UDC 517.9

J. Mawhin

(Inst. de Recherche en Mathématique et Physique Université Catholique de Louvain chemin du cyclotron, Belgium)

VARIATIONS ON SOME FINITE-DIMENSIONAL FIXED-POINT THEOREMS

УЗАГАЛЬНЕННЯ ДЕЯКИХ СКІНЧЕННОВИМІРНИХ ТЕОРЕМ ПРО НЕРУХОМУ ТОЧКУ

We give rather elementary topological proofs of some generalizations of fixed-point theorems in $\mathbb{R}^n$ due to Pireddu – Zanolin and Zgliczyński, which are useful in various questions related to ordinary differential equations.

Наведено елементарні топологічні доведення деяких узагальнень теорем Піредду – Заноліна та Зглічинського про нерухому точку в $\mathbb{R}^n$, які можуть бути використані при розгляді різних питань, пов'язаних із звичайними диференціальними рівняннями.

1. Introduction. Fixed-point theorems in finite-dimensional spaces have applications, for example, in mathematical economy, nonlinear difference equations, periodic solutions of ordinary differential equations using Poincaré's operator, and in chaos theory.

The present paper finds its inspiration in the interesting papers [11] and [12] of Pireddu – Zanolin, where applications to topological chaos theory and a large bibliography can be found. Its aim is to give simple proofs of generalizations of some fixed-point theorems in $\mathbb{R}^n$.

In Section 2, we show that a reduction theorem for Brouwer degree theory developed by the author [8] (Lemma 1), provides a particular short and natural proof for a result (Theorem 1) containing as a special case a slight generalization (Theorem 2) of a fixed-point theorem of Pireddu – Zanolin [11] for mappings of which are expansive on the boundary of a ball in a vector subspace of $\mathbb{R}^n$ and compressive on the boundary of a ball in a direct summand. The precise meaning of 'expansive' and 'compressive' is given in Section 3. This Theorem 2 is deduced from Theorem 1 in Section 3, and some special cases are considered and compared to older fixed-point theorems. Possible extensions to infinite-dimensional normed vector spaces are mentioned.

In Section 4, we describe a generalization of a fixed-point theorem due to Zgliczyński [15] (Theorem 3), whose original proof is rather complicated and based upon more sophisticated topological tools. The proof we give here is inspired by the one given in [12], but uses directly Poincaré – Miranda's theorem (Lemma 2), instead of an intersection result deduced from this theorem. We also relate a special case of Theorem 3 with Theorem 2 and its corollaries.

2. A fixed-point theorem for some mappings in a direct sum of vector spaces. For the reader's convenience, we first recall a reduction theorem for the Brouwer degree stated and proved as Theorem 3.1 in [8], and which is essentially a finite-dimensional version of Proposition II.2 in [7]. If U and V are oriented n-dimensional topological vector spaces, $\Omega \subset X$ an open bounded set with closure $\overline{\Omega}$ and boundary $\partial\Omega$, and $g\colon \overline{\Omega} \subset U \to V$ a continuous mapping such that $0 \notin g(\partial\Omega)$, we denote the *Brouwer degree of* g *with respect to* $\overline{\Omega}$ *and* 0 by $d_B[g, \Omega, 0]$ (see e.g. [3] for its definition and properties). If z is an isolated zero of g, the *Brouwer index* $i_B[g, z]$ *of* g *at* z is defined as the Brouwer degree $d_B[g, B_r(z), 0]$ for sufficiently $r > 0$, where $B_r(z)$ denotes the open ball in U of center z and radius r. The symbol $\subset$ will always mean 'non strict inclusion'.

© J. MAWHIN, 2013

Lemma 1. *Let X and Z be n-dimensional topological vector spaces (oriented if $X \neq Z$), $L\colon X \to Z$ be a linear mapping with kernel $N(L) \neq \{0\}$, let $Y \subset Z$ be a vector subspace such that $Z = Y \oplus R(L)$ (with $R(L)$ the range of L), $D \subset X$ be an open bounded set, and let $r\colon \overline{D} \to Y \times \{0\}$ be a continuous mapping such that $0 \notin (L+r)(\partial D)$. Then, for each isomorphism $J\colon N(L) \to Y$, and each projector $P\colon X \to X$ such that $R(P) = N(L)$, one has*

$$d_B[L+r, D, 0] = i_B[L+JP, 0] \cdot d_B[J^{-1}r|_{N(L)}, D \cap N(L), 0].$$

Let now $p \geq 0, q \geq 0$ be integers such that $p + q = n \geq 1$, so that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, let $D \subset \mathbb{R}^n$ be a nonempty open bounded set, and let

$$f\colon \overline{D} \to \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q, \qquad (x,y) \mapsto (f^e(x,y), f^c(x,y))$$

be a continuous mapping.

