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NOTE ON SEMIALGEBRAICALLY PROPER MAPS *

ПРО НАПІВАЛГЕБРАЇЧНО ВЛАСНІ ВІДОБРАЖЕННЯ

In this paper we prove that a semialgebraic map is semialgebraically proper if and only if it is proper. As an application of this, we compare the semialgebraically proper actions with the proper actions in the sense of Palais.

Доведено, що напівалгебраїчне відображення є напівалгебраїчно власним тоді і тільки тоді, коли воно є власним. Як застосування цього факту, ми порівнюємо напівалгебраїчно власні дії з власними діями в сенсі Пале.

1. Introduction. A semialgebraic space is an object obtained by pasting finitely many semialgebraic sets together along open (or closed) semialgebraic subsets. For this reason, the topologies of semialgebraic spaces are of no interest, so we will only treat the semialgebraic sets over the real numbers in this paper.

A *semialgebraic set* is a subset of some $\mathbb{R}^n$ defined by finite number of polynomial equations and inequalities. Throughout this paper we consider the semialgebraic sets in $\mathbb{R}^n$ equipped with the subspace topology induced by the usual topology of $\mathbb{R}^n$. A continuous map $f\colon X \to Y$ between semialgebraic sets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ is called *semialgebraic* if its graph is a semialgebraic set in $\mathbb{R}^{m+n}$. Usually, semialgebraic map just a map, not necessarily continuous, whose graph is semialgebraic. However, since all semialgebraic maps occurring in this paper are continuous, for simplicity, we will assume that all semialgebraic maps are continuous.

The purpose of this paper is to find the equivalence conditions for semialgebraically proper maps. Recall that a map is called proper if the preimage of any compact set is compact. Similarly, a semialgebraic map $f\colon X \to Y$ is called *semialgebraically proper* if the preimage $f^{-1}(C)$ is compact for every compact and semialgebraic subset C of Y. Since C should be semialgebraic in the definition, this notion is weaker than the condition that f is proper. However, in Section 2, we prove that a semialgebraic map is semialgebraically proper if and only if it is proper. Note that this notion of semialgebraically proper is slightly different from that of Delfs and Knebusch in [3].

As an application of the above result, we also discuss semialgebraically proper actions of semialgebraic groups on semialgebraic sets. Let M be a semialgebraic set and let G be a semialgebraic group. We say M is a semialgebraic G-set if the action $\theta\colon G \times M \to M$ is semialgebraic. A semialgebraic action of G on M is called *semialgebraically proper* if the map

$$\vartheta_*\colon G \times M \to M \times M, \qquad (g, x) \mapsto (\theta(g, x), x)$$

is semialgebraically proper. In Section 3 we compare the semialgebraically proper actions with the proper actions in the sense of Palais [6].

2. Semialgebraically proper maps. We first gather some properties concerning semialgebraic sets and maps without proofs. For the details, we refer the reader to [1] and [4].

The class of semialgebraic sets in $\mathbb{R}^n$ is the smallest collection of subsets containing all subsets of the form $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for a real valued polynomial $p(x) = p(x_1, \ldots, x_n)$, which is stable under finite union, finite intersection and complement.

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Proposition 2.1. (1) *Every semialgebraic set has a finite number of path connected components, which are also semialgebraic.*

(2) *Let X be a semialgebraic set. If A is a semialgebraic subset of X, then the closure $\overline{A}$, the complement A^c and the interior A° in X are semialgebraic.*

(3) *Composition of two semialgebraic maps is semialgebraic.*

(4) *Let $f: X \to Y$ be a semialgebraic map between semialgebraic sets. If A is a semialgebraic subset of X, then its image $f(A)$ is semialgebraic. If B is a semialgebraic subset of Y, then its preimage $f^{-1}(B)$ is semialgebraic.*

(5) *Let $f: X \to Q$ and $g: X \to Y$ be semialgebraic. Assume f is surjective. If $h: Q \to Y$ is a continuous map such that $h \circ f = g$, then h is semialgebraic.*

(6) *Let X be a semialgebraic set. If V is a neighborhood of a point x in X, then there is a semialgebraic neighborhood U of X with $x \in \overline{U} \subset V$.*

As a sequence plays an important role in the category of metric spaces, a curve germ plays in the semialgebraic category, see [1] or [2]. A *curve germ* in a semialgebraic set X is represented by a semialgebraic map $\alpha: (0, \varepsilon] \to X$ for some $\varepsilon > 0$. Two curve germs are considered the same if, after possible reparameterization of the intervals, they agree on a common subinterval $(0, \delta]$ for some $\delta > 0$. Thus, a curve germ is determined by the collection of images sets $\alpha((0, \varepsilon]) \subset X$ for $\varepsilon > 0$. If a curve germ α extends to a continuous map $\alpha: [0, \varepsilon] \to X$, we say the extension is the *completion* of α, and α is *completable*. We write $\alpha \to x$ if α has a completion with $\alpha(0) = x$.

