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ON COUNTABLE ALMOST INVARIANT PARTITIONS OF G-SPACES

ПРО ЗЛІЧЕННІ МАЙЖЕ ІНВАРІАНТНІ РОЗБИТТЯ G-ПРОСТОРІВ

For any σ-finite G-quasiinvariant measure μ given in a G-space which is G-ergodic and possesses the Steinhaus property, it is shown that every nontrivial countable μ-almost G-invariant partition of the G-space has a μ-nonmeasurable member.

Для будь-якої σ-скінченної G-квазіінваріантної міри μ, що задана на G-просторі, є G-ергодичною та має властивість Штейнхауса, показано, що кожне нетривіальне розбиття μ-майже G-інваріантного розбиття G-простору має μ-невимірний член.

Several interesting countable partitions of the real line $\mathbf{R}$ into pairwise congruent subsets are known (see, e.g., [1, 2]).

Historically, the first example of such a partition was presented by Vitali [3] in 1905. With the aid of an uncountable form of the Axiom of Choice, Vitali constructed a set $V \subset \mathbf{R}$ having the following properties:

(a) $(V+p) \cap (V+q) = \varnothing$ for any two distinct rational numbers p and q;

(b) the union of the sets $V+q$, where q ranges over the field $\mathbf{Q}$ of all rational numbers, coincides with $\mathbf{R}$.

Recall that the set V is, in fact, the first example of a Lebesgue nonmeasurable subset of $\mathbf{R}$. V is usually called a Vitali subset of $\mathbf{R}$. Notice that Vitali type subsets of uncountable groups are discussed in many articles, surveys and books (among relatively recent works, see, e.g., [4 – 9]).

After Vitali's result [3], Sierpiński [10] gave another example of a countable partition of $\mathbf{R}$ into pairwise congruent sets. Namely, he constructed a disjoint countable family $\{E_i : i \in I\}$ of subsets of $\mathbf{R}$ such that:

(1) all sets E_i, $i \in I$, are translates of each other and collectively cover $\mathbf{R}$;

(2) all sets E_i, $i \in I$, are thick with respect to the Lebesgue measure on $\mathbf{R}$; in particular, each E_i is nonmeasurable in the Lebesgue sense.

The main purpose of this paper is to present some related results concerning countable almost invariant partitions of the real line (of a finite-dimensional Euclidean space equipped with an appropriate transformation group).

Throughout the paper we use the following fairly standard notation:

ω = the set of all natural numbers, i.e.,

$$\omega = \mathbf{N} = \{0, 1, 2, \ldots, n, \ldots\};$$

simultaneously, ω stands for the least infinite ordinal (cardinal) number.

$\mathbf{c}$ = the cardinality of the continuum; naturally, we identify $\mathbf{c}$ with the least ordinal number which is equinumerous with $\mathbf{R}$.

$\mathbf{R}^m$ = the Euclidean space whose dimension is m, where $m \in \mathbf{N}$.

λ_m = the standard m-dimensional Lebesgue measure on $\mathbf{R}^m$. If $m = 1$, then, for the sake of brevity, we write λ instead of λ_1.

If E is a set, then the symbol Id_E denotes the identity transformation of E.

If X and Y are any two sets, then $X \triangle Y$ stands for the symmetric difference of X and Y.

Let E be a ground (base) set equipped with a transformation group G. In this case, for the sake of brevity, we say that E is a G-space.

Let μ be a nonzero σ-finite measure on E. In the sequel, we denote by $\mathrm{dom}(\mu)$ the σ-algebra of all μ-measurable subsets of E. The symbol $\mathcal{I}(\mu)$ stands for the σ-ideal generated by the family of all μ-measure zero sets in E.

A subset X of E is called μ-thick (in E) if the equality $\mu_*(E \setminus X) = 0$ holds true, where μ_* denotes, as usual, the inner measure associated with μ.

We say that μ is a G-quasiinvariant measure on E if both $\mathrm{dom}(\mu)$ and $\mathcal{I}(\mu)$ are G-invariant classes of sets.