Theorem 1. *Assume that the following conditions hold:*

1) $\forall (x,y) \in \partial D\ \forall \lambda \in [0,1)\colon (\lambda x - f^e(x,y), y - \lambda f^c(x,y)) \neq (0,0)$;

2) $p = 0,\ 0 \in D$;

3) $p \geq 1,\ d_B[f^e(\cdot,0), D \cap (\mathbb{R}^p \times \{0\}), 0] \neq 0$.

Then f has at least one fixed-point in $\overline{D}$.

Proof. Assume first that $p \geq 1$. Because of assumption 1 with $\lambda = 0$, the Brouwer degree $d_B[f^e(\cdot,0), D \cap (\mathbb{R}^p \times \{0\}), 0]$ is well defined. Define

$$L\colon \mathbb{R}^p \oplus \mathbb{R}^q, \quad (x,y) \mapsto (0,y), \tag{1}$$

and the continuous mapping $\mathcal{H}\colon \overline{D} \times [0,1] \to \mathbb{R}^p \times \mathbb{R}^q$ by

$$\mathcal{H}(x,y,\lambda) = (\lambda x - f^e(x,y), -\lambda f^c(x,y)), \tag{2}$$

so that the fixed points of f are the zeros of $L + \mathcal{H}(\cdot,\cdot,1)$.

If f has a fixed point on ∂D, the result is proved. If it is not the case, then assumption 1 holds for $\lambda \in [0,1]$. From the definition of $\mathcal{H}$ and the homotopy invariance of Brouwer degree, we get

$$d_B[I - f, D, 0] = d_B[L + \mathcal{H}(\cdot,\cdot,1), D, 0] = d_B[L + \mathcal{H}(\cdot,\cdot,0), D, 0]. \tag{3}$$

Now, $N(L) = \mathbb{R}^p \times \{0\}$ and $R(L) = \{0\} \times \mathbb{R}^q$, so that we can write $\mathbb{R}^n = N(L) \oplus R(L)$. So

$$L + \mathcal{H}(\cdot,\cdot,0) = (0,\cdot) + (-f^u(\cdot,\cdot), 0)$$

has the structure requested by Lemma 1. If we define the projector $P\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$ by $P(x,y) = (x,0)$, then $N(L) = R(P) = \mathbb{R}^p \times \{0\}$. Thus we can choose the identity on $\mathbb{R}^p \times \{0\}$ for the isomorphism $J\colon N(L) \to N(L)$ involved in Lemma 1. This implies that

$$L + JP = L + P = I \tag{4}$$

(I the identity in $\mathbb{R}^n$). Therefore, from Theorem 3.1 of [8], (4) and elementary properties of Brouwer degree, we obtain

$$d_B[L + \mathcal{H}(\cdot,\cdot,0), D, 0] = i_B[L+P, 0] \cdot d_B[-f^e(\cdot,0), D \cap (\mathbb{R}^p \times \{0\}), 0] =$$

$$= (-1)^p d_B[f^e(\cdot, 0), D \cap (\mathbb{R}^p \times \{0\}), 0]. \tag{5}$$

Using (3) and (5) we obtain

$$d_B[I - f, D, 0] = (-1)^p d_B[f^e(\cdot, 0), D \cap (\mathbb{R}^p \times \{0\}), 0]$$

and the existence of a fixed point of f in D follows from assumption 3 and the existence property of Brouwer degree.

If now $p = 0$, so that $f = f^c$, either f has a fixed point on ∂D or, using assumption 1, for any $\lambda \in [0, 1]$,

$$y + \mathcal{H}(y, \lambda) := y - \lambda f(y) \neq 0$$

for any $y \in \partial D$. By the homotopy invariance of Brouwer degree and assumption 2, we get

$$d_B[I - f, D, 0] = d_B[I - \mathcal{H}(\cdot, 1), 0] = d_B[I - \mathcal{H}(\cdot, 0), 0] = d_B[I, D, 0] = 1.$$

The existence of a fixed point of f in D follows from the existence property of Brouwer degree.

Theorem 1 is proved.

