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ON THE BEHAVIOR OF THE ALGEBRAIC POLYNOMIAL IN UNBOUNDED REGIONS WITH PIECEWISE DINI-SMOOTH BOUNDARY

ПОВЕДІНКА АЛГЕБРАЇЧНОГО ПОЛІНОМА В НЕОБМЕЖЕНИХ ОБЛАСТЯХ З КУСКОВИМИ ДІНІ-ГЛАДКИМИ МЕЖАМИ

Let $G \subset \mathbb{C}$ be a finite region bounded by a Jordan curve $L := \partial G$, $\Omega := \operatorname{ext} \overline{G}$ (with respect to $\overline{\mathbb{C}}$), $\Delta := \{w \colon |w| > 1\}$; $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ, normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. Let $h(z)$ be a weight function, and $A_p(h, G)$, $p > 0$, denote the class of functions f which are analytic in G and satisfying the condition

$$\|f\|^p_{A_p(h,G)} := \iint\limits_G h(z)\, |f(z)|^p \, d\sigma_z < \infty,$$

where σ is a two-dimensional Lebesgue measure.

Let $P_n(z)$ be an arbitrary algebraic polynomial of degree at most $n \in \mathbb{N}$. Well known Bernstein – Walsh lemma shown that:

$$|P_n(z)| \le |\Phi(z)|^n \, \|P_n\|_{C(\overline{G})}\,, \quad z \in \Omega. \qquad (*)$$

In this present work we continue studying the estimation $(*)$, when we replace the norm $\|P_n\|_{C(\overline{G})}$ by $\|P_n\|_{A_p(h,G)}$, $p > 0$, for Jacobi type weight function in regions with piecewise Dini-smooth boundary.

Нехай $G \subset \mathbb{C}$ — скінченна множина, обмежена жордановою кривою $L := \partial G$, $\Omega := \operatorname{ext} \overline{G}$ (відносно $\overline{\mathbb{C}}$), $\Delta := \{w \colon |w| > 1\}$; $w = \Phi(z)$ — однолисте конформне відображення Ω на Δ, нормоване так, що $\Phi(\infty) = \infty$ та $\Phi'(\infty) > 0$. Також нехай $h(z)$ — вагова функція, а $A_p(h, G)$, $p > 0$, — клас функцій f, аналітичних в G, що задовольняють умову

$$\|f\|^p_{A_p(h,G)} := \iint\limits_G h(z)\, |f(z)|^p \, d\sigma_z < \infty,$$

де σ — двовимірна міра Лебега.

Нехай $P_n(z)$ — довільний алгебраїчний поліном степеня не більшого за $n \in \mathbb{N}$. Відома лема Бернштейна – Волша стверджує, що

$$|P_n(z)| \le |\Phi(z)|^n \, \|P_n\|_{C(\overline{G})}\,, \quad z \in \Omega. \qquad (*)$$

У даній роботі продовжено дослідження оцінки $(*)$, в якій норму $\|P_n\|_{C(\overline{G})}$ замінено на $\|P_n\|_{A_p(h,G)}$, $p > 0$, для вагової функції типу Якобі в областях з кусковими Діні-гладкими межами.

1. Introduction and main results. Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $G \subset \mathbb{C}$ be a bounded Jordan region with $0 \in G$ and the boundary $L := \partial G$ in the form of a simple closed Jordan curve, $\Omega := \overline{\mathbb{C}} \backslash \overline{G}$, $B := B(0,1) := \{z \colon |z| < 1\}$, and $\Delta := \Delta(0,1) := \{w \colon |w| > 1\}$. Also let $w = \Phi(z) (w = \varphi(z))$ be a univalent conformal mapping of $\Omega(G)$ onto the $\Delta(B)$ normalized by $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \dfrac{\Phi(z)}{z} > 0$ ($\varphi(0) = 0$, $\varphi'(0) > 0$), and $\Psi := \Phi^{-1}$ ($\psi := \varphi^{-1}$).

By $\wp_n$ we denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $h(z)$ be a weight function. By $A_p(h, G)$, $p > 0$, we denote a class of functions f analytic in G and satisfying the condition

$$\|f\|_{A_p(h,G)} := \left(\iint\limits_G h(z)\, |f(z)|^p \, d\sigma_z \right)^{1/p} < \infty,$$

where σ_z is the two-dimensional Lebesgue measure and $A_p(1, G) \equiv A_p(G)$.

In case where L is rectifiable, let $\mathcal{L}_p(h,L)$, $p>0$, denote a class of functions f integrable on L and satisfying the condition

$$\|f\|_{\mathcal{L}_p(h,L)} := \left(\int\limits_L h(z)\,|f(z)|^p\,|dz| \right)^{1/p} < \infty,$$

and $\mathcal{L}_p(1,L) \equiv \mathcal{L}_p(L)$.

From well known Bernstein – Walsh lemma [14], we see that

$$|P_n(z)| \leq |\Phi(z)|^n \, \|P_n\|_{C(\overline{G})}, \quad z \in \Omega. \tag{1.1}$$

For $R>1$, we set $L_R := \{z\colon\ |\Phi(z)| = R\}$, $G_R := \operatorname{int} L_R$, $\Omega_R := \operatorname{ext} L_R$. Then (1.1) can be rewritten as follows:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \, \|P_n\|_{C(\overline{G})}. \tag{1.2}$$

Hence, setting $R = 1 + \dfrac{1}{n}$, according to (1.2), we see that the C-norm of polynomials $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ is equivalent, i.e., the norm $\|P_n\|_{C(\overline{G})}$ increases with no more than a constant in $\overline{G}_R$.

In the case where L is rectifiable, a similar estimate of the (1.2)-type in the space $\mathcal{L}_p(L)$ was investigated in [9] and obtained in the following form:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \, \|P_n\|_{\mathcal{L}_p(L)}, \quad p>0. \tag{1.3}$$

To give an inequality similar to (1.3) for the case of theA_p-norm, we first present the following definitions and notation:

Definition 1.1 [10, p. 97, 12]. *A Jordan arc* (*curve*) L *is called* K*-quasiconformal* $(K \geq 1)$, *if there is a* K*-quasiconformal mapping* f *of the region* $D \supset L$ *such that* $f(L)$ *is a line segment* (*or circle*).

We denote by $F(L)$ *the set of all sense-preserving plane homeomorphisms* f *of the region* $D \supset L$ *such that* $f(L)$ *is a line segment* (*or circle*) *and let*

$$K_L := \inf\{K(f)\colon\ f \in F(L)\},$$

where $K(f)$ *is the maximal dilatation of a mapping* f *of this kind,* L *is a quasiconformal curve if* $K_L < \infty$, *and* L *is a* K*-quasiconformal curve if* $K_L \leq K$.

We know that there exist quasiconformal curves that are not rectifiable [10, p. 104].

Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points in the curve L located in the positive direction. Consider a so-called generalized Jacobi weight function $h(z)$ defined as follows:

$$h(z) := \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \quad z \in G_R, \tag{1.4}$$

where $\gamma_j > -2$ for all $j = \overline{1,m}$ (i.e., $j = 1, 2, \ldots, m$).

The Bernstein – Walsh type estimation for the regions G with quasiconformal boundary and for the weight function $h(z)$ of (1.4) type in the space $A_p(h,G)$, $p>0$, was contained in [3]. In particular, for $h(z) \equiv 1$,

$$\|P_n\|_{A_p(G_R)} \leq c_2 \, {R^*}^{n+\frac{1}{p}} \, \|P_n\|_{A_p(G)}, \quad p > 0, \tag{1.5}$$

where $R^* := 1 + c_3(R-1)$. Therefore, if we choose $R = 1 + \dfrac{c_1}{n}$, then (1.5) can be shown like that A_p-norm of polynomials $P_n(z)$ in G_R and G is equivalent. Note that, here and throughout this paper we denote by c, c_0, c_1, $c_2, \ldots$ positive constants (in general, different in different relations) that depend on G in general and on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

N. Stylianopoulos in [13] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (1.1) and so has found a new version of the Bernstein – Walsh lemma. Before we give the corresponding result of N. Stylianopoulos, we will give the following:

Definition 1.2. *A bounded Jordan region G is called a k-quasidisk, $0 \leq k < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \dfrac{1+k}{1-k}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In this case, the curve $L := \partial G$ is called a k-quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is k-quasidick (k-quasicircle) with some $0 \leq k < 1$.*

***Remark* 1.1.** It is well known that if we are not interested in the coefficients of quasiconformality of the curve, then the definitions of “quasicircle” and “quasiconformal curve” (in the case of $D = \overline{\mathbb{C}}$) are identical. In the case where we are also interested in the coefficients of quasiconformality of the given curve, we consider that if the curve L is K-quasiconformal, then it is a k-quasicircle with $k = \dfrac{K^2-1}{K^2+1}$.