We recall that μ is G-ergodic (or G-metrically transitive) if, for every μ-measurable set $X \subset E$ with $\mu(X) > 0$, there exists a family $\{g_j : j \in \omega\} \subset G$ such that

$$\mu(E \setminus \cup\{g_j(X) : j \in \omega\}) = 0.$$

Let G be a group of transformations of E and suppose that G is endowed with some topology. In general, we will not assume in our further consideration that this topology is compatible with the algebraic structure of G.

We shall say that μ has (or possesses) the Steinhaus property if, for each μ-measurable set $X \subset E$ with $\mu(X) > 0$, there exists a neighborhood U of the identity transformation Id_E such that

$$(\forall g \in U)(\mu(g(X) \cap X) > 0).$$

***Remark* 1.** Let $(G, \cdot)$ be a σ-compact locally compact topological group, μ be the left Haar measure on G and let μ' denote the completion of μ. Further, let X be a μ'-measurable subset of G satisfying the relations $0 < \mu'(X) < +\infty$. Then, as is well known, the equality

$$\lim_{g \to e} \mu((g \cdot X) \triangle X) = 0$$

holds true, where e stands for the neutral element of G. Obviously, the above equality implies the Steinhaus property of μ'.

Let E be a G-space equipped with some σ-finite measure μ and let $\{X_i : i \in I\}$ be a family of subsets of E.

We say that $\{X_i : i \in I\}$ is a μ-almost disjoint family if

$$\mu(X_i \cap X_{i'}) = 0, \quad i \in I, i' \in I, i \neq i'.$$

We say that $\{X_i : i \in I\}$ is μ-almost G-invariant if, for any $g \in G$, the family $\{g(X_i) : i \in I\}$ almost (more precisely, μ-almost) coincides with $\{X_i : i \in I\}$.

The latter phrase means that, for any index $i \in I$, there exists an index $i' = i'(i)$ such that $\mu(X_{i'} \triangle g(X_i)) = 0$.

Recall that an abstract group $(G, \cdot)$ (not necessarily commutative) is divisible if, for each element $g \in G$ and for each natural number $n > 0$, the equation $x^n = g$ is solvable in G.

***Remark* 2.** The structure of all commutative divisible groups is well known (see, for instance, § 23 in monograph [11]). On the other hand, the structure of noncommutative divisible groups is still unclear and may be very complicated. A simple example of a noncommutative divisible group is provided by the group Is_2^+ of all orientation preserving isometric transformations of the Euclidean plane $\mathbf{R}^2$. It is a widely known fact that any (commutative) group can be isomorphically embedded in a (commutative) divisible group (see § 23 and § 67 in [11]).

***Remark* 3.** It should be noticed that if $(G, +)$ is an infinite commutative divisible group and X is a nonempty proper subset of G, then the family $\{X + g \colon g \in G\}$ is necessarily infinite. Actually, this fact was first proved by Sierpiński (cf. [1]). Also, it worth noticing that there exists a Vitali set V in the divisible commutative group $(\mathbf{R}, +)$ such that the family $\{V + t \colon t \in \mathbf{R}\}$ is countably infinite.

Let E be a G-space and suppose that the group G is endowed with some topology.

We shall say that G is admissible if, for any neighborhood U of Id_E and for any element $g \in G$, there exists a natural number n such that the equation $x^n = g$ has a solution belonging to U.

***Example* 1.** Clearly, the topological group T_m of all translations of $\mathbf{R}^m$ is admissible. Also, the topological group $\mathrm{O}^+(m)$ of all rotations of $\mathbf{R}^m$ about its origin is admissible. These two simple facts will be substantially exploited below.

Let E be again a G-space such that G is endowed with some topology.

We shall say that G is weakly admissible if there are finitely many admissible subgroups $G_1, G_2, \ldots, G_r$ of G such that

$$G = G_1 \circ G_2 \circ \ldots \circ G_r.$$

***Example* 2.** Consider the topological group Is_m^+ of all orientation preserving isometric transformations of $\mathbf{R}^m$. If g is any element of Is_m^+, then g can be uniquely represented in the form $g = h \circ g_0$, where $h \in \mathrm{T}_m$ and $g_0 \in \mathrm{O}_m^+$. In view of Example 1, one may conclude that the group Is_m^+ is weakly admissible. On the other hand, if $m \geq 1$, then the topological group Is_m of all isometric transformations of $\mathbf{R}^m$ is not weakly admissible (cf. Example 5 at the end of this paper).

