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THE ENERGY OF A DOMAIN ON THE SURFACE

ЕНЕРГІЯ ОБЛАСТІ НА ПОВЕРХНІ

We compute the energy of a unit normal vector field on a Riemannian surface M. It is shown that the energy of the unit normal vector field is independent of the choice of an orthogonal basis of the tangent space. We also define the energy of the surface. Moreover, we compute the energy of spheres, domains on a right circular cylinder, torus, and more generally, of the surfaces of revolution.

Розраховано енергію одиничного нормального векторного поля на рімановій поверхні M. Показано, що енергія одиничного нормального векторного поля не залежить від вибору ортогонального базиса в дотичному просторі. Визначено енергію поверхні. Більш того, розраховано енергію сфер, областей на прямому круговому циліндрі та торі і, більш загально, поверхонь обертання.

1. Introduction. The energy of a unit vector field X on a Riemannian manifold M is defined as the energy of the section into the unit tangent bundle T^1M determined by X. In this respect, the energy of distributions of Riemannian manifolds and the energy of unit vector fields on the sphere S^3 were considered in papers by P. M. Chacon, A. M. Naveira [1] and A. Higuichi, B. S. Kay and C. M. Wood [2]. Further, the energy of differentiable maps has been also studied by C. M. Wood [3].

Generally, every geometric problem about curves and surfaces can be solved by means of the Frenet vectors field of the curve and the normal vector field of the surface. Therefore, in [4], we focus on the curve C instead of the manifold M. For a given curve C with a pair (I, α) of parametric unit speeds in a space R^n_v on which we take a fixed point $a \in I$, we denote Frenet frames at the points $\alpha(a)$ and $\alpha(s)$ by $\{V_1(\alpha(a)), \ldots, V_r(\alpha(a))\}$ and $\{V_1(\alpha(s)), \ldots, V_r(\alpha(s))\}$, respectively. We calculate the energy of a Frenet vectors fields as well as the pseudo-angle between the vectors $V_i(\alpha(a))$ and $V_i(\alpha(s))$, where $1 \leq i \leq r$. We observed that both energy and pseudo-angle depend on the curvature functions of the curve C.

In this paper, we calculate the energy of a unit normal vector field on a Riemannian surface M. We achieve that energy of a unit normal vector field depends on the norm of the matrix of the shape operator and the area $A_S(M)$ of M. We also prove that the energy can be expressed in terms of the Gaussian and mean curvatures of the surface. Hence, the energy of the unit normal vector fields of a Riemannian surface M can be expressed via principal curvatures of M, which are independent of the choice of the orthogonal basis of the tangent space of M. Taking into account the above fact we define the energy of the surface by ignoring the constant term obtained from area of the surface. We prove that the energy of a sphere is independent of radius and is equal to 4π. Furthermore, we calculate the energy of the domain on right circular cylinder of height h and radius r as $\dfrac{1}{r}\pi h$. We also compute the energy of the torus. Finally, we establish a formula for the energy of a surface of revolution, which verifies the above computations.

Definition 1. *Let M is a Riemannian surface in R^3. If p is a point of M, then for each tangent vector v to M at p, let*

$$S_p(v) = -\nabla_{v_p} N,$$

where N is a unit normal vector field on a neighborhood of p in M. S_p is called the shape operator of M at p (derived from N) (see [6]). The Gaussian curvature of M is the real-valued function $G = \det S$ on M. Explicitly, for each point p of M, the Gaussian curvature $G(p)$ of M at p is the determinant of the shape operator S of M at p. The mean curvature of M is the function $H = \dfrac{1}{2}\operatorname{trace} S$. Let u be a unit vector tangent to M at a point p. Then the number $k(u) = \langle S(u), u\rangle$ is called the normal curvature of M in the u direction. The maximum and minimum values of the normal curvature $k(u)$ of M at p are called the principal curvatures of M at p, and are denoted by k_1 and k_2 respectively. The directions in which these extreme values occur are called principal directions of M at p. Unit vectors in these directions are called principal vectors of M at p.