Following Remark 1.1, for the sake of simplicity, we use both terms, depending on the situation.

Lemma A [13]. *Assume that L is quasicircle and rectifiable. Then there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \leq c \frac{\sqrt{n}}{d(z,L)} \, \|P_n\|_{A_2(G)} \, |\Phi(z)|^{n+1}, \quad z \in \Omega, \tag{1.6}$$

where $d(z,L) := \inf\{|\zeta - z| : \zeta \in L\}$, holds for every $P_n \in \wp_n$.

In the present work, we study a problem similar to (1.6) for regions with piecewise Dini-smooth boundary and for generalized Jacobi weight function $h(z)$ defined as in (1.4) in $A_p(h,G)$, $p > 1$.

Let us give corresponding definition and some notation that will be used in what follows.

Definition 1.3 [11, p. 48] (see also [7, p. 32]). *A Jordan curve L is called Dini-smooth if it has a parametrization $z = z(s)$, $0 \leq s \leq |L| := \operatorname{mes} L$, such that $z'(s) \neq 0$, $0 \leq s \leq |L|$ and $|z'(s_2) - z'(s_1)| < g(s_2 - s_1)$, $s_1 < s_2$, where g is an increasing function for which*

$$\int\limits_0^1 \frac{g(x)}{x} dx < \infty.$$

Definition 1.4. *We say that a Jordan region G has a piecewise Dini-smooth boundary if $L := \partial G$ consists of the union of finite Dini-smooth arcs L_j, $j = \overline{1,m}$, such that they have exterior (with respect to $\overline{G}$) angles $\lambda_j \pi$, $0 < \lambda_j < 2$ at the corner points $\{z_j\}$, $j = \overline{1,m}$, where two arcs meet.*

According to the "three-point" criterion [10, p. 100], every piecewise Dini-smooth curve (without cusps) is quasiconformal.

For $0 < \delta_j < \delta_0 := \dfrac{1}{4}\min\left\{|z_i - z_j|\colon i,j = 1,2,\ldots,m,\ i \neq j\right\}$, let $\Omega(z_j,\ \delta_j) := \Omega \cap \cap\left\{z\colon |z-z_j| \leq \delta_j\right\}$; $\delta := \min_{1\leq j\leq m}\delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{m}\Omega(z_j,\ \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$; $\Delta_j := \Phi(\Omega(z_j,\ \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m}\Phi(\Omega(z_j,\ \delta))$, $\widehat{\Delta}(\delta) := \Delta\backslash\Delta(\delta)$. Let $w_j := \Phi(z_j)$ and, for $\varphi_j := \arg w_j$, $j = \overline{1,m}$, we set $\Delta_j' := \left\{t = Re^{i\theta}\colon R > 1,\ \dfrac{\varphi_{j-1}+\varphi_j}{2} \leq \theta < \dfrac{\varphi_j + \varphi_{j+1}}{2}\right\}$, where $\varphi_0 \equiv \varphi_m$, $\varphi_1 \equiv \varphi_{m+1}$; $\Omega_j := \Psi(\Delta_j')$, $L_{R_1}^j := L_{R_1} \cap \Omega_j$. Clearly, $\Omega = \bigcup\limits_{j=1}^{m}\Omega_j$.

Let the points $\{z_j\}_{j=1}^m$ be on the curve L in the positive direction. For $k \leq m$, we define $\lambda_k^* := \max\left\{\lambda_j\colon j = \overline{1,k}\right\}$, $\lambda_{k*} := \min\left\{\lambda_j\colon j=\overline{1,k}\right\}$, $\lambda^* := \lambda_m^*$, $\lambda_* := \lambda_{m*}$, $\widetilde{\lambda}_k := \begin{cases}\lambda_{k*}, & \text{if } p \geq 2,\\ \lambda_k^*, & \text{if } p < 2,\end{cases}$ and $\widetilde{\lambda} := \begin{cases}\lambda_*, & \text{if } p\geq 2,\\ \lambda^*, & \text{if } p < 2.\end{cases}$ For any $j = \overline{1,m}$, let us $\mu_j := \dfrac{1}{\lambda_j} + (p-2)$; $\eta_j := \dfrac{1}{\lambda_j} - (2-p)$, $\omega_j := \dfrac{p-1}{\lambda_j}$, $\gamma_k^* := \max\left\{\gamma_j,\ j = \overline{1,k}\right\}, \gamma^* := \gamma_m^*$. $\Gamma := \left\{\gamma_j,\ j = \overline{1,m}\right\}$, $\Gamma_{j,k} := \left\{\gamma_j \in \Gamma\colon \gamma_j \leq \mu_k,\ k,j = \overline{1,m}\right\}$, $\widetilde{\Gamma}_{j,k} := \Gamma\backslash\Gamma_{j,k}$. Let $w_j := \Phi(z_j)$.

We can now state our new results.

Theorem 1.1. *Let $p > 1$. Assume that a Jordan region G has a piecewise Dini-smooth boundary $L := \partial G$ and $h(z)$ is defined as in* (1.4). *Then, for any $P_n \in \wp_n$ and $R_1 = 1 + \dfrac{1}{n}$*

$$|P_n(z)| \leq c_1 \frac{D_{n,1}}{d(z, L_{R_1})} \left\|P_n\right\|_{A_p(h,G)} |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{1.7}$$

where $c_1 = c_1(G,\ p) > 0$ and

$$D_{n,1} = \begin{cases} n^{\frac{1}{p}}, & \begin{array}{l} \text{if } p \geq 2,\ 0 < \lambda_j < 2,\ -2 < \gamma_j < \dfrac{1}{\lambda_j} + (p-2), \\ \text{or } p < 2,\ 1 \leq \lambda_j < 2,\ -2 < \gamma_j < \dfrac{1}{\lambda_j} - (2-p), \\ \text{or } p < 2, 0 < \lambda_j < 1,\ -2 < \gamma_j < \dfrac{p-1}{\lambda_j} \\ \text{for all } j = \overline{1,m}; \end{array} \\ \sum\limits_{j=1}^{m} n^{\frac{\gamma_j\lambda_j}{p} + \left(\frac{2}{p}-1\right)\lambda_j}, & \begin{array}{l} \text{if } p \geq 2,\ 0<\lambda_j<2,\ \gamma_j \geq \dfrac{1}{\lambda_j} + (p-2), \\ \text{or } p < 2,\ 1 \leq \lambda_j < 2,\ \gamma_j \geq \dfrac{1}{\lambda_j} - (2-p) \\ \text{for all } j = \overline{1,m}; \end{array} \\ \sum\limits_{j=1}^{m} n^{\frac{\gamma_j\lambda_j}{p} + \left(\frac{2}{p}-1\right)}, & \begin{array}{l} \text{if } p < 2,\ 0 < \lambda_j < 1,\ \gamma_j \geq \dfrac{p-1}{\lambda_j} \\ \text{for all } j = \overline{1,m}. \end{array} \end{cases}$$

Theorem 1.1 is local, i.e., each term in the sum on the right-hand side shows the growth of $|P_n(z)|$, depending on the behavior of the weight function $h(z)$ and the outside corner λ_j in the neighborhood of a single point z_j for any $j = \overline{1,m}$.

Comparing the terms in the sum for each point $\{z_j\}$, $j = \overline{1,m}$, and using the above notation, we can obtain following result of the global character.

