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HYPERSURFACES WITH NONZERO CONSTANT GAUSS – KRONECKER CURVATURE IN $M^{n+1}(\pm 1)$ *

ГІПЕРПОВЕРХНІ З НЕНУЛЬОВОЮ ПОСТІЙНОЮ КРИВИЗНОЮ ГАУСА – КРОНЕКЕРА В $M^{n+1}(\pm 1)$

We study hypersurfaces in a unit sphere and in a hyperbolic space with nonzero constant Gauss – Kronecker curvature and two distinct principal curvatures one of which is simple. Denoting by K the nonzero constant Gauss – Kronecker curvature of hypersurfaces, we obtain some characterizations of the Riemannian products $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, $a^2 = 1/(1 + K^{\frac{2}{n-2}})$ or $S^{n-1}(a) \times H^1(-\sqrt{1+a^2})$, $a^2 = 1/(K^{\frac{2}{n-2}} - 1)$.

Вивчаються гіперповерхні в одиничній сфері та в гіперболічному просторі з ненульовою постійною кривизною Гауса – Кронекера та двома різними головними кривизнами (???), одна з яких є простою. Якщо K — це ненульова постійна кривизна Гауса – Кронекера гіперповерхонь, то деякі характеристики ріманових добутків можна отримати у вигляді $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, $a^2 = 1/(1 + K^{\frac{2}{n-2}})$ або $S^{n-1}(a) \times H^1(-\sqrt{1+a^2})$, $a^2 = 1/(K^{\frac{2}{n-2}} - 1)$.

1. Introduction. Let M^n be an n-dimensional immersed hypersurface in a real space form $M^{n+1}(c)$, $c = \pm 1$. If $c = 1$ or $c = -1$, we call $M^{n+1}c$ a unit sphere or a hyperbolic space. We notice that there are many important rigidity results for hypersurfaces with constant mean curvature and two distinct principal curvatures, see [1, 9], or with constant scalar curvature and two distinct principal curvatures, see [5, 6]. Since the Gauss – Kronecker curvature of M^n is also an important rigidity invariant under the isometric immersion, it is natural for us to ask such a question: if the nonzero Gauss – Kronecker curvature is constant, can we obtain any rigidity results?

In this note we try to study hypersurfaces in $M^{n+1}(c)$ $(c = \pm 1)$ with nonzero constant Gauss – Kronecker curvature and two distinct principal curvatures one of which is simple. We introduce the well-known standard models of complete hypersurfaces with constant Gauss – Kronecker curvature in $M^{n+1}(c)$ $(c = \pm 1)$.

When $c = 1$, we consider the standard immersions $S^{n-k}(\sqrt{1-a^2}) \hookrightarrow R^{n-k+1}$ and $S^k(a) \hookrightarrow R^{k+1}$, where $0 < a < 1$, $1 \le k \le n-1$, and take the Riemannian product immersion $S^k(a) \times S^{n-k}(\sqrt{1-a^2}) \hookrightarrow S^{n+1}(c) \subset R^{n+2}$, then it has two distinct constant principal curvatures

$$\lambda_1 = \ldots = \lambda_k = \frac{\sqrt{1-a^2}}{a}, \quad \lambda_{k+1} = \ldots = \lambda_n = -\frac{a}{\sqrt{1-a^2}},$$

respectively. We easily see that the Riemannian product $S^k(a) \times S^{n-k}(\sqrt{1-a^2})$ has constant Gauss – Kronecker curvature $K = \left(\dfrac{\sqrt{1-a^2}}{a}\right)^k \left(-\dfrac{a}{\sqrt{1-a^2}}\right)^{n-k}$. The square of the norm of the second fundamental form and the mean curvature of $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, where $a^2 = 1/(1 + K^{\frac{2}{n-2}})$, are

