$

with the Dirichlet boundary conditions

$$u_1(a) = u_1(b) = 0 = u_2(a) = u_2(b), \quad u_1(t) > 0, \quad u_2(t) > 0 \quad \forall t \in (a,b),$$

where r_1, r_2 are real-valued positive continuous functions for all $x \in R$, the exponents satisfy $1 < p_1,\, p_2 < \infty$, and the positive parameters α_1, α_2 satisfy $\dfrac{\alpha_1}{p_1} + \dfrac{\alpha_2}{p_2} = 1$. They gave the following Lyapunov-type inequality:

$$(b-a)^{\alpha_1+\alpha_2-1}\left(\int_a^b r_1(t)dt\right)^{\frac{\alpha_1}{p_1}}\left(\int_a^b r_2(t)dt\right)^{\frac{\alpha_2}{p_2}} \geq 2^{\alpha_1+\alpha_2}. \tag{1.3}$$

More recently, by adopting the method used in Napoli and Pinasco [13], Cakmak and Tiryaki [2] generalized Lyapunov-type inequality (1.3) to the following more general quasilinear systems:

$$\begin{aligned} &-(|u_1'(t)|^{p_1-2}u_1'(t))' = r_1(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}\ldots|u_n(t)|^{\alpha_n}u_1(t),\\ &-(|u_2'(t)|^{p_2-2}u_2'(t))' = r_2(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2-2}\ldots|u_n(t)|^{\alpha_n}u_2(t),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &-(|u_n'(t)|^{p_n-2}u_n'(t))' = r_n(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2}\ldots|u_n(t)|^{\alpha_n-2}u_n(t) \end{aligned} \tag{1.4}$$

with Dirichlet boundary conditions

$$u_i(a) = 0 = u_i(b), \quad u_i(t) \not\equiv 0 \quad \forall t \in (a,b), \quad i = 1,2,\ldots,n.$$

They established the Lyapunov-type inequality

$$\prod_{i=1}^{n}\left(\int_{a}^{b} r_i^{+}(t)dt\right)^{\frac{\alpha_i}{p_i}} \geq 2^m(b-a)^{1-m},$$

where $m=\sum_{i=1}^{n}\alpha_i$ and $r_i^{+}(t)=\max\{r_i(t),0\}$ for $i=1,2,\cdots,n$.

Motivated by the paper [2] and [19], the purpose of this paper is to get three types of Lyapunov inequalities for one-dimensional p-Laplacian system. In Section 2, we show Lyapunov inequality for one-dimensional p-Laplacian problem

$$\begin{aligned} &-(|u'(t)|^{p-2}u'(t))' = r(t)|u(t)|^{p-2}u(t), \quad t\in(a,b),\\ &u(a)+u(b)=0, \quad u'(a)+u'(b)=0, \end{aligned} \tag{1.5}$$

where $r\colon [a,b]\to(0,\infty)$ is a continuous function, $p>1$, q be a conjugate exponent of p, i.e., $1/p+1/q=1$.

In Section 3, the more general system than (1.2)

$$\begin{aligned} &-(|u_1'(t)|^{p_1-2}u_1'(t))' = r_1(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}u_1(t),\\ &-(|u_2'(t)|^{p_2-2}u_2'(t))' = r_2(t)|u_1(t)|^{\beta_1}|u_2(t)|^{\beta_2-2}u_2(t),\\ &u_i(a)+u_i(b)=0, \quad u_i'(a)+u_i'(b)=0, \quad i=1,2, \end{aligned} \tag{1.6}$$

will be studied, we establish new Lyapunov inequality for this system. Where r_1, r_2 are real-valued positive continuous functions for all $t\in R$, the exponents satisfy $1<p_1,p_2<\infty$, $\alpha_1,\alpha_2,\beta_1,\beta_2>0$ satisfy $\dfrac{\alpha_1}{p_1}+\dfrac{\alpha_2}{p_2}=1$ and $\dfrac{\beta_1}{p_1}+\dfrac{\beta_2}{p_2}=1$.