We will be dealing with countable μ-almost G-invariant partitions of a G-space E, where μ is a nonzero σ-finite G-quasiinvariant measure on E and G is a weakly admissible group of transformations of E.

Let us formulate and prove the main statement of this paper.

Theorem 1. *Let G be a weakly admissible group of transformations of E, let μ be a nonzero σ-finite G-ergodic measure on E having the Steinhaus property, and let $\{X_i \colon i \in \omega\}$ be a μ-almost disjoint μ-almost G-invariant covering of E. Then one of the following two assertions holds*:

(1) *there exists an index $i \in \omega$ such that the set X_i is not μ-measurable*;

(2) *there exists an index $k \in \omega$ such that $\mu(E \setminus X_k) = 0$, i.e., the given covering is trivial in the sense of μ.*

Proof. Suppose that (1) is not satisfied, i.e., all sets X_i, $i \in \omega$, are μ-measurable. Then, since μ is not identically equal to zero, there exists an index $k \in \omega$ such that $\mu(X_k) > 0$. We assert that $\mu(E \setminus X_k) = 0$.

Suppose the contrary that $\mu(E \setminus X_k) > 0$. Since μ is G-ergodic, we can find a family $\{g_j \colon j \in \in \omega\} \subset G$ for which the equality

$$\mu(E \setminus \cup\{g_j(X_k)\colon j \in \omega\}) = 0$$

is valid. This equality implies the existence of an index $k' \in \omega \setminus \{k\}$ such that

$$\mu(g_j(X_k) \cap X_{k'}) > 0$$

for some $j \in \omega$. Therefore, taking into account the μ-almost disjointness and μ-almost G-invariance of $\{X_i\colon i \in \omega\}$, we must have

$$\mu(X_{k'} \triangle g_j(X_k)) = 0.$$

Further, let us denote

$$G_k = \{g \in G\colon \mu(g(X_k) \triangle X_k) = 0\}.$$

Clearly, G_k is a subgroup of G and $g_j \notin G_k$. Keeping in mind the inequality $\mu(X_k) > 0$ and remembering that μ has the Steinhaus property, we can find a neighborhood U of Id_E such that

$$(\forall g \in U)(\mu(g(X_k) \cap X_k) > 0).$$

Further, since the group G is weakly admissible, we may write

$$G = H_1 \circ H_2 \circ \ldots \circ H_r$$

for some groups $H_1 \subset G$, $H_2 \subset G$, $\ldots$, $H_r \subset G$, all of which are admissible. In particular, we have the equality

$$g_j = h_1 \circ h_2 \circ \ldots \circ h_r,$$

where $h_1 \in H_1,\ h_2 \in H_2, \ldots, h_r \in H_r$. Since $g_j \notin G_k$, at least one of the transformations $h_1, h_2, \ldots, h_r$ does not belong to G_k. Let p be a natural number from the set $\{1, 2, \ldots, r\}$ such that $h_p \notin G_k$.

Now, there exist a natural number n and an element $h_0 \in U$ for which we have $h_0^n = h_p$. Since $h_p \notin G_k$, we also have $h_0 \notin G_k$. On the other hand, the relation

$$\mu(h_0(X_k) \cap X_k) > 0$$

holds true by virtue of the Steinhaus property. Using once again the μ-almost disjointness and μ-almost G-invariance of the family $\{X_i\colon i \in \omega\}$, we obtain

$$\mu(h_0(X_k) \triangle X_k) = 0, \quad h_0 \in G_k.$$

So we come to a contradiction with the relation $h_0 \notin G_k$. The obtained contradiction finishes the proof.

The following statement is a direct consequence of the above theorem.