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$$|A|^2 = (n-1)K^{\frac{1}{n-2}} + K^{-\frac{2}{n-2}},$$

$$H = \frac{1}{n}\{(n-1)|K|^{\frac{1}{n-2}} - |K|^{-\frac{1}{n-2}}\}.$$

When $c = -1$, we consider the standard immersions $H^{n-k}(-\sqrt{1+a^2}) \hookrightarrow R_1^{n-k+1}$ and $S^k(a) \hookrightarrow R^{k+1}$, where $a > 0$, $1 \leq k \leq n-1$, and take the Riemannian product immersion $S^k(a) \times H^{n-k}(-\sqrt{1+a^2}) \hookrightarrow H^{n+1}(c) \subset R_1^{n+2}$, then it has two distinct constant principal curvatures

$$\lambda_1 = \ldots = \lambda_k = \frac{\sqrt{1+a^2}}{a}, \quad \lambda_{k+1} = \ldots = \lambda_n = \frac{a}{\sqrt{1+a^2}},$$

respectively. We easily see that the Riemannian product $S^k(a) \times H^{n-k}(-\sqrt{1+a^2})$ has constant Gauss – Kronecker curvature $K = \left(\frac{\sqrt{1+a^2}}{a}\right)^k \left(\frac{a}{\sqrt{1+a^2}}\right)^{n-k}$. The square of the norm of the second fundamental form and the mean curvature of $S^{n-1}(a) \times H^1(-\sqrt{1+a^2})$, where $a^2 = 1/(K^{\frac{2}{n-2}} - 1)$, $K > 1$, and $S^1(a) \times H^{n-1}(-\sqrt{1+a^2})$, where $a^2 = K^{\frac{2}{n-2}}/(1 - K^{\frac{2}{n-2}})$, $K < 1$, are

$$|A|^2 = (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

$$H = \frac{1}{n}\{(n-1)K^{\frac{1}{n-2}} + K^{-\frac{1}{n-2}}\}.$$

We obtain some characterizations of $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, $a^2 = 1/(1 + K^{\frac{2}{n-2}})$ and $S^{n-1}(a) \times \times H^1(-\sqrt{1+a^2})$, $a^2 = 1/(K^{\frac{2}{n-2}} - 1)$:

Theorem 1.1. *Let M^n be an n-dimensional with $n \geq 3$ complete smooth connected and oriented hypersurface in $M^{n+1}(c)$ $(c = \pm 1)$ with nonzero constant Gauss – Kronecker curvature K and two distinct principal curvatures one of which is simple.*

(1) *When $c = 1$, $K < 0$, if*

$$|A|^2 \leq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

or

$$|A|^2 \geq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

then M^n is isometric to the Riemannian product $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, $a^2 = 1/(1 + K^{\frac{2}{n-2}})$.

(2) *When $c = -1$, $K > 1$, if*

$$|A|^2 \leq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

or

$$|A|^2 \geq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}},$$

then M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(-\sqrt{1+a^2})$, $a^2 = 1/(K^{\frac{2}{n-2}} - 1)$.

Theorem 1.2. *Let M^n be an n-dimensional with $n \geq 3$ complete smooth connected and oriented hypersurface in $M^{n+1}(c)$ $(c = \pm 1)$ with nonzero constant Gauss–Kronecker curvature K and two distinct principal curvatures one of which is simple.*

(1) *When $c = 1$, $K < 0$, if*

$$H \leq \frac{1}{n}\{(n-1)|K|^{\frac{1}{n-2}} - c|K|^{-\frac{1}{n-2}}\},$$

or

$$H \geq \frac{1}{n}\{(n-1)|K|^{\frac{1}{n-2}} - c|K|^{-\frac{1}{n-2}}\},$$

then M^n is isometric to the Riemannian product $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, $a^2 = 1/(1 + K^{\frac{2}{n-2}})$.