In Section 4, we consider the Lyapunov inequality for system (1.4) with antiperiodic conditions, i.e.,

$$\begin{aligned} &-(|u_1'(t)|^{p_1-2}u_1'(t))' = r_1(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}\ldots|u_n(t)|^{\alpha_n}u_1(t),\\ &-(|u_2'(t)|^{p_2-2}u_2'(t))' = r_2(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2-2}\ldots|u_n(t)|^{\alpha_n}u_2(t),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &-(|u_n'(t)|^{p_n-2}u_n'(t))' = r_n(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2}\ldots|u_n(t)|^{\alpha_n-2}u_n(t),\\ &u_i(a)+u_i(b)=0, \quad u_i'(a)+u_i'(b)=0, \quad i=1,2,\ldots,n. \end{aligned} \tag{1.7}$$

As an application of our Lyapunov-type inequalities, in Section 5, we focus on the estimate of lower bound for eigenvalues.

2. Lyapunov-type inequality for system (1.5). In the proof of our results, the following lemma is very important.

Lemma 2.1. *If* $u'\in L^p(a,b)$ *and* $u(t)$ *satisfying the condition* $u(a)+u(b)=0$, *then*

$$|u(t)| \leq \frac{1}{2}(b-a)^{1/q}\left(\int_{a}^{b}|u'(x)|^p dx\right)^{1/p}.$$

Proof. Let us define,

$$H(x,y)=\begin{cases}\dfrac{1}{2}, & a\le x\le y,\\ -\dfrac{1}{2}, & y\le x\le b.\end{cases}$$

Then, we have

$$u(y)=\int\limits_a^b u'(x)H(x,y)dx,\quad a\le y\le b.$$

So, by Hölder's inequality, we have

$$|u(y)|\le\left(\int\limits_a^b |H(x,y)|^q dx\right)^{1/q}\left(\int\limits_a^b |u'(x)|^p dx\right)^{1/p}=$$

$$=\frac{1}{2}(b-a)^{1/q}\left(\int\limits_a^b |u'(x)|^p dx\right)^{1/p}.$$

The main result of this section is the following Lyapunov-type inequality.

Theorem 2.1. *If $u(t)$ is a solution of system* (1.5) *and* $u(t)\not\equiv 0, t\in[a,b]$, *then*

$$\int\limits_a^b r(t)dt>\frac{2^p}{(b-a)^{p-1}}.$$

Proof. Multiplying the equation in (1.5) by u and integrating over $[a,b]$, yields

$$-\int\limits_a^b(|u'|^{p-2}u')'udt=\int\limits_a^b r|u|^p dt,$$

$$|u'(a)|^{p-2}u'(a)u(a)-|u'(b)|^{p-2}u'(b)u(b)+\int\limits_a^b|u'|^p dt=\int\limits_a^b r|u|^p dt,$$

by the antiperiodic boundary condition in (1.5), we get $|u'(b)|^{p-2}u'(b)u(b)=|u'(a)|^{p-2}u'(a)u(a)$, so we have

$$\int\limits_a^b|u'|^p dt=\int\limits_a^b r|u|^p dt,$$

by Lemma 2.1, we obtain

$$\int\limits_a^b|u'|^p dt=\int\limits_a^b r|u|^p dt<(\max_{a\le t\le b}|u|)^p\int\limits_a^b r(t)dt\le\frac{(b-a)^{p-1}}{2^p}\int\limits_a^b|u'(t)|^p dt\int\limits_a^b r(t)dt. \tag{2.1}$$

Now we claim that

$$\int_a^b |u'(t)|^p dt > 0. \tag{2.2}$$

If not, we have $\int_a^b |u'(t)|^p dt = 0$, then $u'(t) = 0$, $t \in [a,b]$. By the antiperiodic boundary condition $u(a) + u(b) = 0$, we can obtain $u(t) \equiv 0, t \in [a,b]$, which contradicts $u(t) \not\equiv 0, t \in [a,b]$. Therefore (2.2) holds. Divided the inequality (2.1) by $\int_a^b |u'(t)|^p dt$, we have

$$\int_a^b r(t)dt > \frac{2^p}{(b-a)^{p-1}}.$$

3. Lyapunov-type inequality for system (1.6). In this section, the main result is as follows.

Theorem 3.1. *If system* (1.6) *has a solution* $u_1(t), u_2(t)$ *and* $u_i(t) \not\equiv 0$ $\forall t \in [a,b]$, $i = 1,2$, *then*

$$(b-a)^{\beta_1+\alpha_2-\frac{\beta_1}{p_1}-\frac{\alpha_2}{p_2}} \left(\int_a^b r_1(t)dt\right)^{\frac{\beta_1}{p_1}} \left(\int_a^b r_2(t)dt\right)^{\frac{\alpha_2}{p_2}} \geq 2^{\beta_1+\alpha_2}. \tag{3.1}$$