If $\alpha: (0, \varepsilon] \to X$ has a completion with $\alpha \to x$, then given any neighborhood U of x in X, by restricting to a smaller interval whenever necessary, we may assume that $\alpha([0, \varepsilon]) \subset U$.

We state the following elementary propositions because it will be used several times in this paper.

Proposition 2.2 [2, p. 73]**.** *Let X and Y be semialgebraic sets.*

(1) *Every curve germ in a compact and semialgebraic set has a completion.*

(2) *If x belongs to the closure of a semialgebraic subset A of X, then there is a curve germ α in A with $\alpha \to x$.*

(3) *If $f: X \to Y$ is semialgebraic and surjective, then every curve germ α in Y lifts to a curve germ $\tilde{\alpha}$ in X, that is $f \circ \tilde{\alpha} = \alpha$.*

(4) *Every curve germ in $X \cup Y$ is a curve germ in either X or Y.*

Proposition 2.3 [2, p. 73]**.** *Let $f: X \to Y$ be a map whose graph is semialgebraic.*

(1) *f is continuous if and only if for any completable curve germ α in X with $\alpha \to x$, the curve germ $f \circ \alpha$ is also completable in Y with $f \circ \alpha \to f(x)$.*

(2) *Suppose f is continuous. Then f is semialgebraically proper if and only if the following condition holds; if a curve germ $\tilde{\alpha}$ in X such that $f \circ \tilde{\alpha}$ is completable in Y, then $\tilde{\alpha}$ is completable in X.*

The following lemma is valid if Y is a compactly generated Hausdorff space. Every metric space is first-countable and therefore compactly generated Hausdorff. Since semialgebraic sets are (usual) metric spaces, we have the following lemma.

Lemma 2.1. *Let $f: X \to Y$ be a continuous map between semialgebraic sets. Then the following are equivalent*:

(1) *f is proper*;

(2) *f is closed and its fibers are compact.*

The above lemma still valid in the semialgebraic category as in the following proposition. A semialgebraic map $f: X \to Y$ is called *semialgebraically closed* if f maps every closed and semialgebraic subset of X to a closed and semialgebraic subset of Y.

Proposition 2.4. *Let $f\colon X \to Y$ be a semialgebraic map between semialgebraic sets. Then the following are equivalent:*

(1) *f is semialgebraically proper*;

(2) *f is semialgebraically closed and its fibers are compact.*

Proof. Let f be a semialgebraically proper. Clearly $f^{-1}(y)$ is compact for all $y \in Y$. Suppose C is a closed and semialgebraic subset of M and let $\overline{f(C)}$ denote the closure of $f(C)$ in Y. Given $y \in \overline{f(C)}$, it follows from Proposition 2.2(2) that there exists a curve germ α in $f(C)$ with $\alpha \to y$. Since the restriction $f|_C\colon C \to f(C)$ is surjective, by Proposition 2.2(3), there is a curve germ $\tilde{\alpha}$ in C such that $f \circ \tilde{\alpha} = \alpha$. Thus, since α is completable in Y and f is semialgebraically proper, by Proposition 2.3(2), $\tilde{\alpha}$ is completable in X. Let $\tilde{\alpha} \to x$, then $x \in C$ because C is closed in X. By Proposition 2.3(1), $\alpha = f \circ \tilde{\alpha} \to f(x)$, and hence $y = f(x) \in f(C)$. Therefore $f(C)$ is closed in Y.