(2) *When $c = -1$, $K > 1$, if*

$$H \leq \frac{1}{n}\{(n-1)|K|^{\frac{1}{n-2}} - c|K|^{-\frac{1}{n-2}}\},$$

or

$$H \geq \frac{1}{n}\{(n-1)|K|^{\frac{1}{n-2}} - c|K|^{-\frac{1}{n-2}}\},$$

then M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(-\sqrt{1+a^2})$, $a^2 = 1/(K^{\frac{2}{n-2}} - 1)$.

2. Preliminaries. Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature c $(c = \pm 1)$. Let M^n be an n-dimensional complete smooth connected and oriented hypersurface in $M^{n+1}(c)$. We choose a local orthonormal frame $e_1, \ldots, e_{n+1}$ in $M^{n+1}(c)$ such that $e_1, \ldots, e_n$ are tangent to M^n. Let $\omega_1, \ldots, \omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+1, \quad 1 \leq i, j, k, \ldots \leq n.$$

The structure equations of $M^{n+1}(c)$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \quad \Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D,$$

$$K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}),$$

where Ω_{AB} and K_{ABCD} denote the curvature form and the components of the curvature tensor of $M^{n+1}(c)$, respectively.

Restricting to M^n,

$$\omega_{n+1} = 0, \tag{2.1}$$

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}, \tag{2.2}$$

where h_{ij} denotes the components of the second fundamental form of M^n. The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \quad \Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.3}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.4}$$

where Ω_{ij} and R_{ijkl} denote the curvature form and the components of the curvature tensor of M^n, respectively. From (2.4), we have

$$n(n-1)(r-c) = n^2H^2 - |A|^2,$$

where $n(n-1)r = R$ is the scalar curvature, H is the mean curvature and $|A|^2$ is the squared norm of the second fundamental form of M^n.

The function $K = \det(h_{ij})$ is called the Gauss – Kronecker curvature of M^n. We choose $e_1, \ldots, e_n$ such that $h_{ij} = \lambda_i \delta_{ij}$, then we see that $K = \det(h_{ij}) = \lambda_1 \lambda_2 \ldots \lambda_n$. From (2.2) we obtain

$$\omega_{n+1i} = \lambda_i \omega_i, \quad i = 1, 2, \ldots, n.$$

Hence, we get from the structure equations of M^n,

$$d\omega_{n+1i} = d\lambda_i \wedge \omega_i + \lambda_i d\omega_i = d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j. \tag{2.5}$$

On the other hand, we have on the curvature forms of $M^{n+1}(c)$,

$$\Omega_{n+1i} = -\frac{1}{2} \sum_{C,D} K_{n+1iCD} \omega_C \wedge \omega_D =$$

$$= -\frac{1}{2} \sum_{C,D} c(\delta_{n+1C}\delta_{iD} - \delta_{n+1D}\delta_{iC}) \omega_C \wedge \omega_D = -c\omega_{n+1} \wedge \omega_i = 0.$$

Therefore, from the structure equations of $M^{n+1}(c)$, we obtain

$$d\omega_{n+1i} = \sum_j \omega_{n+1j} \wedge \omega_{ji} + \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i} = \sum_j \lambda_j \omega_{ij} \wedge \omega_j. \tag{2.6}$$

From (2.5) and (2.6), we get

$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j) \omega_{ij} \wedge \omega_j = 0. \tag{2.7}$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j) \omega_{ij}, \tag{2.8}$$

we have $\psi_{ij} = \psi_{ji}$. Hence (2.7) can be written as

$$\sum_j (\psi_{ij} + \delta_{ij} d\lambda_j) \wedge \omega_j = 0.$$

By E. Cartan's Lemma, we get

$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k Q_{ijk}\omega_k, \tag{2.9}$$

where Q_{ijk} are uniquely determined functions such that

$$Q_{ijk} = Q_{ikj}.$$

3. Proofs of theorems. The following Proposition 3.1 original due to Otsuki [7] is useful.

Proposition 3.1. *Let M^n be a hypersurface in a real space form $M^{n+1}(c)$ $(c = \pm 1)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1, *then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Let M^n be an n-dimensional complete smooth connected and oriented hypersurface with two distinct principal curvatures one of which is simple and $n \geq 3$, that is, without loss of generality, we may assume