***Remark* 1.** If $p = 0$, Theorem 1 just reduces to a finite-dimensional version of Schaefer's version [13] of the Leray – Schauder fixed-point theorem [6] (see also [3]).

***Remark* 2.** It is easily checked that Theorem 2 remains true if $\mathbb{R}^q$ is replaced by an infinite-dimensional normed vector space and f^c is assumed to be compact on $\overline{D}$. The proof is exactly the same, except that the use of Lemma 1 has to be replaced by that of Proposition II.12 of [7].

3. Fixed-point theorems for expansive-compressive mappings. Let B_r be the open ball of center 0 and radius $r > 0$ in any Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$, and let $\overline{B_r}$ denote its closure. Let $p \geq 0, q \geq 0$ be integers such that $p + q = n \geq 1$, let $a > 0$, $b > 0$ be real numbers, and let $f = (f^e, f^c) \colon \overline{B}_a \times \overline{B}_b \to \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$ be a continuous mapping. The following result is a slight extension of a fixed-point theorem of [11], which is motivated by and generalizes some results of [1].

Theorem 2. *Assume that the following conditions hold*:

(i) $\forall (x, y) \in \mathbb{R}^p \times \mathbb{R}^q \colon \|x\| = a, \ \|y\| \leq b \colon \|f^e(x, y)\| \geq a$;

(ii) $\forall (x, y) \in \mathbb{R}^p \times \mathbb{R}^q \colon \|x\| \leq a, \ \|y\| = b \colon \langle y, f^c(x, y) \rangle \leq b^2$;

(iii) $p \geq 1, \ d_B[f^e(\cdot, 0), B_a, 0] \neq 0$.

Then f has at least one fixed point in $\overline{B}_a \times \overline{B}_b$.

Proof. Notice that, when $p \geq 1$, the Brouwer degree $d_B[f^e(\cdot, 0), B_a, 0]$ is well defined because, if $\|x\| = a$, then $\|f^e(x, 0)\| \geq a \neq 0$. Also, assumption 2 of Theorem 1 is trivially satisfied. If f has a fixed point on $\partial(B_a \times B_b)$, the result is proved. Assume therefore that f has no fixed point on $\partial(B_a \times B_b)$. Notice that

$$\partial(B_a \times B_b) = (\partial B_a \times \overline{B_b}) \cup (\overline{B_a} \times \partial B_b).$$

To apply Theorem 1, let $\lambda \in [0, 1)$. If $(x, y) \in \partial B_a \times \overline{B_b}$, then, using assumption (i),

$$\|\lambda x\| = \lambda a < a \leq \|f^e(x, y)\|,$$

so that (x, y) is not a zero of $L + \mathcal{H}(\cdot, \cdot, \lambda)$ with L defined in (1) and $\mathcal{H}$ defined in (2). If $\lambda \in [0, 1)$ and $(x, y) \in \overline{B_a} \times \partial B_b$, then, using assumption (ii),

$$\|y\|^2 = b^2 > \lambda b^2 \geq \lambda \langle y, f^c(x, y)\rangle,$$

so that (x, y) is not a zero of $L + \mathcal{H}(\cdot, \cdot, \lambda)$. Thus, assumption 1 of Theorem 1 is satisfied. Assumption (iii) is just assumption 3 of Theorem 1, and the result follows.

An first consequence of Theorem 2 is the following fixed-point theorem, where assumptions (i) and (ii) of Theorem 2 have a more symmetric structure, which justifies the name of 'expansive-compressive' mapping given to f.

***Corollary* 1.** *Assume that the following conditions hold*:

(i) $\forall (x, y) \in \mathbb{R}^p \times \mathbb{R}^q\colon \|x\| = a,\ \|y\| \leq b\colon \|f^e(x, y)\| \geq a;$

(ii$'$) $\forall (x, y) \in \mathbb{R}^p \times \mathbb{R}^q\colon \|x\| \leq a,\ \|y\| = b\colon \|f^c(x, y)\| \leq b;$

(iii) $p \geq 1,\ d_B[f^e(\cdot, 0), B_a, 0] \neq 0.$

Then f has at least one fixed point in $\overline{B}_a \times \overline{B}_b$.

Proof. It suffices to notice that, for all $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $\|x\| \leq a$ and $\|y\| = b$, one has

$$\langle y, f^c(x, y)\rangle \leq \|y\| \|f^c(x, y)\| \leq \|y\| b = b^2,$$

so that assumption (ii$'$) implies assumption (ii) of Theorem 2.