Lemma 1. *If k_1 and k_2 are principal curvatures at a point p, then we have*

$$S_p = \begin{bmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{bmatrix}.$$

Proposition 1. *The connection map $K : T(T^1M) \to T^1M$ satisfies the following*:

1) $\pi\circ K = \pi\circ d\pi$ *and* $\pi\circ K = \pi\circ\widetilde{\pi}$. *Here* $\widetilde{\pi} : T(T^1M) \to T^1M$ *is the tangent bundle projection.*

2) *For $\omega\in T_xM$ and a section $\xi : M \to T^1M$, we get*

$$K(d\xi(\omega)) = \nabla_\omega \xi.$$

Here ∇ is the Levi–Civita covariant derivative (*see* [5]).

Definition 2. *For $\eta_1, \eta_2 \in T_\xi(T^1M)$ define*

$$g_S(\eta_1, \eta_2) = \langle d\pi(\eta_1), d\pi(\eta_2)\rangle + \langle K(\eta_1), K(\eta_2)\rangle. \tag{1}$$

This gives a Riemannian metric on TM. Recall that g_S is called the Sasaki metric. The metric g_s makes the projection $\pi : T^1M \to M$ a Riemannian submersion (*see* [5]).

We use the classical notation of surface theory; for this purpose we can give [7] as a general reference. Let $\varphi : U \to R^3$, $\varphi(U) = M$, $\varphi(U) = (\varphi_1(u,v), \varphi_2(u,v), \varphi_3(u,v))$ and $\varphi(u,v)$ be a local parametrization of surface M in R^3.

Let $\Re$ be a domain on surface M. Area of $\Re$ is

$$A_s(\Re) = \int\int\limits_{\varphi^{-1}(\Re)} \sqrt{EG - F^2}\,dudv,$$

where $\upsilon = \sqrt{EG - F^2}dudv$ is the area form in M and $E = \langle\varphi_u, \varphi_u\rangle$, $F = \langle\varphi_u, \varphi_v\rangle$, $G = \langle\varphi_v, \varphi_v\rangle$ (see [6]).

Definition 3. *The energy of a differentiable map $f\colon (M,\langle,\rangle)\to(N,h)$ between Riemannian manifolds is given by*

$$\mathcal{E}(f)=\frac{1}{2}\int\limits_M\left(\sum_{a=1}^{n}h(df(e_a),df(e_a)\right)\upsilon, \tag{2}$$

where υ is the canonical volume form in M and $\{e_a\}$ is a local basis of the tangent space (*see, for example,* [1, 3]).

The energy of a unit vector field X is defined to be the energy of the section $X\colon M\to T^1M$, where T^1M is the unit tangent bundle equipped with the restriction of the Sasaki metric on TM. Now let $\pi\colon T^1M\to M$ be the bundle projection, and let $T(T^1M)=\mathcal{V}\oplus\mathcal{H}$ denote the vertical/horizontal splitting induced by the Levi–Civita connection (see [3]).

Since we provide some background material in the previous section, we are in a position to calculate the energy of a unit normal vector field on the Riemannian surface M.

2. The energy of the unit normal vector field of a surface.

Theorem 1. *Let N be unit normal vector field of M. Then for the energy of N the following formula hold*:

$$\mathcal{E}(N)=\int\limits_M(2H^2-G)\upsilon+A_s(M),$$

where υ is the area form in M, G and H are Gaussian and mean curvature of M respectively, $A_s(M)$ is area of M.

Proof. Let $\{e_u,e_v\}$ be an local orthonormal basis of the tangent space, N be unit normal vector field of M and NM be normal bundle. Thus, we have $N\colon M\to NM$, where $NM=\bigcup_{q\in U}N_{\varphi(q)}M$, and $N_{\varphi(q)}M$ is the straight line through the point $\varphi(q)$ in the N direction. By using equation (2), we obtain

$$\mathcal{E}(N)=\frac{1}{2}\int\limits_M(g_{\mathcal{S}}(dN(e_u),dN(e_u))+g_{\mathcal{S}}(dN(e_v),dN(e_v)))\upsilon. \tag{3}$$