Theorem 1.2. *Let $p > 1$. Assume that a Jordan region G has a piecewise Dini-smooth boundary $L := \partial G$ and $h(z)$ is defined as in* (1.4). *Then, for any $P_n \in \wp_n$ and $R_1 = 1 + \dfrac{1}{n}$*

$$|P_n(z)| \le c_2 \frac{D_{n,2}}{d(z, L_{R_1})} \|P_n\|_{A_p(h,G)} \, |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{1.8}$$

where $c_2 = c_2(G, p, m) > 0$ and

$$D_{n,2} = \begin{cases} n^{\frac{1}{p}}, & \text{if } p \ge 2,\ 0 < \lambda_j < 2, \quad -2 < \gamma_j < \mu_1, \\ & \text{or } p < 2,\ 1 \le \lambda_j < 2, \quad -2 < \gamma_j < \eta_1, \\ & \text{or } p < 2,\ 0 < \lambda_j < 1, \quad -2 < \gamma_j < \omega_1 \\ & \text{for all } j = \overline{1,m}; \\ n^{\frac{\gamma^* \lambda^*}{p} + \left(\frac{2}{p} - 1\right)\widetilde{\lambda}}, & \text{if } p \ge 2,\ 0 < \lambda_j < 2,\ \gamma_j \ge \mu_m, \\ & \text{or } p < 2,\ 1 \le \lambda_j < 2,\ \gamma_j \ge \eta_m \\ & \text{for all } j = \overline{1,m}; \\ n^{\frac{\gamma_k^* \lambda_k^*}{p} + \left(\frac{2}{p} - 1\right)\widetilde{\lambda}_k}, & \text{if } p \ge 2,\ 0 < \lambda_j < 2,\ \mu_k \le \gamma_j < \mu_{k+1}, \\ & \text{or } p < 2,\ 1 \le \lambda_j < 2,\ \eta_k \le \gamma_j < \eta_{k+1} \\ & \text{for all } k = \overline{1,m-1} \text{ and } j = \overline{1,m}; \\ n^{\frac{\gamma^* \lambda^*}{p} + \left(\frac{2}{p} - 1\right)}, & \text{if } p < 2,\ 0 < \lambda_j < 1,\ \gamma_j \ge \omega_m \\ & \text{for all } j = 1, 2, \ldots, m; \\ n^{\frac{\gamma_k^* \lambda_k^*}{p} + \left(\frac{2}{p} - 1\right)}, & \text{if } p < 2,\ 0 < \lambda_j < 1,\ \omega_k \le \gamma_j < \omega_{k+1} \\ & \text{for all } k = 1, 2, \ldots, m-1 \text{ and } j = 1, 2, \ldots, m. \end{cases}$$

In particular, in the case of one singular point $(m = 1)$ on the boundary curve L, we assume, for simplicity, that $\lambda_1 =: \lambda$ and obtain the following corollary.

***Corollary* 1.1.** *Let $p > 1$, $m = 1$. Assume that a Jordan region G has a piecewise Dini-smooth boundary $L := \partial G$ and $h(z)$ is defined as in* (1.4) *for $m = 1$. Then, for any $P_n \in \wp_n$ and $R_1 = 1 + \dfrac{1}{n}$ we have*

$$|P_n(z)| \le c_3 \frac{D_{n,3}}{d(z, L_{R_1})} \|P_n\|_{A_p(h,G)} \, |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{1.9}$$

where $c_3 = c_3(G,\ p) > 0$ and

$$D_{n,3} = \begin{cases} n^{\frac{1}{p}}, & \textit{if } p \geq 2,\ 0 < \lambda < 2,\ -2 < \gamma < \dfrac{1}{\lambda} + (p-2), \\ & \textit{or } p < 2,\ 1 \leq \lambda < 2,\ -2 < \gamma < \dfrac{1}{\lambda} - (2-p), \\ & \textit{or } p < 2,\ 0 < \lambda < 1,\ -2 < \gamma < \dfrac{p-1}{\lambda}; \\ n^{\frac{\gamma\lambda}{p} + \left(\frac{2}{p}-1\right)\lambda}, & \textit{if } p \geq 2,\ 0 < \lambda < 2,\ \gamma \geq \dfrac{1}{\lambda} + (p-2), \\ & \textit{or } p < 2,\ 1 \leq \lambda < 2,\ \gamma \geq \dfrac{1}{\lambda} - (2-p); \\ n^{\frac{\gamma\lambda}{p} + \left(\frac{2}{p}-1\right)}, & \textit{if } p < 2,\ 0 < \lambda < 1,\ \gamma \geq \dfrac{p-1}{\lambda}. \end{cases}$$

***Corollary* 1.2.** *Let* $p = 2$, $m = 1$. *Assume that a Jordan region* G *has a piecewise Dini-smooth boundary* $L := \partial G$ *and* $h(z)$ *is defined as in* (1.4). *Then, for any* $P_n \in \wp_n$ *we have*

$$|P_n(z)| \leq c_4 \frac{D_{n,4}}{d(z, L_{R_1})} \|P_n\|_{A_2(h,G)} |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{1.10}$$

where $c_4 = c_4(G) > 0$ *and*

$$D_{n,4} < \begin{cases} n^{\frac{1}{2}}, & -2 < \gamma < \dfrac{1}{\lambda}, \quad 0 < \lambda < 2, \\ n^{\frac{\gamma\lambda}{2}}, & \gamma \geq \dfrac{1}{\lambda}, \quad 0 < \lambda < 2. \end{cases}$$

The sharpness of (1.7) – (1.10) can be seen from the following remark.

***Remark* 1.2.** For any $n \in \mathbb{N}$ there exists $P_n^* \in \wp_n$ and $G^* \subset \mathbb{C}$ such that

$$|P_n^*(z)| \geq c_5 \frac{\sqrt{n}}{d(z, L)} \|P_n^*\|_{A_2(G^*)} |\Phi(z)|^{n+1}, \quad z \in F \Subset \Omega^* := C\overline{G}^*, \tag{1.11}$$

where $c_5 = c_5(G^*) > 0$.

2. Some auxiliary results. Let $G \subset \mathbb{C}$ be a finite region bounded by Jordan curve L. The interior level curve can be defined for $t > 0$ as $L_t := \{z\colon\ |\varphi(z)| = t, \text{ if } t < 1\}$, $L_1 \equiv L$, and let $G_t := \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$.

Throughout this paper we also denote by ε, ε_1, $\varepsilon_2, \ldots$ sufficiently small positive constants (in general, different in different relations) that depend on G in general and on parameters inessential for the argument. For the $a > 0$ and $b > 0$, we use the expression "$a \preceq b$" (order inequality), if $a \leq cb$ and the expression "$a \asymp b$" means that $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and$b$) respectively.

Let L is a K-quasiconformal curve. Then [5] there exists a quasiconformal reflection $y(\cdot)$ across L such that $y(G) = \Omega$, $y(\Omega) = G$ and $y(\cdot)$ fixes the points of L. The quasiconformal reflection $y(\cdot)$ can be chosen such that it satisfies the following conditions [5, 6, p. 26]:

$$\begin{gathered} |y(\zeta) - z| \asymp |\zeta - z|, \quad z \in L, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ |y_{\overline{\zeta}}| \asymp |y_\zeta| \asymp 1, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ |y_{\overline{\zeta}}| \asymp |y(\zeta)|^2, \quad |\zeta| < \varepsilon, \quad |y_{\overline{\zeta}}| \asymp |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon}, \end{gathered} \tag{2.1}$$

and for the Jacobian $J_y = |y_z|^2 - |y_{\overline{z}}|^2$ of $y(\cdot)$ the relation $\left|y_{\overline{\zeta}}\right|^2 \le \dfrac{1}{1-\kappa^2} J_y$ is hold, where $\kappa = \dfrac{K^2-1}{K^2+1}$.

Lemma 2.1 [1]. *Let L be a K-quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z\colon |z - z_1| \preceq \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

(a) *the statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent; so are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$;*

(b) *if $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon < 1,\ c > 1$, $0 < r_0 < 1$ are constants, depending on G.