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu,$$

where λ_i for $i = 1, 2, \ldots, n$ are the principal curvatures of M^n. Thus, we have

$$K = \lambda^{n-1}\mu.$$

From $K \neq 0$, we conclude that $\lambda \neq 0$. By changing the orientation for M^n and renumbering $e_1, \ldots, e_n$ if necessary, we may assume that $\lambda > 0$. Thus

$$\mu = \frac{K}{\lambda^{n-1}}, \tag{3.1}$$

$$0 \neq \lambda - \mu = \frac{\lambda^n - K}{\lambda^{n-1}}. \tag{3.2}$$

We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$d\lambda = \sum_{k=1}^n \lambda_{,k}\,\omega_k, \quad d\mu = \sum_{k=1}^n \mu_{,k}\,\omega_k,$$

from Proposition 3.1, we get

$$\lambda_{,1} = \lambda_{,2} = \ldots = \lambda_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.3}$$

From (3.1), we obtain

$$d\mu = -\frac{(n-1)K}{\lambda^n} d\lambda. \tag{3.4}$$

Thus, we also have

$$\mu_{,1} = \mu_{,2} = \ldots = \mu_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x). \tag{3.5}$$

In this case, we may consider locally λ as a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.9) and (3.3), we get, for $1 \le j \le n-1$,

$$\lambda_{,n}\,\omega_n = \sum_{i=1}^{n} \lambda_{,i}\,\omega_i = d\lambda = d\lambda_j = \sum_{k=1}^{n} Q_{jjk}\omega_k = \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n.$$

Therefore, we obtain

$$Q_{jjk} = 0, \;\; 1 \le k \le n-1, \quad \text{and} \quad Q_{jjn} = \lambda_{,n}\,. \tag{3.6}$$

By (2.9) and (3.5), we have

$$\mu_{,n}\,\omega_n = \sum_{i=1}^{n} \mu_{,i}\,\omega_i = d\mu = d\lambda_n = \sum_{k=1}^{n} Q_{nnk}\omega_k = \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n.$$

Hence, we obtain

$$Q_{nnk} = 0, \quad 1 \le k \le n-1, \quad \text{and} \quad Q_{nnn} = \mu_{,n}\,. \tag{3.7}$$

From (3.4), we get

$$Q_{nnn} = \mu_{,n} = -\frac{(n-1)K}{\lambda^n}\lambda_{,n}\,.$$

From the definition of ψ_{ij}, if $i \ne j$, we have $\psi_{ij} = 0$ for $1 \le i \le n-1$ and $1 \le j \le n-1$. Therefore, from (2.9), if $i \ne j$ and $1 \le i \le n-1$ and $1 \le j \le n-1$ we have

$$Q_{ijk} = 0, \quad \text{for any} \quad k. \tag{3.8}$$

By (2.9), (3.6), (3.7) and (3.8), for $j < n$, we get

$$\psi_{jn} = \sum_{k=1}^{n} Q_{jnk}\omega_k = Q_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda_{,n}\,\omega_j. \tag{3.9}$$

From (2.8), (3.2) and (3.9), for $j < n$, we obtain

$$\omega_{jn} = \frac{\psi_{jn}}{\lambda - \mu} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_j = \frac{\lambda^{n-1}\lambda_{,n}}{\lambda^n - K}\omega_j.$$

Thus, from the structure equations of M^n we have

$$d\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0.$$

Therefore, we may put $\omega_n = ds$. By (3.3), we get

$$d\lambda = \lambda_{,n}\,ds, \quad \lambda_{,n} = \frac{d\lambda}{ds}.$$

Thus, we obtain

$$\omega_{jn} = \frac{\lambda^{n-1}\frac{d\lambda}{ds}}{\lambda^n - K}\omega_j = \frac{d(\log|\lambda^n - K|^{\frac{1}{n}})}{ds}\omega_j. \tag{3.10}$$