From (1) it follows that

$$\mathcal{E}(N)=\frac{1}{2}\int\limits_M[\langle d\pi(dN(e_u)),d\pi(dN(e_u))\rangle+\langle K(dN(e_u)),K(dN(e_u))\rangle+$$

$$+\langle d\pi(dN(e_v)),d\pi(dN(e_v))\rangle+\langle K(dN(e_v)),K(dN(e_v))\rangle]\,\upsilon,$$

where $\pi\colon NM\to M$ be the bundle projection and $K\colon T(NM)\to NM$. Since N is a section, we have $d(\pi)\circ d(N)=d(\pi\circ N)=d(id_M)=id_{TM}$. We also have by Proposition 1 that $K(dN(e_u))=$ $=\nabla_{e_u}N=-S(e_u)\ \forall p\in M$. Combining all these, we get

$$\mathcal{E}(N)=\frac{1}{2}\int\limits_M[\langle e_u,e_u\rangle+\langle S(e_u),S(e_u)\rangle+\langle e_v,e_v\rangle+\langle S(e_v),S(e_v)\rangle]\,\upsilon. \tag{4}$$

On the other hand, we have that

$$S(e_u) = -\nabla_{e_u} N = ae_u + be_v, \qquad S(e_v) = -\nabla_{e_v} N = ce_u + de_v \qquad \forall p \in M, \tag{5}$$

where a, b, c, d are real-valued functions.

Therefore, the matrix which corresponds to the shape operator of M is $S = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Using equalities in (5) and putting them in equation (4), we obtain

$$\mathcal{E}(N) = \frac{1}{2} \int\limits_M [a^2 + b^2 + c^2 + d^2 + \langle e_u, e_u \rangle + \langle e_v, e_v \rangle] \upsilon. \tag{6}$$

Since S is a symmetric matrix, then the Gaussian curvature of M is $G = \det S = ad - b^2$ and the mean curvature of M is $H = \frac{1}{2}(a + d)$.

Hence,

$$a^2 + b^2 + c^2 + d^2 = a^2 + 2b^2 + d^2 = 4H^2 - 2G. \tag{7}$$

Putting (7) in (6), we get

$$\mathcal{E}(N) = \frac{1}{2} \int\limits_M [4H^2 - 2G + \langle e_u, e_u \rangle + \langle e_v, e_v \rangle] \upsilon. \tag{8}$$

Since $\{e_u, e_v\}$ is orthonormal basis of the tangent space of M, then (8) becomes

$$\mathcal{E}(N) = \int\limits_M (2H^2 - G)\upsilon + \int\limits_M \upsilon = \int\limits_M (2H^2 - G)\upsilon + A_s(M).$$

Theorem 1 is proved.

Consequently, combining Lemma 1 and equation (6) we can give the following corollary.

***Corollary* 1.** *If k_1 and k_2 are principal curvatures of M, then we have*

$$\mathcal{E}(N) = \frac{1}{2} \int\limits_M (k_1^2 + k_2^2)\upsilon + A_s(M).$$

The Gaussian curvature and mean curvature of M are independent from the choice of the basis $\{e_u, e_v\}$, thus the energy of a unit normal vector field is independent of the choice of the orthogonal basis of the tangent space of M. We may ignore the constant term of $A_s(M)$ and we can give the following definition.

Definition 4. *The integral*

$$\frac{1}{2} \int\limits_M (a^2 + b^2 + c^2 + d^2)\upsilon = \frac{1}{2} \int\limits_M (4H^2 - 2G)\upsilon = \int\limits_M (2H^2 - G)\upsilon$$

is called the energy of surface M and is denoted by $\mathcal{E}(M)$.

Once we have the Definition 4, we may be able to compute the energy of the surface of a sphere but not compute the surface of a cylinder. However, we can only calculate the energy of a region of cylinder (see Example 2). By using Definition 4, we can calculate the energy of a domain on surface as follow.

Let $\varphi\colon U \to R^3$, $\varphi(U) = M$, $\varphi(u,v)$ be a local parametrization of surface M in R^3, and $\Re$ be a domain on surface M.

$$\mathcal{E}(\Re) = \int\int_{\varphi^{-1}(\Re)} (2H^2 - G)\sqrt{EG - F^2}dudv,$$

where $\upsilon = \sqrt{EG - F^2}dudv$ is the area form in M.