The following lemma is a consequence of the results given in [7, p. 32–36; 11, p. 48].

Lemma 2.2. *Assume that a Jordan region G has a piecewise Dini-smooth boundary $L := \partial G$. Then*

(i) *for any $w \in \Delta_j$, and $0 < \lambda_j < 2$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|^{\lambda_j}$, $|\Psi'(w)| \asymp |w - w_j|^{\lambda_j - 1}$;*

(ii) *for any $w \in \overline{\Delta}\backslash\Delta_j$, $|\Psi(w) - \Psi(w_j)| \asymp |w - w_j|$, $|\Psi'(w)| \asymp 1$.*

Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on curve L located in the positive direction and the weight function $h(z)$ is defined as (1.4).

Lemma 2.3 [3]. *Let L be a K-quasiconformal curve and $h(z)$ be defined in* (1.4). *Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$, and $n = 1, 2, \ldots$*

$$\|P_n\|_{A_p(h,G_R)} \preceq \widetilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \quad p > 0, \tag{2.2}$$

where $\widetilde{R} = 1 + c(R-1)$ and c is independent from n and R.

Lemma 2.4. *Let L be a K-quasiconformal curve, $R = 1 + \dfrac{c}{n}$. Then, for any fixed $\varepsilon \in (0,1)$ there exists a level curve $L_{1+\varepsilon(R-1)}$ such that the following holds for any polynomial $P_n(z) \in \wp_n$, $n \in \mathbb{N}$:*

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{\Phi'}, L_{1+\varepsilon(R-1)}\right)} \preceq n^{\frac{1}{p}} \|P_n\|_{A_p(h,\, G)}, \quad p > 0. \tag{2.3}$$

Proof. Without loss of generality, we can take $\varepsilon = \dfrac{1}{2}$. Then $R_1 := 1 + \dfrac{R-1}{2}$. We have

$$A_n := \int\limits_{L_{R_1}} h(z) \, |P_n(z)|^p \frac{|dz|}{|\Phi'(z)|} =$$

$$= \int\limits_{|w|=R_1} \prod_{j=1}^{m} |\Psi(w) - \Psi(w_j)|^{\gamma_j} \left| P_n(\Psi(w))(\Psi'(w))^{\frac{2}{p}} \right|^p |dw| = \int\limits_{|w|=R_1} |f_{n,p}(w)|^p \, |dw|, \tag{2.4}$$

where

$$f_{n,p}(w) := \prod_{j=1}^{m} (\Psi(w) - \Psi(w_j))^{\frac{\gamma_j}{p}} P_n(\Psi(w)) \left(\Psi'(w)\right)^{\frac{2}{p}}, \quad w \in \Delta.$$

We now split the circle $|t| = R_1$ into n equal parts δ_n with $\operatorname{mes}\delta_n = \dfrac{2\pi R_1}{n}$. Applying the mean-value theorem to the integral A_n, we get

$$A_n = \int\limits_{|t|=R_1} |f_{n,p}(w)|^p \, |dw| = \sum_{k=1}^{n} \int\limits_{\delta_k} |f_{n,p}(w)|^p \, |dw| =$$

$$= \sum_{k=1}^{n} \left| f_{n,p}\left(t'_k\right)\right|^p \operatorname{mes}\delta_k, \ t'_k \in \delta_k.$$

On the other hand, by applying the mean-value estimation,

$$\left| f_{n,p}\left(t'_k\right)\right|^p \leq \frac{1}{\pi\left(\left|t'_k\right| - 1\right)^2} \iint\limits_{\left|\xi - t'_k\right| < \left|t'_k\right| - 1} |f_{n,p}(\xi)|^p \, d\sigma_\xi,$$

we obtain

$$A_n \preceq \sum_{k=1}^{n} \frac{\operatorname{mes}\delta_k}{\pi\left(\left|t'_k\right| - 1\right)^2} \iint\limits_{\left|w - t'_k\right| < \left|t'_k\right| - 1} |f_{n,p}(w)|^p \, d\sigma_w, \quad t'_k \in \delta_k.$$

Taking into account that discs with origin at the points t'_k at most two may be crossing, we have

$$A_n \preceq \frac{\operatorname{mes}\delta_1}{\left(\left|t'_1\right| - 1\right)^2} \iint\limits_{1<|w|<R} |f_{n,p}(w)|^p \, d\sigma_w \preceq n \cdot \iint\limits_{1<|w|<R} |f_{n,p}(w)|^p \, d\sigma_w =$$

$$= n \cdot \iint\limits_{1<|w|<R} \prod_{j=1}^{m} |\Psi(w) - \Psi(w_j)|^{\gamma_j} \, |P_n(\Psi(w))|^p \left|\Psi'(w)\right|^2 d\sigma_w \leq$$

$$\leq n \cdot \iint\limits_{1<|w|<R} \prod_{j=1}^{m} |\Psi(w) - \Psi(w_j)|^{\gamma_j} \, |P_n(\Psi(w))|^p \left|\Psi'(w)\right|^2 d\sigma_w \preceq$$

$$\preceq n \cdot \iint\limits_{G_R \setminus G} h(z) \, |P_n(z)|^p \, d\sigma_z.$$

According to (2.3), for A_n, we get

$$A_n \preceq n \cdot \iint\limits_{G_R \setminus G} h(z) \, |P_n(z)|^p \, d\sigma_z \preceq n \cdot \|P_n\|^p_{A_p(h,G)}. \tag{2.5}$$

Combining (2.4), (2.5), we prove estimate (2.3).

Lemma 2.4 is proved.

3. Proofs. ***Proof of Theorem* 1.1.** Let for $z \in \Omega$:

$$T_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}. \tag{3.1}$$

For any $R > 1$ and $R_1 := 1 + \dfrac{R-1}{2}$, the Cauchy integral representation for the region Ω_{R_1} gives

$$T_n(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} T_n(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in \Omega_{R_1}.$$

Since $|\Phi(\zeta)| > 1$, for $\zeta \in L_{R_1}$, then we have

$$|P_n(z)| = \frac{|\Phi(z)|^{n+1}}{2\pi} \int\limits_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} \leq \frac{|\Phi(z)|^{n+1}}{2\pi d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)| \, |d\zeta| . \tag{3.2}$$

Let's

$$A_n := \int\limits_{L_{R_1}} |P_n(\zeta)| \, |d\zeta| = \sum_{i=1}^{m} \int\limits_{L^i_{R_1}} |P_n(\zeta)| \, |d\zeta| =$$

$$= \sum_{i=1}^{m} \int\limits_{F^i_{R_1}} |P_n(\Psi(\tau))| \left|\Psi'(\tau)\right| |d\tau| , \tag{3.3}$$

where $F^i_{R_1} := \Phi(L^i_{R_1}) = \Delta'_i \cap \{\tau \colon |\tau| = R_1\}$, $i = \overline{1, m}$. Replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the by multiplier $\prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}} \left|\Psi'(\tau)\right|^{\frac{2}{p}}$, after then, according to the H$\acute{}$older inequality, we obtain

$$A_n = \sum_{i=1}^{m} \int\limits_{F^i_{R_1}} \frac{\prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}} \left| P_n(\Psi(\tau)) \left(\Psi'(\tau)\right)^{\frac{2}{p}} \right| \left|\Psi'(\tau)\right|^{1-\frac{2}{p}}}{\prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}}} |d\tau| \leq$$

$$\leq \sum_{i=1}^{m} \left(\int\limits_{F^i_{R_1}} \prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\gamma_j} |P_n(\Psi(\tau))|^p \left|\Psi'(\tau)\right|^2 |d\tau| \right)^{\frac{1}{p}} \times$$

$$\times \left(\int\limits_{F^i_{R_1}} \left(\frac{\left|\Psi'(\tau)\right|^{1-\frac{2}{p}}}{\prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}}} \right)^q |d\tau| \right)^{\frac{1}{q}} \leq$$

$$\leq \sum_{i=1}^{m} A^i_n, \tag{3.4}$$

where

$$A^i_n := \left(\int\limits_{F^i_{R_1}} |f_{n,p}(\tau)|^p \, |d\tau| \right)^{\frac{1}{p}} \left(\int\limits_{F^i_{R_1}} \frac{\left|\Psi'(\tau)\right|^{2-q}}{\prod_{j=1}^{m} |\Psi(\tau) - \Psi(w_j)|^{\gamma_j(q-1)}} |d\tau| \right)^{\frac{1}{q}} =: J^i_{n,1} \cdot J^i_{n,2},$$

$$f_{n,p}(\tau):=\prod_{j=1}^{m}\left(\Psi(\tau)-\Psi(w_j)\right)^{\frac{\gamma_j}{p}}P_n\left(\Psi(\tau)\right)\left(\Psi'(\tau)\right)^{\frac{2}{p}},\ |\tau|=R_1.$$