Another consequence comes from the other way of 'symmetrizing' assumptions (i) and (ii) of Theorem 2.

***Corollary* 2.** *Assume that the following conditions hold*:

(i$'$) $\forall (x, y) \in \mathbb{R}^p \times \mathbb{R}^q\colon \|x\| = a,\ \|y\| \leq b\colon \langle x, f^e(x, y)\rangle \geq a^2;$

(ii) $\forall (x, y) \in \mathbb{R}^p \times \mathbb{R}^q\colon \|x\| \leq a,\ \|y\| = b\colon \langle y, f^e(x, y)\rangle \leq b^2.$

Then f has at least one fixed point in $\overline{B}_a \times \overline{B}_b$.

Proof. If condition (i$'$) holds, then for all $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $\|x\| = a$ and $\|y\| \leq b$, one has

$$a\|f^e(x, y)\| = \|x\| \|f^e(x, y)\| \geq \langle x, f^e(x, y)\rangle \geq a^2,$$

and hence assumption (i$'$) implies assumption (i) of Theorem 2. Furthermore, if we define the homotopy

$$\mathcal{F}\colon \overline{B_a} \times [0, 1] \to \mathbb{R}^p, \qquad (x, \lambda) \mapsto (1 - \lambda)x + \lambda f^e(x, 0),$$

we have, for all x such that $\|x\| = a$ and all $\lambda \in [0, 1]$, using again assumption (i$'$),

$$(1 - \lambda)\|x\|^2 + \lambda\langle x, f^e(x, 0)\rangle \geq (1 - \lambda)a^2 + \lambda a^2 = a^2 > 0$$

so that $\mathcal{F}(\cdot, \lambda)$ has no zero on ∂B_a when $\lambda \in [0, 1]$. Hence, the homotopy invariance of Brouwer degree implies that

$$d_B[f^e(\cdot, 0), B_a, 0] = d_B[\mathcal{F}(\cdot, 1), B_a, 0] = d_B[\mathcal{F}(\cdot, 0), B_a, 0] = d_B[I, B_a, 0] = 1,$$

and assumption (iii) of Theorem 2 is satisfied.

***Remark* 3.** For $p = 0$, Theorem 2 reduces to a fixed point generally attributed to Krasnosel'skii, and which, in the finite-dimensional case, follows directly from the proof of Brouwer fixed-point theorem given in 1910 by Hadamard [5, p. 472]. For $p = 0$, Corollary 1 reduces to Rothe's fixed-point theorem [3] in a finite-dimensional space, already mentioned in 1922 by Birkhoff–Kellogg ([2, p. 100], footnote). For $q = 0$ and $n = 2$, Theorem 2 was already obtained by Dolcher [4] in 1948. In this case, condition (i) is not sufficient to get a fixed point, as shown by the simple example of the one-dimensional mapping $f\colon [-1, 1] \to \mathbb{R}, x \mapsto x + 1$, which has no fixed point and is such that

$$|f(-1)| = 1, \qquad |f(1)| = 3 > 1,$$

so that assumption (i) holds. On the other hand,

$$d_B[I + 2, (-1, 1), 0] = 0.$$

***Remark* 4.** *It follows from Remark* 2 *that Theorem* 2 *and its corollaries remain valid if one replaces* $\mathbb{R}^q$ *by an infinite-dimensional normed vector space under the assumption that* f^c *is compact on* $\overline{B}_a \times \overline{B}_b$.

4. Poincaré–Miranda's and fixed-point theorems. Let $I = [-1, 1]$, $n \geq 1$ an integer, and for $j \in \{1, \ldots, n\}$ and $\varepsilon = \pm 1$, denote by $[x_j = \epsilon]$ the set $\{x \in I^n \colon x_j = \varepsilon\}$, i.e., the j-th opposite faces of I^n. For the reader's convenience, let us recall *Poincaré–Miranda's theorem* [10, 14], a n-dimensional generalization of Bolzano's theorem, for which an elementary proof is given in [9].

Lemma 2. *Let* $g = I^n \to \mathbb{R}^n$ *be a continuous mapping such that there exists a finite sequence* $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ *in* $\{-1, +1\}$ *with the property that, for each* $j \in \{1, \ldots, n\}$, *one has*

$$\epsilon_j g_j(x) \leq 0 \quad \forall x \in [x_j = -1] \qquad \textit{and} \qquad \epsilon_j g_j(x) \geq 0 \quad \forall x \in [x_j = 1].$$

Then g *has at least one zero in* I^n.