Theorem 2. *Let $G = \mathrm{Is}_m^+$ be the topological group of all orientation preserving isometric transformations of the space $\mathbf{R}^m$, where $m \geq 1$, and let μ be a nonzero σ-finite G-ergodic measure on $\mathbf{R}^m$ possessing the Steinhaus property. Suppose that $\{X_i\colon i \in \omega\}$ is a μ-almost disjoint and μ-almost G-invariant covering of $\mathbf{R}^m$. Then either there exists at least one index $i \in \omega$ such that the set X_i is not μ-measurable or there exists an index $k \in \omega$ such that $\mu(\mathbf{R}^m \setminus X_k) = 0$.*

Proof. It suffices to apply Theorem 1, keeping in mind the fact that G is a weakly admissible group of transformations of $\mathbf{R}^m$ (see Example 2).

Now, we are going to give an application of Theorem 2 to that case where the role of μ is played by the standard Lebesgue measure λ_m on $\mathbf{R}^m$.

For this purpose, we need one auxiliary statement (probably, it is well known but, for the sake of completeness, we present its proof here).

Theorem 3. *Let G be a subgroup of the group Is_m. The following three assertions are equivalent*:

(1) *the measure λ_m is G-metrically transitive*;

(2) *there exists a point $y \in \mathbf{R}^m$ whose G-orbit $G(y)$ is dense in $\mathbf{R}^m$*;

(3) *for any point $z \in \mathbf{R}^m$, its G-orbit $G(z)$ is dense in $\mathbf{R}^m$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) is easy to show and, actually, this equivalence remains true in a much more general situation (e.g., in the case of a metric space E equipped with some group G of isometric transformations of E). So, in our further consideration, we will be focused only on the proof of the equivalence (1) $\Leftrightarrow$ (2).

Suppose that (1) holds and consider an arbitrary point $y \in \mathbf{R}^m$. Let ε be a strictly positive real number and let $U(y)$ be the open ε-neighborhood of y. Let us denote

$$V(0) = U(y) - y.$$

Then $V(0)$ is the open ε-neighborhood of 0. Since $\lambda_m(V(0)) > 0$ and λ_m is G-metrically transitive, there exists a family $\{g_i \colon i \in \omega\} \subset G$ such that

$$\lambda_m(\mathbf{R}^m \setminus \cup\{g_i(V(0)) \colon i \in \omega\}) = 0.$$

Further, since $\lambda_m(U(y)) > 0$, there is an index $i \in \omega$ such that

$$\lambda_m(g_i(V(0)) \cap U(y)) > 0$$

and, consequently,

$$g_i(V(0)) \cap U(y) \neq \varnothing.$$

So we infer that the point $g_i(0)$ belongs to the (2ε)-neighborhood of y. Since ε was taken arbitrarily small, we conclude that the orbit $G(0)$ is dense in the space $\mathbf{R}^m$, i.e., (2) holds true.

Suppose now that (2) is satisfied for a group $G \subset \mathrm{Is}_m$. Without loss of generality, we may assume that G is at most countable and the orbit $G(0)$ is dense in $\mathbf{R}^m$.

Let X be a λ_m-measurable subset of $\mathbf{R}^m$ with $\lambda_m(X) > 0$. We assert that

$$\lambda_m(\mathbf{R}^m \setminus G(X)) = 0.$$

Suppose otherwise, i.e., $\lambda_m(\mathbf{R}^m \setminus G(X)) > 0$, and denote $Z = \mathbf{R}^m \setminus G(X)$. Since $\lambda_m(X) > 0$, there exists a density point x of X. Analogously, since $\lambda_m(Z) > 0$, there exists a density point z of Z. Let B be a ball in $\mathbf{R}^m$ whose center is 0 and whose radius is so small that

$$\lambda_m(X \cap (B + x)) \geq (2/3)\lambda_m(B), \quad \lambda_m(Z \cap (B + z)) \geq (2/3)\lambda_m(B).$$

Further, let $g \in G$ be such that

$$\lambda_m(g(B+x) \cap (B+z)) > (2/3)\lambda_m(B).$$

Then a straightforward calculation enables one to conclude that

$$\lambda_m(g(X \cap (B+x)) \cap (Z \cap (B+z))) > 0$$

and, therefore,

$$\lambda_m(g(X) \cap Z) > 0,$$

which contradicts the obvious equality $G(X) \cap Z = \varnothing$. The obtained contradiction establishes the validity of the implication (2) $\Rightarrow$ (1).