Applying to Lemma 2.4, we get

$$J_{n,1}^{i}\preceq n^{\frac{1}{p}}\left\|P_n\right\|_{A_p(h,\ G)},\quad i=\overline{1,m}. \tag{3.5}$$

For the estimation the integral $J_{n,2}^{i}$

$$\left(J_{n,2}^{i}\right)^{q}:=\int\limits_{F_{R_1}^{i}}\frac{\left|\Psi'(\tau)\right|^{2-q}}{\prod_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j(q-1)}}\left|d\tau\right|\asymp\int\limits_{F_{R_1}^{i}}\frac{\left|\Psi'(\tau)\right|^{2-q}}{\left|\Psi(\tau)-\Psi(w_i)\right|^{\gamma_i(q-1)}}\left|d\tau\right|, \tag{3.6}$$

since the points $w_j:=\Phi(z_j)$ are distinct.

For simplicity, we may take $i=1$, $J_{n,1}^{i}=:J_1$; $J_{n,2}^{1}=:J_2$. Let us set: $w_1:=\Phi(z_1)$,

$$E_{R_1}^{11}:=\left\{\tau\colon\tau\in F_{R_1}^{1},\ |\tau-w_1|<c_1(R_1-1)\right\},$$
$$E_{R_1}^{12}:=\left\{\tau\colon\tau\in F_{R_1}^{1},\ c_1(R_1-1)\ \le|\tau-w_1|<c_2\right\},$$
$$E_{R_1}^{13}:=\left\{\tau\colon\tau\in F_{R_1}^{1},\ |\tau-w_1|\ge c_2\right\},\qquad j=1,2,3,$$
$$F_{R_1}^{1}=\bigcup_{k=1}^{3}E_{R_1}^{1k}.$$

Taking into consideration these designations, (3.6) can be written as

$$J_2=J_2(E_{R_1}^{11})+J_2(E_{R_1}^{12})+J_2(E_{R_1}^{13})=:J_2^1+J_2^2+J_2^3 \tag{3.7}$$

and, consequently,

$$A_n^1=:J_1\cdot(J_2^1+J_2^2+J_2^3)=:A_{n,1}^1+A_{n,2}^1+A_{n,3}^1, \tag{3.8}$$

where

$$A_{n,k}^{1}:=n^{\frac{1}{p}}\left\|P_n\right\|_{A_p(h,G)}\int\limits_{E_{R_1}^{1k}}\frac{\left|\Psi'(\tau)\right|^{2-q}}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma_1(q-1)}}\left|d\tau\right|,\qquad k=1,2,3. \tag{3.9}$$

Given the possible values q ($q>2$ and $q<2$), λ_1 ($0<\lambda_1<1$ and $1<\lambda_1<2$), and γ_1 ($-2<\gamma_1<0$ and $\gamma_1\ge 0$), we will consider separately the cases.

Case 1. Let $1<q<2$ ($p>2$). Then

$$\left(J_2^1\right)^q\asymp\int\limits_{E_{R_1}^{11}}\frac{\left|\Psi'(\tau)\right|^{2-q}}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma_1(q-1)}}\left|d\tau\right|.$$

1.1. Let $1\le\lambda_1<2$.

1.1.1. If $\gamma_1\ge 0$, applying Lemma 2.2 to (3.9), we get

$$\left(J_2^1\right)^q\preceq\int\limits_{E_{R_1}^{11}}\frac{|\tau-w_1|^{(\lambda_1-1)(2-q)}}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}}\left|d\tau\right|\preceq\left(\frac{1}{n}\right)^{(\lambda_1-1)(2-q)}\int\limits_{E_{R_1}^{11}}\frac{|d\tau|}{(|\tau|-1)^{\gamma_1\lambda_1(q-1)}}\preceq$$

$$\preceq n^{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1,$$

$$J_2^1 \preceq n^{\frac{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1,$$

$$\left(J_2^2\right)^q \preceq \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(\lambda_1-1)(2-q)}}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}}\,|d\tau| \preceq \left(\frac{1}{n}\right)^{(\lambda_1-1)(2-q)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{(|\tau|-1)^{\gamma_1\lambda_1(q-1)}} \preceq$$

$$\preceq n^{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1,$$

$$J_2^2 \preceq n^{\frac{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1.$$

In this case, from (3.7) and (3.8), we obtain

$$A_{n,1}^1 \preceq n^{\frac{1}{p}+\frac{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}{q}} \|P_n\|_{A_p(h,G)} =$$

$$= n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \gamma_1\lambda_1(q-1)>1, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1, \tag{3.10}$$

$$A_{n,2}^1 \preceq n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1. \tag{3.11}$$

1.1.2. If $\gamma_1<0$, analogously we have

$$\left(J_2^1\right)^q \asymp \int\limits_{E_{R_1}^{11}} \frac{|\tau-w_1|^{(\lambda_1-1)(2-q)}}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}}\,|d\tau| \preceq \int\limits_{E_{R_1}^{11}} |\tau-w_1|^{(\lambda_1-1)(2-q)+(-\gamma_1)\lambda_1(q-1)}\,|d\tau| \preceq$$

$$\preceq \left(\frac{1}{n}\right)^{(\lambda_1-1)(2-q)+(-\gamma_1)\lambda_1(q-1)} \cdot \operatorname{mes} E_{R_1}^{11},$$

$$J_2^1 \preceq n^{\frac{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}{q}},$$

$$\left(J_2^2\right)^q \asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(\lambda_1-1)(2-q)}}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}}\,|d\tau| \preceq \int\limits_{E_{R_1}^{12}} |\tau-w_1|^{(\lambda_1-1)(2-q)+(-\gamma_1)\lambda_1(q-1)}\,|d\tau| \preceq$$

$$\preceq \int\limits_{E_{R_1}^{12}} |d\tau| \preceq 1.$$

Also

$$\begin{gathered} A_{n,1}^1 \preceq n^{\frac{1}{p}+\frac{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}{q}} \|P_n\|_{A_p(h,G)} = n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \\ A_{n,2}^1 \preceq n^{\frac{1}{p}} \|P_n\|_{A_p(h,G)}. \end{gathered} \tag{3.12}$$

1.2. Let $0<\lambda_1<1$.

1.2.1. If $\gamma_1 \geq 0$, applying Lemma 2.2 to (3.9), we get

$$\left(J_2^1\right)^q \preceq \int\limits_{E_{R_1}^{11}} \frac{|\tau - w_1|^{(\lambda_1-1)(2-q)}}{|\tau - w_1|^{\gamma_1\lambda_1(q-1)}} |d\tau| \preceq \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{(|\tau|-1)^{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)}} \preceq$$

$$\preceq n^{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q) > 1,$$

$$J_2^1 \preceq n^{\frac{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q) > 1,$$

$$\left(J_2^2\right)^q \preceq \int\limits_{E_{R_1}^{12}} \frac{|\tau - w_1|^{(\lambda_1-1)(2-q)}}{|\tau - w_1|^{\gamma_1\lambda_1(q-1)}} |d\tau| \preceq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{(|\tau|-1)^{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)}} \preceq$$

$$\preceq n^{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q) > 1,$$

$$J_2^2 \preceq n^{\frac{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q) > 1.$$