The following result generalizes a fixed-point theorem due to Zgliczyński [15], with a proof inspired by [12], but using directly Lemma 2 instead of some of its topological consequences.

Theorem 3. *Let* $f\colon I^n \to \mathbb{R}^n$ *be a continuous map, and supppose there exists a finite sequence of indexes* $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, *such that the following conditions hold*:

(a) *for every* $j \in \{i_1, i_2, \ldots, i_k\}$,

(a$_1$) $[-1, 1] \subset \Big[\max_{[x_j=-1]} f_j, \min_{[x_j=+1]} f_j\Big]$

or

(a$_2$) $[-1, 1] \subset \Big[\max_{[x_j=1]} f_j, \min_{[x_j=-1]} f_j\Big]$;

(b) *for every* $j \in \{1, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$,

$$\min_{[x_j=-1]} f_j \geq -1 \qquad \textit{and} \qquad \max_{[x_j=1]} f_j \leq 1$$

Then f *has at least a fixed point in* I^n.

Proof. Let $g\colon I^n \to \mathbb{R}^n$ be the continuous map defined by $g = I - f$. It suffices to prove that g has a zero in I^n. From assumption (a), we see that, for $j \in \{i_1, i_2, \ldots, i_k\}$, we have either

$$\max_{[x_j=-1]} f_j \leq -1 \leq 1 \leq \min_{[x_j=1]} f_j, \tag{6}$$

or

$$\max_{[x_j=1]} f_j \leq -1 \leq 1 \leq \min_{[x_j=-1]} f_j. \tag{7}$$

Now, (6) is equivalent to

$$f_j(x) \leq -1 \quad \forall x \in [x_j = -1], \qquad f_j(x) \geq 1 \quad \forall x \in [x_j = 1],$$

and hence to

$$\begin{gathered} g_j(x) = x_j - f_j(x) = -1 - f_j(x) \geq 0 \quad \forall\, x \in [x_j = -1], \\ g_j(x) = x_j - f_j(x) = 1 - f_j(x) \leq 0 \quad \forall x \in [x_j = 1]. \end{gathered} \tag{8}$$

Similarly, (7) is equivalent to

$$f_j(x) \leq -1 \quad \forall x \in [x_j = 1], \qquad f_j(x) \geq 1 \quad \forall x \in [x_j = -1],$$

$$\begin{gathered} g_j(x) = x_j - f_j(x) = 1 - f_j(x) \geq 2 \quad \forall x \in [x_j = 1], \\ g_j(x) = x_j - f_j(x) = -1 - f_j(x) \leq -2 \quad \forall x \in [x_j = -1]. \end{gathered} \tag{9}$$

Now, for $j \in \{1, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$, it follows from assumption (b) that

$$f_j(x) \geq -1 \quad \forall x \in [x_j = -1] \qquad \text{and} \qquad f_j(x) \leq 1 \quad \forall x \in [x_j = 1].$$

Hence,

$$\begin{gathered} g_j(x) = x_j - f_j(x) = -1 - f_j(x) \leq 0 \quad \forall x \in [x_j = -1], \\ g_j(x) = x_j - f_j(x) = 1 - f_j(x) \geq 0 \quad \forall x \in [x_j = 1]. \end{gathered} \tag{10}$$

Conditions (8), (9) and (10) show that, for each $j \in \{1, \ldots, n\}$, g_j takes opposite signs on the opposite faces $[x_j = -1]$ and $[x_j = 1]$ of I^n. It follows from Lemma 2 that g has a zero in I^n.

Remark **5.** Assumption (b) is obviously satisfied if

$$\left[\min_{[x_j=-1]} f_j, \max_{[x_j=1]} f_j\right] \subset [-1, 1] \qquad \text{or} \qquad \left[\max_{[x_j=1]} f_j, \min_{[x_j=-1]} f_j\right] \subset [-1, 1].$$

In this case, one sees that the interval constructed on the values of f_j on the opposite j^{th}-faces of I^n covers $[-1, 1]$ if $j \in \{i_1, i_2, \ldots, i_k\}$, and is contained in $[-1, 1]$ if $j \in \{1, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$.