From (3.10) and the structure equations of $M^{n+1}(c)$, for $j < n$, we have

$$d\omega_{jn} = \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn}\wedge\omega_{nn} + \omega_{jn+1}\wedge\omega_{n+1n} + \Omega_{jn} =$$

$$= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn+1}\wedge\omega_{n+1n} - c\omega_j\wedge\omega_n =$$

$$= \frac{d(\log|\lambda^n - K|^{1/n})}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k - (\lambda\mu + c)\omega_j\wedge ds.$$

Differentiating (3.10), we obtain

$$d\omega_{jn} = \frac{d^2(\log|\lambda^n - K|^{1/n})}{ds^2}ds\wedge\omega_j + \frac{d(\log|\lambda^n - K|^{1/n})}{ds}d\omega_j =$$

$$= \frac{d^2(\log|\lambda^n - K|^{1/n})}{ds^2}ds\wedge\omega_j + \frac{d(\log|\lambda^n - K|^{1/n})}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k =$$

$$= \left\{-\frac{d^2(\log|\lambda^n - K|^{1/n})}{ds^2} + \left[\frac{d(\log|\lambda^n - K|^{1/n})}{ds}\right]^2\right\}\omega_j\wedge + ds$$

$$+\frac{d(\log|\lambda^n - K|^{1/n})}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k.$$

From the previous two equalities, we get

$$\frac{d^2(\log|\lambda^n - K|^{1/n})}{ds^2} - \left\{\frac{d(\log|\lambda^n - K|^{1/n})}{ds}\right\}^2 - (\lambda\mu + c) = 0. \tag{3.11}$$

If we define $\varpi = |\lambda^n - K|^{-1/n}$, from (3.11) we have

$$\frac{d^2\varpi}{ds^2} + \varpi(\lambda\mu + c) = 0. \tag{3.12}$$

On the other hand, from (3.10), we have $\nabla_{e_n}e_n = \sum_{i=1}^{n}\omega_{ni}(e_n)e_i = 0$. By the definition of geodesic, we know that any integral curve of the principal vector field corresponding to the principal curvature μ is a geodesic. Thus, we see that $\varpi(s)$ is a function defined in $(-\infty, +\infty)$ since M^n is complete and any integral curve of the principal vector field corresponding to μ is a geodesic.

We can prove the following lemma.

Lemma 3.1. *If $c = 1$ or $c = -1$ and $K > 1$, then the positive function ϖ is bounded from above.*

Proof. From (3.2), we know that $\lambda^n - K \neq 0$. Thus (3.1) and (3.12) imply that

$$\frac{d^2\varpi}{ds^2} + \varpi\frac{c\lambda^{n-2} + K}{\lambda^{n-2}} = 0, \tag{3.13}$$

that is

$$\frac{d^2\varpi}{ds^2} + \varpi\left[c + K(K \pm \varpi^{-n})^{2/n-1}\right] = 0. \tag{3.14}$$

Multiplying (3.14) by $2\frac{d\varpi}{ds}$ and integrating, we get

$$\left(\frac{d\varpi}{ds}\right)^2 + c\varpi^2 + \varpi^2(K \pm \varpi^{-n})^{2/n} = C,$$

where C is a constant. Thus, we obtain

$$c + (K \pm \varpi^{-n})^{2/n} \leq \frac{C}{\varpi^2}. \tag{3.15}$$

If the positive function ϖ is not bounded from above, that is, $\varpi \to +\infty$. From (3.15), we have that $c + K^{2/n} \leq 0$, a contradiction with the assumption. Thus we conclude.

We can also prove the following lemma.