Proof. Multiplying the first equation in (1.6) by u_1 and integrating over $[a,b]$ together with the antiperiodic condition, yields

$$\int_a^b |u_1'(t)|^{p_1} dt = \int_a^b r_1(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2} dt, \tag{3.2}$$

similarly,

$$\int_a^b |u_2'(t)|^{p_2} dt = \int_a^b r_2(t)|u_1(t)|^{\beta_1}|u_2(t)|^{\beta_2} dt, \tag{3.3}$$

by Lemma 2.1, we get

$$|u_1(t)|^{p_1} \leq \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b |u_1'(x)|^{p_1} dx. \tag{3.4}$$

Now, it follows from (3.2), (3.4) and the Hölder inequality that

$$\int_a^b r_1(t)|u_1(t)|^{p_1} dt \leq \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_1(t)dt \int_a^b |u_1'(t)|^{p_1} dt =$$

$$= \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_1(t)dt \int_a^b r_1(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2} dt \leq$$

$$\leq \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_1(t)dt \left(\int_a^b r_1(t)|u_1(t)|^{p_1} dt \right)^{\frac{\alpha_1}{p_1}} \left(\int_a^b r_1(t)|u_2(t)|^{p_2} dt \right)^{\frac{\alpha_2}{p_2}} =$$

$$= M_1 \left(\int_a^b r_1(t)|u_1(t)|^{p_1} dt \right)^{\frac{\alpha_1}{p_1}} \left(\int_a^b r_1(t)|u_2(t)|^{p_2} dt \right)^{\frac{\alpha_2}{p_2}} \tag{3.5}$$

and

$$\int_a^b r_2(t)|u_1(t)|^{p_1} dt \leq \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_2(t)dt \int_a^b |u_1'(t)|^{p_1} dt =$$

$$= \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_2(t)dt \int_a^b r_1(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2} dt \leq$$

$$\leq \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_2(t)dt \left(\int_a^b r_1(t)|u_1(t)|^{p_1} dt \right)^{\frac{\alpha_1}{p_1}} \left(\int_a^b r_1(t)|u_2(t)|^{p_2} dt \right)^{\frac{\alpha_2}{p_2}} =$$

$$= M_2 \left(\int_a^b r_1(t)|u_1(t)|^{p_1} dt \right)^{\frac{\alpha_1}{p_1}} \left(\int_a^b r_1(t)|u_2(t)|^{p_2} dt \right)^{\frac{\alpha_2}{p_2}}, \tag{3.6}$$

where

$$M_1 = \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_1(t)dt, \quad M_2 = \frac{(b-a)^{p_1-1}}{2^{p_1}} \int_a^b r_2(t)dt. \tag{3.7}$$

Similarly, we also have

$$|u_2(t)|^{p_2} \leq \frac{(b-a)^{p_2-1}}{2^{p_2}} \int_a^b |u_2'(x)|^{p_2} dx. \tag{3.8}$$

It follows from (3.3), (3.8) and the Hölder inequality that

$$\int_a^b r_1(t)|u_2(t)|^{p_2} dt \leq \frac{(b-a)^{p_2-1}}{2^{p_2}} \int_a^b r_1(t)dt \int_a^b |u_2'(t)|^{p_2} dt =$$

$$= \frac{(b-a)^{p_2-1}}{2^{p_2}} \int_a^b r_1(t)dt \int_a^b r_2(t)|u_1(t)|^{\beta_1}|u_2(t)|^{\beta_2} dt \leq$$

$$\leq \frac{(b-a)^{p_2-1}}{2^{p_2}} \int\limits_a^b r_1(t)dt \left(\int\limits_a^b r_2(t)|u_1(t)|^{p_1} dt \right)^{\frac{\beta_1}{p_1}} \left(\int\limits_a^b r_2(t)|u_2(t)|^{p_2} dt \right)^{\frac{\beta_2}{p_2}} =$$