Conversely, let f be a semialgebraically closed map such that $f^{-1}(y)$ is compact for all $y \in Y$. Let K be compact and semialgebraic in Y. By Proposition 2.1(4) $f^{-1}(K)$ is semialgebraic. We will show that $f^{-1}(K)$ is compact. Let $\{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $f^{-1}(K)$. For $x \in U_\alpha$ we can take an open and semialgebraic set $W_{\alpha,x}$ such that $x \in W_{\alpha,x} \subset U_\alpha$. Then the collection $\mathcal{B} = \bigcup_{\alpha\in\Lambda}\{W_{\alpha,x} \mid x \in U_\alpha\}$ is a refinement of $\{U_\alpha \mid \alpha \in \Lambda\}$. It is enough to show that $\mathcal{B}$ contains a finite subcollection that also covers $f^{-1}(K)$. For all $z \in K$, $\mathcal{B}$ is also an open cover of $f^{-1}(z)$. Since the latter is compact, it has a finite subcover. In other words, for each $z \in K$, there is a finite set $\mathcal{A}_z \subset \mathcal{B}$ such that $f^{-1}(z) \subset \bigcup_{W\in\mathcal{A}_z} W$. The set $X - \cup_{W\in\mathcal{A}_z} W$ is closed and semialgebraic in X. Its image is closed in Y, because f is a semialgebraically closed map. Hence the set $V_z = Y - f\left(X - \bigcup_{W\in\mathcal{A}_z} W\right)$ is open in Y. It is easy to check that V_z contains the point z. Since $K \subset \cup_{z\in K} V_z$ and K is compact, there are finitely many points $z_1, \ldots, z_n$ such that $K \subset \bigcup_{i=1}^n V_{z_i}$. Furthermore the set $\mathcal{B}_* = \bigcup_{i=1}^n \mathcal{A}_{z_i}$ is a finite union of finite sets, thus $\mathcal{B}_*$ is finite. Since $f^{-1}(K) \subset f^{-1}\left(\bigcup_{i=1}^n V_{z_i}\right) \subset \bigcup_{W\in\mathcal{B}_*} W$, we have found a finite subcover of $f^{-1}(K)$.

Proposition 2.4 is proved.

Theorem 2.1. *Let $f\colon X \to Y$ be a semialgebraic map between semialgebraic sets. Then the following are equivalent:*

(1) *f is semialgebraically proper*;

(2) *f is proper.*

Proof. Let f be a semialgebraically proper map. Clearly, all fibers $f^{-1}(y)$, $y \in Y$, are compact. It suffices to show that f is closed map. Let C be a closed subset of X. Suppose $f(C)$ is not closed in Y. Then there exist a point y in $\overline{f(C)}$ which is not contained in $f(C)$. Since $f^{-1}(y)$ is disjoint from the closed set C, for every point $x \in f^{-1}(y)$ has a semialgebraic neighborhood U_x which does not meet C. Since $f^{-1}(y)$ is compact there exist finitely many points $x_1, \ldots, x_n \in f^{-1}(y)$ such that

$$f^{-1}(y) \subset U_{x_1} \cup \ldots \cup U_{x_n}.$$

Then the set $B = X - (U_{x_1} \cup \cdots \cup U_{x_n})$ is closed and semialgebraic in X and contains C. Since f is semialgebraically closed, the image $f(B)$ is closed in Y. Thus $\overline{f(C)} \subset f(B)$. This contradiction since $y \notin f(B)$. Therefore f is closed.

The converse is trivial.

Theorem 2.1 is proved.

Let $f\colon X \to Y$ and $g\colon Z \to Y$ be semialgebraic maps. Then the pullback $X \times_Y Z = \{(x,z) \in \in X \times Z \mid f(x) = g(x)\}$ is closed and semialgebraic in $X \times Z$. The pullback diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\ p_2\ } & Z \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

commutes, where p_1 and p_2 are the canonical projections.

Theorem 2.2. *Let $f\colon X \to Y$ be a semialgebraic map between semialgebraic sets. Then the following are equivalent:*

(1) *f is semialgebraically proper*;

(2) *for every semialgebraic map $g\colon Z \to Y$, the canonical projection $p_2\colon X \times_Y Z \to Z$ is semialgebraically proper.*

Proof. Let f be a semialgebraically proper map. Let $g\colon Z \to Y$ be a semialgebraic map. Suppose K is compact and semialgebraic in Y. Sice the preimage $p_2^{-1}(K)$ is a closed subset of a compact set $f^{-1}(g(K)) \times K$, it is semialgebraic and compact. Hence p_2 is semialgebraically proper.

Conversely, taking $Z = Y$ and g the identity map on Y, it follows immediately that f is semialgebraically proper.