This finishes the proof of Theorem 3.

Combining Theorems 2 and 3, we obtain the next result.

Theorem 4. *Let G be a subgroup of Is_m^+ with the property that at least one point of the space $\mathbf{R}^m$ has dense G-orbit, and let $\{X_i \colon i \in \omega\}$ be a λ_m-almost disjoint and λ_m-almost G-invariant covering of $\mathbf{R}^m$. Then either there exists an index $i \in \omega$ such that the set X_i is not λ_m-measurable or there exists an index $k \in \omega$ such that $\lambda_m(E \setminus X_k) = 0$.*

We would like to underline that the proof of Theorem 1 uses the following two conditions:

(i) the G-ergodicity of a given measure μ;

(ii) the Steinhaus property of μ.

Now, we are going to show by relevant examples that none of the conditions (i) and (ii) can be omitted.

To present these examples, we must consider some quasiinvariant and invariant extensions of the Lebesgue measure. Notice that various constructions of extensions of such a type are given, e.g., in [6, 8, 9, 12, 13].

***Example* 3.** Let $m \geq 1$ be a natural number. There exists a partition $\{A_n \colon n < \omega\}$ of $\mathbf{R}^m$ such that:

(a) for any $n < \omega$ and $g \in \mathrm{Is}_m^+$, we have $\mathrm{card}(g(A_n) \triangle A_n) < \mathbf{c}$;

(b) if Z is a Borel subset of $\mathbf{R}^m$ with $\lambda_m(Z) > 0$ and n is any natural index, then $\mathrm{card}\,(Z \cap A_n) = \mathbf{c}$; in particular, all sets A_n $(n < \omega)$ are λ_m-thick in $\mathbf{R}^m$.

The transfinite construction of $\{A_n \colon n < \omega\}$ is fairly standard and may be found in several works (see, e.g., [12, 13]).

Consider the family $\mathcal{F}$ of all those sets which admit a representation in the form

$$\cup\{A_n \cap Z_n \colon (\forall n < \omega)(Z_n \in \mathrm{dom}(\lambda_m))\}.$$

Notice that $\mathcal{F}$ is a σ-algebra of subsets of $\mathbf{R}^m$ containing $\mathrm{dom}(\lambda_m)$. Define a functional μ on $\mathcal{F}$ by the formula

$$\mu(\cup\{A_n \cap Z_n \colon n < \omega\}) = \sum\{(1/2^{n+1})\lambda_m(Z_n) \colon n < \omega\}.$$

This μ is well defined and is a measure extending λ_m. Moreover, μ can be uniquely extended to an Is_m^+-quasiinvariant measure μ' by adding to the domain of μ the family of all subsets of $\mathbf{R}^m$ whose cardinalities are strictly less than $\mathbf{c}$. For the extended measure μ', one can readily conclude that:

(i) μ' is not Is_m^+-ergodic;

(ii) μ' has the Steinhaus property;

(iii) $\{A_n : n < \omega\}$ is a nontrivial Is_m^+-invariant partition of $\mathbf{R}^m$ into countably many μ'-measurable subsets of $\mathbf{R}^m$, each of which is of strictly positive μ'-measure.

The next example essentially relies on the existence of a Hamel basis in $\mathbf{R}$.

***Example* 4.** Consider the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers. Let $\{e_\xi : \xi < \mathbf{c}\}$ denote a Hamel basis of $\mathbf{R}$. Now, take the vector space over $\mathbf{Q}$ generated by the partial family $\{e_\xi : 0 < \xi < \mathbf{c}\}$. Denote this vector space by H and observe that H is a hyperplane in $\mathbf{R}$ complementary to the "line" $\mathbf{Q}e_0$. So we come to the countable partition

$$\{H_n : n < \omega\} = \{H + qe_0 : q \in \mathbf{Q}\}$$

of $\mathbf{R}$. Obviously, this partition is T_1-invariant. Moreover, the following relations are satisfied:

(a) for each λ-measurable set Z with $\lambda(Z) > 0$ and for each $n < \omega$, we have $Z \cap H_n \neq \varnothing$;

(b) for any two natural indices n and m, there exists $h \in \mathbf{R}$ such that $h + H_n = H_m$.