$$= M_3 \left(\int\limits_a^b r_2(t)|u_1(t)|^{p_1} dt \right)^{\frac{\beta_1}{p_1}} \left(\int\limits_a^b r_2(t)|u_2(t)|^{p_2} dt \right)^{\frac{\beta_2}{p_2}} \tag{3.9}$$

and

$$\int\limits_a^b r_2(t)|u_2(t)|^{p_2} dt \leq \frac{(b-a)^{p_2-1}}{2^{p_2}} \int\limits_a^b r_2(t)dt \int\limits_a^b |u_2'(t)|^{p_2} dt =$$

$$= \frac{(b-a)^{p_2-1}}{2^{p_2}} \int\limits_a^b r_2(t)dt \int\limits_a^b r_2(t)|u_1(t)|^{\beta_1}|u_2(t)|^{\beta_2} dt \leq$$

$$\leq \frac{(b-a)^{p_2-1}}{2^{p_2}} \int\limits_a^b r_2(t)dt \left(\int\limits_a^b r_2(t)|u_1(t)|^{p_1} dt \right)^{\frac{\beta_1}{p_1}} \left(\int\limits_a^b r_2(t)|u_2(t)|^{p_2} dt \right)^{\frac{\beta_2}{p_2}} =$$

$$= M_4 \left(\int\limits_a^b r_2(t)|u_1(t)|^{p_1} dt \right)^{\frac{\beta_1}{p_1}} \left(\int\limits_a^b r_2(t)|u_2(t)|^{p_2} dt \right)^{\frac{\beta_2}{p_2}}, \tag{3.10}$$

where

$$M_3 = \frac{(b-a)^{p_2-1}}{2^{p_2}} \int\limits_a^b r_1(t)dt, \quad M_4 = \frac{(b-a)^{p_2-1}}{2^{p_2}} \int\limits_a^b r_2(t)dt. \tag{3.11}$$

Next, we prove that

$$\int\limits_a^b r_1(t)|u_1(t)|^{p_1} dt > 0. \tag{3.12}$$

In fact, if (3.12) is not true, then

$$\int\limits_a^b r_1(t)|u_1(t)|^{p_1} dt = 0. \tag{3.13}$$

From (3.2) and (3.13), we have

$$0 \leq \int\limits_a^b |u_1'(t)|^{p_1} dt = \int\limits_a^b r_1(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2} dt \leq$$

$$\leq \left(\int_a^b r_1(t)|u_1(t)|^{p_1}dt\right)^{\frac{\alpha_1}{p_1}} \left(\int_a^b r_1(t)|u_2(t)|^{p_2}dt\right)^{\frac{\alpha_2}{p_2}} = 0.$$

So $u_1'(t) \equiv 0$, $a \leq t \leq b$, by the antiperiodic boundary condition, we obtain $u_1(t) \equiv 0$ for $a \leq t \leq b$, which contradicts $u_1(t) \not\equiv 0$ $\forall t \in [a, b]$. Therefore, (3.12) holds. Similarly, we get

$$\int_a^b r_1(t)|u_2(t)|^{p_2}dt > 0, \quad \int_a^b r_2(t)|u_2(t)|^{p_2}dt > 0, \quad \int_a^b r_2(t)|u_1(t)|^{p_1}dt > 0.$$

From (3.5), (3.6), (3.9), (3.10), we obtain

$$M_1^{\frac{\alpha_1\beta_1}{p_1^2}} M_2^{\frac{\beta_1\alpha_2}{p_1p_2}} M_3^{\frac{\beta_1\alpha_2}{p_1p_2}} M_4^{\frac{\alpha_2\beta_2}{p_2^2}} \geq 1.$$

It follows from (3.7), (3.11) that (3.1) holds.

4. Lyapunov-type inequality for system (1.7). In this section, we establish new Lyapunov-type inequality for system (1.7). Assume that

(H1) r_i, $i = 1, 2, \ldots, n$, are real-valued positive continuous functions for all $t \in R$,

(H2) the exponents satisfy $1 < p_i < \infty$ and the positive parameters α_i satisfy $\sum_{i=1}^n \dfrac{\alpha_i}{p_i} = 1$.