Theorem 2.2 is proved.

3. Semialgebraically proper actions. The definition of a semialgebraic group is similar to that of a Lie group, i.e., a semialgebraic set G is called a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion are semialgebraic. Every semialgebraic group has a Lie group structure, and hence locally compact. Moreover, every semialgebraic subgroup of a semialgebraic group is closed (see, [7, 8]). In this section G always a semialgebraic group.

By a *semialgebraic transformation group* we mean a triple (G, X, θ), where G is a semialgebraic group, X is a semialgebraic set, and $\theta\colon G \times X \to X$ is a semialgebraic map such that

$\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $g, h \in G$ and $x \in X$;

$\theta(e, x) = x$ for all $x \in X$, where e is the identity of G.

In this case X is called a *semialgebraic G-set*, and θ is called the *semialgebraic action* of G on X. As usual we shortly write gx for $\theta(g, x)$. A semialgebraic G-set X is called *semialgebraically proper* if the map

$$\vartheta_*\colon G \times X \to X \times X, \quad (g, x) \mapsto (gx, x)$$

is semialgebraically proper. Clearly, if G is compact, then every semialgebraic G-set is semialgebraically proper. If G is not compact but X is compact, then X is not semialgebraically proper. Moreover, if X is a semialgebraically proper G-set, then the orbit space X/G is Hausdorff. Similarly, a G-space is called *proper* if the map ϑ_* is proper.

Theorem 2.1 implies that ϑ_* is semialgebraically proper if and only if it is proper. Thus we have the following theorem.

Theorem 3.1. *Let X be a semialgebraic G-set. Then the following are equivalent*:

(1) *X is a semialgebraically proper G-set*;

(2) *X is a proper G-space.*

Proposition 3.1. *Let X be a semialgebraic G-set. Then the following are equivalent:*

(1) *X is a semialgebraically proper G-set*;

(2) *if two curve germs α in X and γ in G such that α and $\gamma\alpha$ are completable, then γ is completable.*

Proof. This follows from Proposition 2.3(2).

Lemma 3.1. *Let X be a semialgebraically proper G-set. If H is a semialgebraic subgroup of G and K is compact and semialgebraic in X, then the set $HK = \{hx \mid h \in H,\ x \in K\}$ is closed and semialgebraic in X.*

Proof. Clearly $HK = \theta(H \times K)$. It follows that the set HK is semialgebraic by Proposition 2.1(4). We now prove that HK is closed in X. For $x \in \overline{HK}$, there exists a curve germ α in HK with $\alpha \to x$. Since the restriction $\theta| \colon H \times K \to HK$ is surjective, we can find curve germs $\gamma \subset H$ and $\beta \subset K$ such that $\gamma\beta = \alpha$. By Proposition 3.1, β is completable in K, say $\beta \to y$. By Proposition 3.1, γ is completable in G, say $\gamma \to g$. Since H is closed, $g \in H$, and hence $x = gy \in HK$.

Lemma 3.1 is proved.

In particular, for each $x \in X$, the orbit $G(x)$ is closed in X.

If the map θ is semialgebraically proper, then the map ϑ_* is also semialgebraically proper by Propositions 2.3 and 3.1. But the converse does not hold.

***Example* 3.1.** We consider the semialgebraic group $\mathbb{R}^*$ of nonzero real numbers under multiplication. Let $X = \mathbb{R}^2 - \{\mathbf{0}\}$. We can view X as a semialgebraic $\mathbb{R}^*$-set with the action

$$\theta \colon \mathbb{R}^* \times X \to X, \quad \theta(t, \mathbf{x}) = t\mathbf{x}.$$

Then ϑ_* is semialgebraically proper, and hence X is semialgebraically proper. Indeed, given two curve germs α in X and γ in $\mathbb{R}^*$ such that α and $\gamma\alpha$ are completable, say $\alpha \to \mathbf{a}$ and $\gamma\alpha \to \mathbf{b}$, then $\gamma \to \dfrac{\langle \mathbf{a}, \mathbf{b} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \in \mathbb{R}^*$ where $\langle , \rangle$ denotes the inner product, and thus γ is completable. So it follows that ϑ_* is semialgebraically proper by Proposition 3.1.