***Example* 1.** Let $\varphi\colon U \to R^3$, $U = \left]-\pi, \pi\right[\times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, $\varphi(u,v) = (r\text{cos}u\text{cos}v, r\text{cos}v\text{sin}u, r\text{sin}v)$ be a local parametrization of the sphere S^2 of radius r. The matrix of the shape operator of sphere is

$$S = \begin{bmatrix} -\dfrac{1}{r} & 0 \\ 0 & -\dfrac{1}{r} \end{bmatrix} \quad \text{and} \quad \sqrt{EG - F^2} = r^2\text{cos}v.$$

Then

$$\mathcal{E}(S^2) = \frac{1}{2}\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\left(-\frac{1}{r}\right)^2 \sqrt{EG - F^2}dudv = 4\pi.$$

***Example* 2.** Let $\varphi(u,v) = (r\text{cos}u, r\text{sin}u, v)$ be a local parametrization of a right circular cylinder, $\varphi\colon U \to R^3$, $U =]0, 2\pi[\times]0, h[$, $\varphi(U) = \Re \subset M$. The matrix of the shape operator of cylinder is

$$S = \begin{bmatrix} -\dfrac{1}{r} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \sqrt{EG - F^2} = r^2\text{cos}v.$$

Therefore,

$$\mathcal{E}(\Re) = \frac{1}{2}\int_{0}^{2\pi}\int_{0}^{h} \left(-\frac{1}{r}\right)^2 \sqrt{EG - F^2}dudv = \frac{1}{r}\pi h.$$

***Example* 3.** Let $\varphi\colon R \times R \to R^3$, $\varphi(u,v) = (u, v, au + bv)$ be a local parametrization of a plane M. The matrix of the plane is $S = 0$ and $\mathcal{E}(M) = 0$.

***Example* 4.** The energy of a minimal surface M is $\mathcal{E}(M) = -\int_M (G)\upsilon = \int_M (a^2 + b^2)\upsilon$.

***Example* 5.** Torus of revolution T. Suppose that α is the circle in the xz plane with radius $r > 0$ and center $(R, 0, 0)$. We shall rotate about the z axis; hence we must require $R > r$ to keep α from meeting the axis of revolution. A natural parametrization of α is

$$\alpha(u) = (R + r\text{cos}(u), 0, r\text{sin}(u)), \qquad u \in I.$$

A local parametrization of the torus T is then given by

$$\varphi(u,v) = ((R + r\mathrm{cos}u)\mathrm{cos}v, (R + r\mathrm{cos}u)\mathrm{sin}v, r\mathrm{sin}(u)),\ 0 < u < 2\pi,\ 0 < v < 2\pi.$$

The matrix of the shape operator of torus is

$$S = \begin{bmatrix} \dfrac{1}{r} & 0 \\ 0 & \dfrac{\mathrm{cos}u}{R + r\mathrm{cos}u} \end{bmatrix} \quad \text{and} \quad \sqrt{EG - F^2} = r(R + r\mathrm{cos}u).$$

Hence,

$$\mathcal{E}(T) = \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\left(\left(\frac{1}{r}\right)^2 + \left(\frac{\mathrm{cos}u}{R + r\mathrm{cos}u}\right)^2\right) r(R + r\mathrm{cos}u)dudv$$

and

$$\mathcal{E}(T) = \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\left(1 + \left(\frac{\mathrm{cos}u}{\dfrac{R}{r} + \mathrm{cos}u}\right)^2\right)\left(\frac{R}{r} + \mathrm{cos}u\right)dudv.$$

***Example* 6.** Let M be a surfaces of revolution. We suppose the axis of revolution of M is the z-axis of our coordinate system. We denote the profile curve of M by α. We can suppose that α is represented by

$$\alpha(u) = (f(u), 0, g(u)), \qquad u \in I,$$

where f, g are real functions on the open interval I, we suppose that α has arc length parametrization. A local parametrization of the surface M is then given by

$$\varphi(u,v) = (f(u)\mathrm{cos}v, f(u)\mathrm{sin}v, g(u)), \qquad u \in I, \qquad 0 < v < 2\pi.$$

The matrix of the shape operator of surfaces of revolution is

$$S = \begin{bmatrix} \dfrac{g''(u)}{f'(u)} & 0 \\ 0 & \dfrac{g'(u)}{f(u)} \end{bmatrix} \quad \text{and} \quad \sqrt{EG - F^2} = f(u).$$

Let $\Re$ be a domain on surface M, then

$$\mathcal{E}(\Re) = \frac{1}{2}\int\int\limits_{\varphi^{-1}(\Re)}\left[\left(\frac{g''(u)}{f'(u)}\right)^2 + \left(\frac{g'(u)}{f(u)}\right)^2\right] f(u)dudv.$$

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