Theorem 4.1. *If system* (1.7) *has a solution* $(u_1(t), u_2(t), \ldots, u_n(t))$ *with* $u_i(t) \not\equiv 0$, $t \in [a, b]$, $i = 1, 2, \ldots, n$, *then*

$$\prod_{i=1}^n \prod_{j=1}^n \left[\frac{(b-a)^{p_i-1}}{2^{p_i}} \int_a^b r_j(t)dt\right]^{\frac{\alpha_i\alpha_j}{p_ip_j}} \geq 1. \tag{4.1}$$

Proof. Multiplying the ith equation in (1.7) by u_i and integrating over $[a, b]$ together with the antiperiodic condition, yields

$$\int_a^b |u_i'(t)|^{p_i}dt = \int_a^b r_i(t) \prod_{k=1}^n |u_k(t)|^{\alpha_k}dt, \quad i = 1, 2, \ldots, n, \tag{4.2}$$

similarly, by Lemma 2.1, we get

$$|u_i(t)|^{p_i} \leq \frac{(b-a)^{p_i-1}}{2^{p_i}} \int_a^b |u_i'(x)|^{p_i}dx, \quad i = 1, 2, \ldots, n. \tag{4.3}$$

Now, it follows from (4.2), (4.3) and the generalized Hölder inequality that

$$\int_a^b r_j(t)|u_i(t)|^{p_i}dt \leq \frac{(b-a)^{p_i-1}}{2^{p_i}} \int_a^b r_j(t)dt \int_a^b |u_i'(t)|^{p_i}dt =$$

$$= \frac{(b-a)^{p_i-1}}{2^{p_i}} \int_a^b r_j(t)dt \int_a^b r_i(t) \prod_{k=1}^n |u_k(t)|^{\alpha_k}dt =$$

$$= M_{ij} \int\limits_a^b r_i(t) \prod_{k=1}^n |u_k(t)|^{\alpha_k} dt \leq M_{ij} \prod_{k=1}^n \left(\int\limits_a^b r_i(t) |u_k(t)|^{p_k} dt \right)^{\frac{\alpha_k}{p_k}}, \quad i,j = 1,2,\ldots,n, \tag{4.4}$$

where

$$M_{ij} = \frac{(b-a)^{p_i - 1}}{2^{p_i}} \int\limits_a^b r_j(t) dt, \quad i,j = 1,2,\ldots,n. \tag{4.5}$$

Next, we prove that

$$\int\limits_a^b r_i(t) |u_j(t)|^{p_j} dt > 0, \quad i,j = 1,2,\ldots,n. \tag{4.6}$$

If (4.6) is not true, then there exists $i_0, j_0 \in \{1, 2, \ldots, n\}$ such that

$$\int\limits_a^b r_{i_0}(t) |u_{j_0}(t)|^{p_{j_0}} dt = 0. \tag{4.7}$$

From (4.2) and (4.7), we have

$$0 \leq \int\limits_a^b |u'_{i_0}(t)|^{p_{i_0}} dt = \int\limits_a^b r_{i_0}(t) \prod_{k=1}^n |u_k(t)|^{\alpha_k} dt \leq \prod_{k=1}^n \left(\int\limits_a^b r_{i_0}(t) |u_k(t)|^{p_k} dt \right)^{\frac{\alpha_k}{p_k}} = 0.$$

So that

$$u'_{i_0}(t) = 0, \quad a \leq t \leq b. \tag{4.8}$$

Combining the antiperiodic boundary and (4.8), we obtain that $u_{i_0}(t) \equiv 0$ for $a \leq t \leq b$, which contradicts $u_i(t) \not\equiv 0$, $t \in [a,b]$, $i = 1,2,\ldots,n$. Therefore (4.6) holds. From (4.4), we have

$$\prod_{i=1}^n \prod_{j=1}^n M_{ij}^{\frac{\alpha_i \alpha_j}{p_i p_j}} \geq 1.$$

It follows from (4.5) that (4.1) holds.

5. Lower bounds for eigenvalues problem. In this section, we apply our Lyapunov-type inequalities to obtain lower bounds for eigenvalues. Firstly, we investigate the problem

$$\begin{aligned} &-(|u'(t)|^{p-2} u'(t))' = \lambda r(t) |u(t)|^{p-2} u(t), \qquad t \in (a,b), \\ &u(a) + u(b) = 0, \quad u'(a) + u'(b) = 0. \end{aligned} \tag{5.1}$$

As a corollary of Theorem 2.1, we have the result for system (5.1).