On the other hand, let $H = \{t \in \mathbb{R}^* \mid t > 0\}$ and $C = \{(x, y) \in X \mid xy = 1\}$, then $H \times C$ is closed and semialgebraic in $\mathbb{R}^* \times X$. Since the image $\theta(H \times C) = \{(x, y) \in X \mid xy > 0\}$ is not closed in X, θ is not semialgebraically closed, and hence not semialgebraically proper.

For two subsets U and V of X, we set

$$((U, V)) = \{g \in G \mid U \cap gV \neq \emptyset\}.$$

Proposition 3.2. *Let X be a semialgebraic G-set. Then the following are equivalent*:

(1) *X is a semialgebraically proper G-set*;

(2) *for any two compact subsets K_1 and K_2 of M, the subset $((K_1, K_2))$ of G is compact.*

Proof. Suppose X is a semialgebraically proper G-set. Then ϑ_* is semialgebraically proper, and hence proper. For given two compact subsets K_1 and K_2 of M, $\vartheta_*^{-1}(K_1 \times K_2)$ is a compact subset of $G \times X$. Let $p \colon G \times X \to G$ be the canonical projection. Then $((K_1, K_2)) = p(\vartheta_*^{-1}(K_1 \times K_2))$, and hence $((K_1, K_2))$ is compact.

Conversely, suppose K is compact and semialgebraic in $X \times X$. Let p_1 and p_2 be the canonical projections of $X \times X$ onto its first and second factors, respectively. Then $K_1 = p_1(K)$ and $K_2 = p_2(K)$ are compact and semialgebraic in X. Obviously, we see that

$$\vartheta_*^{-1}(K) \subset \vartheta_*^{-1}(K_1 \times K_2) \subset ((K_1, K_2)) \times K_2.$$

Since $\vartheta_*^{-1}(K)$ is closed and $((K_1, K_2)) \times K_2$ is compact, $\vartheta_*^{-1}(K)$ is compact. Hence ϑ_* is semialgebraically proper.

Proposition 3.2 is proved.

In particular, for each $x \in X$, the isotropy subgroup $G_x = ((\{x\}, \{x\}))$ is compact.

Now we want to compare this notion of proper with that of Palais in [6]. Let X be a G-space. A subset U of X is called *thin* if the set $((U, U))$ has compact closure in G. A G-space X is called a *Cartan G-space* if every point $x \in X$ has a thin neighborhood. A subset S of a G-space X is called *small* if each point $x \in X$ has a neighborhood U such that the set $((S, U))$ has compact closure. A G-space X is called by Palais [6] a “proper” if every point $x \in X$ has a small neighborhood. We need to distinguish this notion from the former. To do this, in this case we call X a *Palais-proper* G-space.

Proposition 3.3. *Let X be a semialgebraic G-set. If X is a Palais-proper G-space, then it is a semialgebraically proper G-set.*

Proof. Suppose X is a Palais-proper G-space. Let K_1 and K_2 be two compact subsets of X. We first prove that $((K_1, K_2))$ is a closed subset of G. If g is a point of the closure of $((K_1, K_2))$ in G, then there is a sequence g_n of points of $((K_1, K_2))$ converges to g. For each positive integer n, we can choose $x_n \in K_2$ such that $g_n x_n \in K_1$ because $K_1 \cap g_n K_2 \neq \varnothing$. Since K_2 is compact, there exists a subsequence x_{n_i} of x_n which converges to $x \in K_2$, so that $g_{n_i} x_{n_i} \to gx$. Then $gx \in K_1$ because K_1 is closed, it follows that $g \in ((K_1, K_2))$. Therefore, $((K_1, K_2))$ is closed in G.

We now prove that $((K_1, K_2))$ is compact. Since X is Palais-proper G-space, for each $(x, y) \in \in X \times X$, there is an open neighborhood $U_* \times V_*$ of (x, y) such that $((U_*, V_*))$ has compact closure. By the compactness of $K_1 \times K_2$, there exists a finite open covering $\{U_1 \times V_1, \ldots, U_k \times V_k\}$ of $K_1 \times K_2$ such that $((U_i, V_i))$ has compact closure for all i. Since $((K_1, K_2)) \subset \bigcup_{i=1}^{k} ((U_i, V_i))$, the set $((K_1, K_2))$ has compact closure. Then $((K_1, K_2))$ is compact because it is closed. Hence X is semialgebraically proper by Proposition 3.2.

Proposition 3.3 is proved.