Lemma 3.2. (1) *Let*

$$P_K(t) = ct^{\frac{n-2}{n}} + K, \quad t > 0.$$

(i) *For* $c = 1$, *if* $K < 0$, *then* $P_K(t)$ *is a strictly monotone increasing function of* t *and has a positive real root* $t_0 = (-K)^{\frac{n}{n-2}}$;

(ii) *For* $c = -1$, *if* $K > 0$, *then* $P_K(t)$ *is a strictly monotone decreasing function of* t *and has a positive real root* $t_0 = K^{\frac{n}{n-2}}$.

(2) *Let*

$$|A|^2(t) = \frac{1}{t^{2(n-1)/n}}\{(n-1)t^2 + K^2\}, \ \ t > 0,$$

$t_0 = (-K)^{\frac{n}{n-2}}$ *for* $c = 1$, $K < 0$ *and* $t_0 = K^{\frac{n}{n-2}}$ *for* $c = -1$, $K > 0$. *Then*

(i) *If* $t \geq |K|$ *and* $t_0 \geq |K|$, *then* $t \leq t_0$ *holds if and only if* $|A|^2(t) \leq (n-1)t_0^{2/n} + t_0^{-2/n}$ *and* $t \geq t_0$ *holds if and only if* $|A|^2(t) \geq (n-1)t_0^{2/n} + t_0^{-2/n}$;

(ii) *If* $t \leq |K|$ *and* $t_0 \leq |K|$, *then* $t \leq t_0$ *holds if and only if* $|A|^2(t) \geq (n-1)t_0^{2/n} + t_0^{-2/n}$ *and* $t \geq t_0$ *holds if and only if* $|A|^2(t) \leq (n-1)t_0^{2/n} + t_0^{-2/n}$.

(3) *Let*

$$H(t) = \frac{1}{nt^{(n-1)/n}}\{(n-1)t + K\}, \quad t > 0.$$

Then

(i) *For* $c = 1$, $K < 0$ *and* $t_0 = (-K)^{\frac{n}{n-2}}$, *if* $t \geq K$, *then* $t \leq t_0$ *holds if and only if* $H(t) \leq \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\}$ *and* $t \geq t_0$ *holds if and only if* $H(t) \geq \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\}$;

(ii) *For* $c = -1$, $K > 0$ *and* $t_0 = K^{\frac{n}{n-2}}$,

(a) *if* $t \geq K$ *and* $t_0 \geq K$, *then* $t \leq t_0$ *holds if and only if* $H(t) \leq \dfrac{1}{n}\{(n-1)t_0^{1/n} + t_0^{-1/n}\}$ *and* $t \geq t_0$ *holds if and only if* $H(t) \geq \dfrac{1}{n}\{(n-1)t_0^{1/n} + t_0^{-1/n}\}$;

(b) *if* $t \leq K$ *and* $t_0 \leq K$, *then* $t \leq t_0$ *holds if and only if* $H(t) \geq \dfrac{1}{n}\{(n-1)t_0^{1/n} + t_0^{-1/n}\}$ *and* $t \geq t_0$ *holds if and only if* $H(t) \leq \dfrac{1}{n}\{(n-1)t_0^{1/n} + t_0^{-1/n}\}$.

Proof. (1) Obvious fact.

(2) We have

$$\frac{d|A|^2(t)}{dt} = \frac{2(n-1)t^{(2-3n)/n}}{n}(t^2 - K^2),$$

it follows that the solution of $\dfrac{d|A|^2(t)}{dt} = 0$ is $t = |K|$. Therefore, we know that $t \leq |K|$ if and only if $|A|^2(t)$ is a decreasing function, $t \geq |K|$ if and only if $|A|^2(t)$ is an increasing function and $|A|^2(t)$ obtain its minimum at $t = |K|$.