In this case, from (3.8) for $A_{n,1}^1$ and $A_{n,2}^1$, we obtain

$$A_{n,1}^1 \preceq n^{\frac{1}{p}+\frac{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}{q}} \|P_n\|_{A_p(h,G)} =$$

$$= n^{\frac{\gamma_1\lambda_1}{p}-\left(1-\frac{2}{p}\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q) > 1, \tag{3.13}$$

$$A_{n,2}^1 \preceq n^{\frac{1}{p}+\frac{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}{q}} \|P_n\|_{A_p(h,G)} =$$

$$= n^{\frac{\gamma_1\lambda_1}{p}-\left(1-\frac{2}{p}\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q) > 1. \tag{3.14}$$

1.2.2. If $\gamma_1 < 0$, analogously we have

$$\left(J_2^1\right)^q \asymp \int\limits_{E_{R_1}^{11}} \frac{|\tau - w_1|^{(-\gamma_1)\lambda_1(q-1)}}{|\tau - w_1|^{(1-\lambda_1)(2-q)}} |d\tau| \preceq$$

$$\preceq \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1(q-1)} \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{(1-\lambda_1)(2-q)}} \preceq \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1(q-1)} \preceq 1,$$

$$\left(J_2^2\right)^q \asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau - w_1|^{(-\gamma_1)\lambda_1(q-1)}}{|\tau - w_1|^{(1-\lambda_1)(2-q)}} |d\tau| \preceq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_1|^{(1-\lambda_1)(2-q)}} \preceq 1.$$

Also

$$\begin{aligned} A_{n,1}^1 &\preceq n^{\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \\ A_{n,2}^1 &\preceq n^{\frac{1}{p}} \|P_n\|_{A_p(h,G)}. \end{aligned} \tag{3.15}$$

Case 2. Let $q > 2(p < 2)$. Then, $2 - q < 0$ and, so

$$\left(J_2^1\right)^q \asymp \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\zeta - z_1|^{\gamma_1(q-1)}}. \tag{3.16}$$

2.1. Let $1 \leq \lambda_1 < 2$.

2.1.1. If $\gamma_1 \geq 0$, applying Lemma 2.2 to (3.16), we obtain

$$\left(J_2^1\right)^q \preceq \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda_1-1)(q-2)}\,|\tau - w_1|^{\gamma_1\lambda_1(q-1)}} \preceq \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{(|\tau|-1)^{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)}} \preceq$$

$$\preceq n^{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2) > 1,$$

$$J_2^1 \preceq n^{\frac{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2) > 1,$$

$$\left(J_2^2\right)^q \preceq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda_1-1)(q-2)}\,|\tau - w_1|^{\gamma_1\lambda_1(q-1)}} \preceq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{(|\tau|-1)^{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)}} \preceq$$

$$\preceq n^{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2) > 1,$$

$$J_2^2 \preceq n^{\frac{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2) > 1.$$

In this case, from (3.8), we have

$$A_{n,1}^1 \preceq n^{\frac{1}{p}+\frac{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}{q}} \|P_n\|_{A_p(h,G)} =$$

$$= n^{\frac{\gamma_1\lambda_1}{p}+\left(\frac{2}{p}-1\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2) > 1, \tag{3.17}$$

$$A_{n,2}^1 \preceq n^{\frac{1}{p}+\frac{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}{q}} \|P_n\|_{A_p(h,G)} =$$

$$= n^{\frac{\gamma_1\lambda_1}{p}+\left(\frac{2}{p}-1\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2) > 1. \tag{3.18}$$

2.1.2. If $\gamma_1 < 0$, analogously we have

$$\left(J_2^1\right)^q \asymp \int\limits_{E_{R_1}^{11}} \frac{|\tau - w_1|^{(-\gamma_1)\lambda_1(q-1)}}{|\tau - w_1|^{(\lambda_1-1)(q-2)}}\,|d\tau| \preceq \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1(q-1)} \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda_1-1)(q-2)}} \preceq$$

$$\preceq n^{(\lambda_1-1)(q-2)+\gamma_1\lambda_1(q-1)-1}, \quad \text{if} \quad (\lambda_1-1)(q-2) > 1,$$

$$J_2^1 \preceq n^{\frac{(\lambda_1-1)(q-2)+\gamma_1\lambda_1(q-1)-1}{q}}, \quad \text{if} \quad (\lambda_1-1)(q-2) > 1,$$

$$\left(J_2^2\right)^q \asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau - w_1|^{(-\gamma_1)\lambda_1(q-1)}}{|\tau - w_1|^{(\lambda_1-1)(q-2)}}\,|d\tau| \preceq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda_1-1)(q-2)}} \preceq$$

$$\preceq n^{(\lambda_1-1)(q-2)-1}, \quad \text{if} \quad (\lambda_1-1)(q-2) > 1,$$

$$J_2^2 \preceq n^{\frac{(\lambda_1-1)(q-2)-1}{q}}, \quad \text{if} \quad (\lambda_1-1)(q-2)>1.$$

So,

$$A_{n,1}^1 \preceq n^{\frac{1}{p}+\frac{(\lambda_1-1)(q-2)+\gamma_1\lambda_1(q-1)-1}{q}} \|P_n\|_{A_p(h,G)} =$$

$$= n^{\left(\frac{2}{p}-1\right)\lambda_1+\frac{\gamma_1\lambda_1}{p}} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad (\lambda_1-1)(q-2)>1, \tag{3.19}$$

$$A_{n,2}^1 \preceq n^{\frac{1}{p}+\frac{(\lambda_1-1)(q-2)-1}{q}} \|P_n\|_{A_p(h,G)} = n^{\left(\frac{2}{p}-1\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad (\lambda_1-1)(q-2)>1. \tag{3.20}$$

2.2. Let $0<\lambda_1<1$.

2.2.1. If $\gamma_1 \geq 0$, applying Lemma 2.2 to (3.16) we obtain

$$\left(J_2^1\right)^q \preceq \int\limits_{E_{R_1}^{11}} \frac{|\tau-w_1|^{(1-\lambda_1)(q-2)}\,|d\tau|}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}} \preceq \left(\frac{1}{n}\right)^{(1-\lambda_1)(q-2)} \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}} \preceq$$

$$\preceq n^{-(1-\lambda_1)(q-2)+\gamma_1\lambda_1(q-1)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1,$$

$$J_2^1 \preceq n^{\frac{-(1-\lambda_1)(q-2)+\gamma_1\lambda_1(q-1)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1,$$

$$\left(J_2^2\right)^q \preceq \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(1-\lambda_1)(q-2)}}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}}\,|d\tau| \preceq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}} \preceq$$

$$\preceq n^{\gamma_1\lambda_1(q-1)-1}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1,$$

$$J_2^2 \preceq n^{\frac{\gamma_1\lambda_1(q-1)-1}{q}}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1.$$

In this case, from (3.8), we have

$$A_{n,1}^1 \preceq n^{\frac{1}{p}+\frac{-(1-\lambda_1)(q-2)+\gamma_1\lambda_1(q-1)-1}{q}} \|P_n\|_{A_p(h,G)} =$$

$$= n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1, \tag{3.21}$$

$$A_{n,2}^1 \preceq n^{\frac{1}{p}+\frac{\gamma_1\lambda_1(q-1)-1}{q}} \|P_n\|_{A_p(h,G)} = n^{\left(\frac{2}{p}-1\right)+\frac{\gamma_1\lambda_1}{p}} \|P_n\|_{A_p(h,G)}, \quad \text{if} \quad \gamma_1\lambda_1(q-1)>1. \tag{3.22}$$