Remark **6.** Theorem 3 is distinct from, but related to Theorem 2 and Corollary 2. Indeed, if in Theorem 3 we take $n = 2$, $k = 1$ and $i_1 = 1$, then the assumptions (a) and (b) become

(a′) $f_1(-1, y) \leq -1\ \forall y \in [-1, 1]$, $f_1(1, y) \geq 1\ \forall y \in [-1, 1]$,

or

$f_1(1,y) \le -1 \ \forall y \in [-1,1]$, $f_1(-1,y) \ge 1 \ \forall y \in [-1,1]$;

(b′) $f_2(x,-1) \ge -1 \ \forall x \in [-1,1]$, $f_2(x,1) \le 1 \ \forall x \in [-1,1]$.

The first condition in assumption (a′) can be written

(a″) $xf_1(x,y) \ge 1 \ \forall (x,y) \colon |x| = 1, \ |y| \le 1$,

and assumption (b′) can be written equivalenly

(b″) $yf_2(x,y) \le 1 \ \forall (x,y) \colon |x| \le 1, \ |y| = 1$.

In this case, its statement is a special case of Corollary 2. Now, the second condition in assumption (a′) can be written

(a‴) $xf_1(x,y) \le -1 \ \forall (x,y) \colon |x| = 1, \ |y| \le 1$,

which is not covered by Corollary 2 but implies that

$$|f_1(x,y)| \ge 1 \quad \forall (x,y) \colon |x| = 1, \qquad |y| \le 1,$$

and that

$$d_B[f_1(\cdot,0),(-1,1),0] = -1.$$

So, the existence of a fixed point of $f = (f_1, f_2)$ under the second condition in assumption (a′) and assumption (b′) follows also from Theorem 2.

***Remark* 7.** Lemma 2 and Theorem 3, stated and proved for I^n for simplicity and to make more easy the comparison with Theorem 2 and its Corollaries. They hold equally, with trivial adaptations, when I^n is replaced by $[a_1,b_1] \times \ldots \times [a_n,b_n]$.

1. *Andres J., Gaudenzi M., Zanolin F.* A transformation theorem for periodic solutions of nondissipative systems // Rend. Semin. mat. Univ. politecn. Torino. – 1990. – **48**. – P. 171 – 186.
2. *Birkhoff G. D., Kellogg O. D.* Invariant points in function spaces // Trans. Amer. Math. Soc. – 1922. – **23**. – P. 96 – 115.
3. *Deimling K.* Nonlinear functional analysis. – Berlin: Springer, 1985.
4. *Dolcher M.* Due teoremi sull'esistenza di punti uniti nelle transformazioni piane continue // Rend. Semin. mat. Univ. Padova. – 1948. – **17**. – P. 97 – 101.
5. *Hadamard J.* Sur quelques applications de l'indice de Kronecker // Introduction à la théorie des fonctions d'une variable / J. Tannery. – 2e éd. – Paris: Hermann, 1910. – **2**.
6. *Leray J., Schauder J.* Topologie et équations fonctionnelles // Ann. sci. Ecole norm. supér. – 1934. – **51**. – P. 45 – 78.
7. *Mawhin J.* Topological degree methods in nonlinear boundary value problems // CBMS Region. Conf. – Providence RI: Amer. Math. Soc., 1979. – **40**.
8. *Mawhin J.* Reduction and continuation theorems for Brouwer degree and applications to nonlinear difference equations // Opusc. Math. – 2008. – **28**. – P. 541 – 560.
9. *Mawhin J.* Variations on Poincaré – Miranda's theorem // Adv. Nonlinear Stud. – 2013. – **13** (to appear).
10. *Miranda C.* Un' osservazione su un teorema di Brouwer // Bol. Unione. mat. ital. – 1940–1941. – **3**, № 2. – P. 5 – 7.
11. *Pireddu M., Zanolin F.* Fixed points for dissipative-repulsive systems and topological dynamics of mappings defined on N-dimensional cells // Adv. Nonlinear Stud. – 2005. – **5**. – P. 411 – 440.
12. *Pireddu M., Zanolin F.* Cutting surfaces and applications to periodic points and chaotic-like dynamics // Top. Meth. Nonlinear Anal. – 2007. – **30**. – P. 279 – 319.
13. *Schaefer H. H.* Über die Methode der a-priori Schranken // Math. Ann. – 1955. – **129**. – S. 415 – 416.
14. *Poincaré H.* Sur certaines solutions particulières du problème des trois corps // C. R. Acad. Sci. Paris. – 1883. – **97**. – P. 251 – 252.
15. *Zgliczyński P.* On periodic points for systems of weakly coupled 1-dim maps // Nonlinear Anal. – 2001. – **46**. – P. 1039 – 1062.

Received 10.01.13