Now, we introduce the σ-algebra of sets

$$\mathcal{F} = \{\cup\{H_n \cap Z_n : n < \omega\} : (\forall n < \omega)(Z_n \in \mathrm{dom}(\lambda))\}$$

and define a functional μ on $\mathcal{F}$ by the formula

$$\mu(\cup\{H_n \cap Z_n : n < \omega\}) = \sum\{(1/2^{n+1})\lambda(Z_n) : n < \omega\}.$$

It is not difficult to check that μ is a T_1-quasiinvariant T_1-ergodic extension of λ for which $\{H_n : n < \omega\}$ is a nontrivial μ-almost T_1-invariant countable partition of $\mathbf{R}$, and all sets H_n $(n < \omega)$ are μ-measurable and have strictly positive μ-measure.

Consequently, in view of Theorem 1, μ does not possess the Steinhaus property.

***Remark* 4.** Actually, for the measure μ of Example 4, the Steinhaus property fails in a very strong form. Namely, one can see that

$$H_n \cap (H_n + qe_0) = \varnothing$$

for any $n < \omega$ and $q \in \mathbf{Q} \setminus \{0\}$.

Let us present one more example which shows that the assumption on a transformation group G be weakly admissible is very essential for the validity of Theorem 1.

***Example* 5.** Let $m \geq 1$ be a natural number. There exists a partition $\{A, B, C\}$ of the space $\mathbf{R}^m$ such that:

(a) for any $g \in \mathrm{Is}_m^+$, we have

$$\mathrm{card}(g(A)\triangle A) < \mathbf{c}, \quad \mathrm{card}(g(B)\triangle B) < \mathbf{c};$$

(b) for any $g \in \mathrm{Is}_m \setminus \mathrm{Is}_m^+$, we have

$$\mathrm{card}(g(A)\triangle B) < \mathbf{c}, \quad \mathrm{card}(g(B)\triangle A) < \mathbf{c};$$

(c) for any $g \in \mathrm{Is}_m$, we have

$$\mathrm{card}(g(C)\triangle C) < \mathbf{c};$$

(d) if Z is any Borel subset of $\mathbf{R}^m$ with $\lambda_m(Z) > 0$, then

$$\mathrm{card}(Z \cap A) = \mathrm{card}(Z \cap B) = \mathbf{c},$$

in particular, both sets A and B are λ_m-thick in $\mathbf{R}^m$.

A detailed transfinite construction of the partition $\{A, B, C\}$ is given in [14]. Notice, by the way, that condition (c) directly follows from the conjunction of the conditions (a) and (b).

Now, consider the σ-algebra of sets

$$\mathcal{F} = \{(A \cap X) \cup (B \cap Y) \cup (C \cap Z) \colon \{X, Y, Z\} \subset \mathrm{dom}(\lambda_m)\}$$

and define on $\mathcal{F}$ a functional μ by the formula

$$\mu((A \cap X) \cup (B \cap Y) \cup (C \cap Z)) = 1/2(\lambda_m(X) + \lambda_m(Y)).$$

It can be checked that μ is well defined and is a measure extending λ_m. Furthermore, by adding to $\mathrm{dom}(\mu)$ the class of all those subsets of $\mathbf{R}^m$ whose cardinalities are strictly less than $\mathbf{c}$, we obtain the measure μ' which is Is_m-invariant, Is_m-ergodic and has the Steinhaus property.

However, we see that $\{A, B, C\}$ is a μ'-almost Is_m-invariant partition of $\mathbf{R}^m$ such that all three sets A, B, C are μ'-measurable and

$$\mu'(A) = \mu'(B) = +\infty.$$

This circumstance can be explained, by taking into account the fact that the group Is_m is not weakly admissible.

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