2.2.2. If $\gamma_1<0$, analogously we have

$$\left(J_2^1\right)^q \asymp \int\limits_{E_{R_1}^{11}} \frac{|\tau-w_1|^{(1-\lambda_1)(q-2)}\,|d\tau|}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}} \preceq \int\limits_{E_{R_1}^{11}} |\tau-w_1|^{(1-\lambda_1)(q-2)+(-\gamma_1)\lambda_1(q-1)}\,|d\tau| \preceq$$

$$\preceq \left(\frac{1}{n}\right)^{(1-\lambda_1)(q-2)+(-\gamma_1)\lambda_1(q-1)} \cdot \operatorname{mes} E_{R_1}^1,$$

$$J_2^2 \preceq n^{\frac{-(1-\lambda_1)(q-2)+\gamma_1\lambda_1(q-1)-1}{q}} \preceq 1,$$

$$\left(J_2^2\right)^q \asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau - w_1|^{(1-\lambda_1)(q-2)}|d\tau|}{|\tau - w_1|^{\gamma_1\lambda_1(q-1)}} \preceq \int\limits_{E_{R_1}^{12}} |\tau - w_1|^{(1-\lambda_1)(q-2)+(-\gamma_1)\lambda_1(q-1)}|d\tau| \preceq \int\limits_{E_{R_1}^{12}} |d\tau| \preceq 1, \tag{3.23}$$

$$A_{n,1}^1 \preceq n^{\frac{1}{p}} \|P_n\|_{A_p(h,G)},$$

$$A_{n,2}^1 \preceq n^{\frac{1}{p}} \|P_n\|_{A_p(h,G)}.$$

To estimate $A_{n,3}^1$, note that for each $\zeta \in E_{R_1}^3$ $|\zeta - z_1| \asymp 1$, and so

$$J_2^3 \preceq 1,$$
$$A_{n,3}^1 \preceq n^{\frac{1}{p}} \|P_n\|_{A_p(h,G)}. \tag{3.24}$$

Therefore, combining the (3.8) – (3.24), for A_n, we get

$$A_n^1 = \sum_{k=1}^{3} A_{n,k}^1 \preceq \|P_n\|_{A_p(h,G)} \times$$

$$\times \begin{cases} n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} + n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} + n^{\frac{1}{p}}, & p>2, \quad \lambda_1 \geq 1, \quad \gamma_1\lambda_1(q-1) > 1, \\ n^{\frac{\gamma_1\lambda_1}{p}-\left(1-\frac{2}{p}\right)\lambda_1} + n^{\frac{\gamma_1\lambda_1}{p}-\left(1-\frac{2}{p}\right)\lambda_1} + n^{\frac{1}{p}}, & p>2, \quad \lambda_1 < 1, \quad \gamma_1\lambda_1(q-1) > 1, \\ n^{\frac{\gamma_1\lambda_1}{p}+\left(\frac{2}{p}-1\right)\lambda_1} + n^{\frac{\gamma_1\lambda_1}{p}+\left(\frac{2}{p}-1\right)\lambda_1} + n^{\frac{1}{p}}, & p<2, \quad \lambda_1 \geq 1, \quad \gamma_1\lambda_1(q-1)+ \\ & \qquad\qquad\qquad +(\lambda_1-1)(q-2) > 1, \\ n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} + n^{\left(\frac{2}{p}-1\right)+\frac{\gamma_1\lambda_1}{p}} + n^{\frac{1}{p}}, & p<2, \quad \lambda_1 < 1, \quad \gamma_1\lambda_1(q-1) > 1, \end{cases}$$

if $\gamma_1 \geq 0$, and

$$A_n^1 = \sum_{k=1}^{3} A_{n,k}^1 \preceq \|P_n\|_{A_p(h,G)} \times$$

$$\times \begin{cases} n^{\left(\frac{2+\gamma_1}{p}-1\right)\lambda_1} + n^{\frac{1}{p}} + n^{\frac{1}{p}}, & p>2, \quad \lambda_1 \geq 1, \quad \gamma_1 < 0, \\ n^{\frac{1}{p}} + n^{\frac{1}{p}} + n^{\frac{1}{p}}, & p>2, \quad \lambda_1 < 1, \quad \gamma_1 < 0, \\ n^{\left(\frac{2}{p}-1\right)\lambda_1+\frac{\gamma_1\lambda_1}{p}} + n^{\left(\frac{2}{p}-1\right)\lambda_1} + n^{\frac{1}{p}}, & p<2, \quad \lambda_1 \geq 1, \quad (\lambda_1-1)(q-2) > 1, \\ n^{\left(\frac{2}{p}-1\right)+\frac{\gamma_1\lambda_1}{p}} + n^{\frac{1}{p}} + n^{\frac{1}{p}}, & p<2, \quad \lambda_1 < 1, \end{cases}$$

if $\gamma_1 < 0$.

Hence,

$$A_n^1 \preceq \|P_n\|_{A_p(h,G)} \begin{cases} n^{\frac{1}{p}}, & p>2, \quad \lambda_1 \geq 1, \quad 0 \leq \gamma_1\lambda_1 < 1+\lambda_1(p-2), \\ n^{\frac{\gamma_1\lambda_1}{p}-\left(1-\frac{2}{p}\right)\lambda_1}, & p>2, \quad \lambda_1 \geq 1, \quad \gamma_1\lambda_1 \geq 1+\lambda_1(p-2), \\ n^{\frac{1}{p}}, & p>2, \quad \lambda_1 < 1, \quad 0 \leq \gamma_1\lambda_1 < 1+\lambda_1(p-2), \\ n^{\frac{\gamma_1\lambda_1}{p}-\left(1-\frac{2}{p}\right)\lambda_1}, & p>2, \quad \lambda_1 < 1, \quad \gamma_1\lambda_1 \geq 1+\lambda_1(p-2), \\ n^{\frac{1}{p}}, & p<2, \quad \lambda_1 > 1, \quad 0 \leq \gamma_1\lambda_1 < 1-\lambda_1(2-p), \\ n^{\frac{\gamma_1\lambda_1}{p}+\left(\frac{2}{p}-1\right)\lambda_1}, & p<2, \quad \lambda_1 > 1, \quad \gamma_1\lambda_1 \geq 1-\lambda_1(2-p), \\ n^{\frac{1}{p}}, & p<2, \quad \lambda_1 < 1, \quad 0 \leq \gamma_1\lambda_1 < p-1, \\ n^{\frac{\gamma_1\lambda_1}{p}+\left(\frac{2}{p}-1\right)}, & p<2, \quad \lambda_1 < 1, \quad \gamma_1\lambda_1 \geq p-1, \end{cases} \tag{3.25}$$

if $\gamma_1 \geq 0$, and

$$A_n \preceq \|P_n\|_{A_p(h,G)} \begin{cases} n^{\frac{1}{p}}, & p>2, \quad \lambda_1 \geq 1, \quad \gamma_1 < 0, \\ n^{\frac{1}{p}}, & p>2, \quad \lambda_1 < 1, \quad \gamma_1 < 0, \\ n^{\frac{1}{p}}, & p<2, \quad \lambda_1 \geq 1, \quad \gamma_1 < 0, \\ n^{\frac{1}{p}}, & p<2, \quad \lambda_1 < 1, \quad \gamma_1 < 0, \end{cases} \tag{3.26}$$

if $\gamma_1 < 0$. Therefore, taking into account also the case $p = 2$, and summing over all $j = \overline{1, m}$, from (3.8) and (3.9) we get

$$A_n \leq \sum_{j=1}^{m} A_n^j \preceq \|P_n\|_{A_p(h,G)} \times$$

$$\times \begin{cases} n^{\frac{1}{p}}, & p \geq 2, \quad 0<\lambda_j<2, \quad -2<\gamma_j<\dfrac{1}{\lambda_j}+(p-2), \\ \sum\limits_{j=1}^{m} n^{\frac{\gamma_j\lambda_j}{p}-\left(1-\frac{2}{p}\right)\lambda_j}, & p \geq 2, \quad 0<\lambda_j<2, \quad \gamma_j \geq \dfrac{1}{\lambda_j}+(p-2), \\ n^{\frac{1}{p}}, & p<2, \quad 1 \leq \lambda_j<2, \quad -2<\gamma_j<\dfrac{1}{\lambda_j}-(2-p), \\ \sum\limits_{j=1}^{m} n^{\frac{\gamma_j\lambda_j}{p}+\left(\frac{2}{p}-1\right)\lambda_j}, & p<2, \quad 1 \leq \lambda_j<2, \quad \gamma_j \geq \dfrac{1}{\lambda_j}-(2-p), \\ n^{\frac{1}{p}}, & p<2, \quad 0<\lambda_j<1, \quad -2<\gamma_j<\dfrac{p-1}{\lambda_j}, \\ \sum\limits_{j=1}^{m} n^{\frac{\gamma_j\lambda_j}{p}+\left(\frac{2}{p}-1\right)}, & p<2, \quad 0<\lambda_j<1, \quad \gamma_j \geq \dfrac{p-1}{\lambda_j}. \end{cases}$$