Theorem 3.2. *If X be a semialgebraically proper G-set, then it is a Cartan G-space.*

Proof. Suppose X is a semialgebraically proper G-set. Then ϑ_* is semialgebraically proper, and hence proper. For every $x \in X$, the isotropy subgroup G_x is compact. Since G is locally compact, there exists an open semialgebraic neighborhood W of G_x in G whose closure is compact. By Lemma 2.1, ϑ_* is closed, so that the image $\vartheta_*((G - W) \times X)$ is closed in $X \times X$. Since $(x, x) \notin \vartheta_*((G - W) \times X)$, there exist open neighborhoods U_1 and U_2 of x such that

$$U_1 \times U_2 \cap \vartheta_*((G - W) \times X) = \varnothing.$$

It follows that $((U_1, U_2)) \subset W$. Indeed, if $g \in ((U_1, U_2))$, then there exists $y \in U_2$ such that $gy \in U_1$. Hence $\vartheta_*(g, y) = (gy, y) \in U_1 \times U_2$. Since $U_1 \times U_2 \cap \vartheta_*((G - W) \times X) = \varnothing$, we have $g \notin G - W$, and so $g \in W$. Hence $((U_1, U_2)) \subset W$. Therefore, the intersection $U_1 \cap U_2$ is a thin neighborhood of x.

Theorem 3.2 is proved.

The converse of the above theorem does not hold.

***Example* 3.2.** Let $\mathbb{R}^*$ denote the semialgebraic group of nonzero real numbers under multiplication, and let $X = \mathbb{R}^2 - \{\mathbf{0}\}$. We consider a semialgebraic $\mathbb{R}^*$-set X with the action

$$\theta \colon \mathbb{R}^* \times X \to X, \ \theta(t, (x, y)) = \left(tx, \frac{1}{t} y\right).$$

Then X is a Cartan G-space. Indeed, given $(x, y) \in X$, we set

$$U = (x - |x|/2, x + |x|/2) \times (y - 1, y + 1) \quad \text{if} \quad x \neq 0,$$

$$U = (x - 1, x + 1) \times (y - |y|/2, y + |y|/2) \quad \text{if} \quad y \neq 0,$$

then $((U, U)) \subset \left[\frac{1}{3}, 3\right]$, and hence the set U is a thin neighborhood of (x, y).

But X is not semialgebraically proper. Indeed, let $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1) \in \mathbb{R}^2$, and let K_i denote the closed ball of radius $\frac{1}{2}$ centered at $\mathbf{e}_i$ in Y for $i = 1, 2$. Then $K_1 \times K_2$ is compact and semialgebraic in $X \times X$, but $\vartheta_*^{-1}(K_1 \times K_2)$ is not compact because it contains an unbounded set $\left\{ \left(t, \left(\frac{1}{t}, 1\right)\right) \,\middle|\, t \geq 2 \right\}$. Hence ϑ_* is not semialgebraically proper.

***Corollary* 3.1.** *Assume that X is a locally complete semialgebraic set. If X is a semialgebraically proper G-set, then it is a Palais-proper G-space.*

Proof. Since X is a Cartan G-space, every point $x \in X$ has a thin neighborhood W. Note that a semialgebraic set is locally complete if and only if it is locally compact. Since X is locally compact, we can take a semialgebraic neighborhood U of x such that the closure $\overline{U}$ is compact and $\overline{U} \subset W$

We will show that U is a small neighborhood of x. It is enough to show that, for each $y \in X$, there is a neighborhood V of y in X such that $((U, V))$ has compact closure. In case $G(y) \cap \overline{U} = \emptyset$, put $V = M - G(\overline{U})$, then it is an open neighborhood of y by Lemma 3.1. Clearly $((U, V)) = \varnothing$. In case $G(y) \cap \overline{U} \neq \varnothing$, $gy \in \overline{U}$ for some $g \in G$. Let $V = g^{-1}W$, we have $((U, V)) = ((U, W))g \subset \subset ((W, W))g$. Since $((W, W))$ has compact closure, so does $((U, V))$.

Corollary 3.1 is proved.

There is an example of a proper action of $\mathbb{R}$ on a G_δ-subset X of $\mathbb{R}^2$ which is not Palais-proper(see [5, p. 79]). But, in this case, the $\mathbb{R}$-space X is not semialgebraic.

We conclude this paper with a natural question.

Question. *Does there exists a semialgebraically proper G-set which is not Palais-proper*?

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