Theorem 5.1. *Assume that* $p > 1$, $r(t)$ *is real-valued positive continuous function for all* $t \in R$ *and system* (5.1) *has a solution* $u(t)$ *satisfying* $u(t) \not\equiv 0$, $t \in [a, b]$. *Let* λ *be the eigenvalue of system* (5.1). *Then*

$$\lambda > \frac{2^p}{(b-a)^{p-1} \displaystyle\int_a^b r(x)dx}.$$

Secondly, we consider the eigenvalues problem

$$\begin{aligned}
&-\left(|u_1'(t)|^{p_1-2}u_1'(t)\right)' = \lambda_1\alpha_1 r(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}\ldots|u_n(t)|^{\alpha_n}u_1(t),\\
&-\left(|u_2'(t)|^{p_2-2}u_2'(t)\right)' = \lambda_2\alpha_2 r(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2-2}\ldots|u_n(t)|^{\alpha_n}u_2(t),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (5.2)\\
&-\left(|u_n'(t)|^{p_n-2}u_n'(t)\right)' = \lambda_n\alpha_n r(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2}\ldots|u_n(t)|^{\alpha_n-2}u_n(t),\\
&u_i(a) + u_i(b) = 0, \quad u_i'(a) + u_i'(b) = 0, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be eigenvalue of problem (5.2) and $(u_1(t), u_2(t), \ldots, u_n(t))$ be the eigenfunctions associated with $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then $(u_1(t), u_2(t), \ldots, u_n(t))$ is a solution of system (1.7) with $r_i(t) = \lambda_i\alpha_i r(t) > 0$ for $i = 1, 2, \ldots, n$.

Theorem 5.2. *Assume that* $1 < p_i < \infty, \alpha_i > 0$ *satisfy* $\sum_{i=1}^n \frac{\alpha_i}{p_i} = 1$, $r(t)$ *is a real-valued positive continuous function defined on* R, *system* (5.2) *has a solution* $(u_1(t), u_2(t), \ldots, u_n(t))$ *with* $u_i(t) \not\equiv 0$, $t \in [a, b]$, $i = 1, 2, \ldots, n$. *Then there exists a function* $g(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ *such that* $\lambda_n > g(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ *for every eigenvalue* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *of system* (5.2), *where* $g(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ *is given by*

$$g(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \frac{1}{\alpha_n}\left[\prod_{j=1}^{n-1}(\lambda_j\alpha_j)^{\frac{\alpha_j}{p_j}}\prod_{i=1}^{n}\left(\frac{(b-a)^{p_i-1}}{2^{p_i}}\int_a^b r_j(t)dt\right)^{\frac{\alpha_i}{p_i}}\right]^{-\frac{p_n}{\alpha_n}}.$$

Proof. For system (1.7), $r_i(t) = \lambda_i\alpha_i r(t) > 0$ for $i = 1, 2, \ldots, n$. Hence, it follows from (4.1) that

$$1 \le \prod_{i=1}^{n}\prod_{j=1}^{n}\left[\frac{(b-a)^{p_i-1}}{2^{p_i}}\int_a^b r_j(t)dt\right]^{\frac{\alpha_i\alpha_j}{p_ip_j}} =$$

$$= \prod_{j=1}^{n}(\lambda_j\alpha_j)^{\frac{\alpha_j}{p_j}}\prod_{i=1}^{n}\left[\frac{(b-a)^{p_i-1}}{2^{p_i}}\int_a^b r_j(t)dt\right]^{\frac{\alpha_i}{p_i}} =$$

$$= (\lambda_n\alpha_n)^{\frac{\alpha_n}{p_n}}\prod_{j=1}^{n-1}(\lambda_j\alpha_j)^{\frac{\alpha_j}{p_j}}\prod_{i=1}^{n}\left[\frac{(b-a)^{p_i-1}}{2^{p_i}}\int_a^b r_j(t)dt\right]^{\frac{\alpha_i}{p_i}}.$$

Hence, we have

$$\lambda_n > \frac{1}{\alpha_n}\left[\prod_{j=1}^{n-1}(\lambda_j\alpha_j)^{\frac{\alpha_j}{p_j}}\prod_{i=1}^{n}\left(\frac{(b-a)^{p_i-1}}{2^{p_i}}\int_a^b r_j(t)dt\right)^{\frac{\alpha_i}{p_i}}\right]^{-\frac{p_n}{\alpha_n}}.$$

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