Also

$$A_n \preceq \|P_n\|_{A_p(h,G)} \times$$

$$
\times \begin{cases}
n^{\frac{1}{p}}, & p \ge 2, \quad 0 < \lambda_j < 2, \quad -2 < \gamma_j < \dfrac{1}{\lambda_1} + (p-2) \quad \text{for all} \quad j = \overline{1,m}; \\
n^{\frac{\gamma^*\lambda^*}{p} - \left(1-\frac{2}{p}\right)\lambda_*}, & p \ge 2, \quad 0 < \lambda_j < 2, \quad \gamma_j \ge \dfrac{1}{\lambda_m} + (p-2) \quad \text{for all} \quad j = \overline{1,m}; \\
n^{\frac{\gamma_k^*\lambda_k^*}{p} - \left(1-\frac{2}{p}\right)\lambda_{k*}}, & p \ge 2, \quad 0 < \lambda_j < 2, \quad \mu_k \le \gamma_j < \mu_{k+1} \quad \text{for all} \quad k = \overline{1,m-1} \\
 & \qquad \text{and} \quad j = \overline{1,m}; \\
n^{\frac{1}{p}}, & p < 2, \quad 1 \le \lambda_j < 2, \quad -2 < \gamma_j < \dfrac{1}{\lambda_1} - (2-p) \quad \text{for all} \quad j = \overline{1,m}; \\
n^{\frac{\gamma^*\lambda^*}{p} + \left(\frac{2}{p}-1\right)\lambda^*}, & p < 2, \quad 1 \le \lambda_j < 2, \quad \gamma_j \ge \dfrac{1}{\lambda m} - (2-p) \quad \text{for all} \quad j = \overline{1,m}; \\
n^{\frac{\gamma_k^*\lambda_k^*}{p} + \left(\frac{2}{p}-1\right)\lambda_k^*}, & p < 2, \quad 1 \le \lambda_j < 2, \quad \eta_k \le \gamma_j < \eta_{k+1} \quad \text{for all} \quad k = \overline{1,m-1} \\
 & \qquad \text{and} \quad j = \overline{1,m}; \\
n^{\frac{1}{p}}, & p < 2, \quad 0 < \lambda_j < 1, \quad -2 < \gamma_j < \dfrac{p-1}{\lambda_1} \quad \text{for all} \quad j = \overline{1,m}; \\
n^{\frac{\gamma^*\lambda^*}{p} + \left(\frac{2}{p}-1\right)}, & p < 2, \quad 0 < \lambda_j < 1, \quad \gamma_j \ge \dfrac{p-1}{\lambda_m} \quad \text{for all} \quad j = \overline{1,m}; \\
n^{\frac{\gamma_k^*\lambda_k^*}{p} + \left(\frac{2}{p}-1\right)}, & p < 2, \quad 0 < \lambda_j < 1, \quad \omega_k \le \gamma_j < \omega_{k+1} \quad \text{for all} \quad k = \overline{1,m-1} \\
 & \qquad \text{and} \quad j = \overline{1,m}.
\end{cases}
$$

Combining the formulas (3.2), (3.3), (3.5), (3.8), (3.25) and (3.26) we complete the proof of Theorem 1.1.

***Proof of Corollary* 1.2.** Let $p = 2$, $m = 1$. First of all we note that, from (3.5), (3.8) and (3.9), for $\gamma_1 = 0$ we can obtain easily

$$A_n \preceq n^{\frac{1}{2}} \left\| P_n \right\|_{A_2(G)}. \tag{3.27}$$

1. Let $\gamma_1 > 0$. Then

$$
\begin{aligned}
\left(J_2^1\right)^2 &\preceq \int\limits_{E_{R_1}^1} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1\lambda_1}} \preceq n^{\gamma_1\lambda_1 - 1}, \qquad \gamma_1\lambda_1 > 1, \quad 0 < \lambda_1 < 2, \\
\left(J_2^2\right)^2 &\preceq \int\limits_{E_{R_1}^2} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1\lambda_1}} \preceq n^{\gamma_1\lambda_1 - 1}, \qquad \gamma_1\lambda_1 > 1, \quad 0 < \lambda_1 < 2.
\end{aligned} \tag{3.28}
$$

2. Let now $-2 < \gamma_1 < 0$. Then

$$
\begin{aligned}
\left(J_2^1\right)^2 &\preceq \int\limits_{E_{R_1}^1} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1\lambda_1}} \preceq 1, \quad 0 < \lambda_1 < 2, \\
\left(J_2^2\right)^2 &\preceq \int\limits_{E_{R_1}^1} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1\lambda_1}} \preceq 1, \quad 0 < \lambda_1 < 2,
\end{aligned} \tag{3.29}
$$

$$J_2^2 \preceq 1, \quad 0 < \lambda_1 < 2. \tag{3.30}$$

On the other hand, in case of all $\gamma_1 > -2$ and $0 < \lambda_1 < 2$ we have

$$J_2^3 \preceq 1. \tag{3.31}$$

Therefore, combining (3.2)–(3.5), (3.8), (3.27)–(3.31) , we get

$$A_n \preceq \|P_n\|_{A_2(h,G)} \left\{ n^{\frac{\gamma_1\lambda_1}{2}} + n^{\frac{1}{2}}, \quad \gamma_1\lambda_1 > 1, \preceq \right.$$

$$\preceq \|P_n\|_{A_2(h,G)} \begin{cases} n^{\frac{1}{2}}, & -2 < \gamma_1 < \dfrac{1}{\lambda_1}, \quad 0 < \lambda_1 < 2, \\ n^{\frac{\gamma_1\lambda_1}{2}}, & \gamma_1 \geq \dfrac{1}{\lambda_1}, \quad 0 < \lambda_1 < 2. \end{cases}$$

Corollary 1.2 is proved.

***Proof of Remark* 1.2.** Let the region G bounded by Dini-smooth curve $L = \partial G$. According to the "three-point" criterion [10, p.100], the curve L is quasiconformal. Let $\{K_n(z)\}$, $\deg K_n = n$, denote of the system of Bergman polynomials for region G, i.e., system of polynomials $\{K_n(z)\}$, $K_n(z) := \alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_0$, $\alpha_n > 0$ and satisfying the conditions

$$\iint\limits_G K_n(z)\overline{K_m(z)}d\sigma_z = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker symbol. According to [2], for arbitrary quasidiscs, we have

$$K_n(z) = \alpha_n \rho^{n+1} \Phi^n(z) \Phi'(z) A_n(z), \quad z \in F \Subset \Omega,$$

where

$$\sqrt{\frac{n+1}{\pi}} \leq \alpha_n \rho^{n+1} \leq c_1 \sqrt{\frac{n+1}{\pi}},$$

for some $c_1 = c_1(G) > 1$ and

$$c_2 \leq |A_n(z)| \leq 1 + \frac{c_3}{\sqrt{|\Phi(z)| - 1}},$$

for some $c_i = c_i(G) > 0$, $i = 2, 3$. Therefore, since $\|K_n\|_{A_2(G)} = 1$, we have

$$|K_n(z)| \geq c_2 \sqrt{\frac{n+1}{\pi}} \, |\Phi(z)|^n \, \frac{|\Phi(z)| - 1}{d(z,L)} \geq$$

$$\geq c_3 \frac{\sqrt{n}}{d(z,L)} \, |\Phi(z)|^{n+1} \left(1 - \frac{1}{|\Phi(z)|} \right) \geq$$

$$\geq c_4 \frac{\sqrt{n}}{d(z,L)} \, |\Phi(z)|^{n+1} \, \|K_n\|_{A_2(G)} \, .$$

Remark 1.2 is proved.

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