If $t_0 \geq |K|$, since $t \geq |K|$ if and only if $|A|^2(t)$ is an increasing function, we infer that if $t \geq |K|$, then $t \leq t_0$ holds if and only if

$$|A|^2(t) \leq |A|^2(t_0) = \frac{1}{t_0^{2(n-1)/n}}\{(n-1)t_0^2 + K^2\} =$$

$$= \frac{1}{t_0^{2(n-1)/n}}\left\{(n-1)t_0^2 + \left[\left(ct_0^{\frac{n-2}{n}} + K\right) - ct_0^{\frac{n-2}{n}}\right]^2\right\} =$$

$$= \frac{1}{t_0^{2(n-1)/n}}\left\{(n-1)t_0^2 + \left[-ct_0^{\frac{n-2}{n}}\right]^2\right\} = (n-1)t_0^{2/n} + t_0^{-2/n},$$

where $c = 1$, $t_0 = (-K)^{\frac{n}{n-2}}$ and $K < 0$ or $c = -1$, $t_0 = K^{\frac{n}{n-2}}$ and $K > 0$. By the same reason, the rest of case (2) follows.

(3) Since

$$\frac{dH(t)}{dt} = \frac{n-1}{n^2 t^{(2n-1)/n}}(t - K),$$

we see that $H(t)$ is an increasing function if $t \geq K$ and $H(t)$ is a decreasing function if $t \leq K$, then it follows the result of (3).

***Proof of Theorem* 1.1.** Putting $t = \lambda^n (> 0)$, from (3.1), we see that the square of the norm of second fundamental form $|A|^2 = (n-1)\lambda^2 + \dfrac{K^2}{\lambda^{2(n-1)}} = \dfrac{1}{t^{2(n-1)/n}}\{(n-1)t^2 + K^2\} = |A|^2(t)$. From (3.13), we have

$$\frac{d^2\varpi}{ds^2} + \varpi\frac{P_K(t)}{t^{\frac{n-2}{n}}} = 0. \tag{3.16}$$

(1) When $c = 1$, we consider two cases $t \geq |K|$ and $t \leq |K|$.

Case (i). If $t \geq |K|$, we also consider two subcases $|K| > t_0$ and $|K| \leq t_0$, where $t_0 = (-K)^{\frac{n}{n-2}}$.

If $|K| > t_0$, since $t \geq |K|$, we get $t > t_0$. Since we assume that $K < 0$, by Lemma 3.4, we infer that $P_K(t) > P_K(t_0) = 0$. From (3.16), we have $\dfrac{d^2\varpi}{ds^2} < 0$, this implies that $\dfrac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. If $\dfrac{d\varpi(s)}{ds}$ has no zero point in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in $(-\infty, +\infty)$. If $\dfrac{d\varpi(s)}{ds}$ has exactly one zero point s_0 in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in both $(-\infty, s_0]$ and $[s_0, +\infty)$.

On the other hand, from Lemma 3.2, we know that $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty}\varpi(s)$ and $\lim_{s\to+\infty}\varpi(s)$ exist and then we get

$$\lim_{s\to-\infty}\frac{d\varpi(s)}{ds} = \lim_{s\to+\infty}\frac{d\varpi(s)}{ds} = 0. \tag{3.17}$$

This is impossible because $\dfrac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s. Therefore, we know that the case $|K| > t_0$ does not occur. It follows that $|K| \leq t_0$.

If $|K| \leq t_0$, since $t \geq |K|$, from Lemma 3.2 and (3.16), we have

$$|A|^2 \leq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}} = (n-1)t_0^{2/n} + t_0^{-2/n},$$

holds if and only if $t \leq t_0$ if and only if $P_K(t) \leq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \geq 0$. Also

$$|A|^2 \geq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}} = (n-1)t_0^{2/n} + t_0^{-2/n},$$

holds if and only if $t \geq t_0$ if and only if $P_K(t) \geq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \leq 0$. Thus $\dfrac{d\varpi}{ds}$ is a monotonic decreasing function of $s \in (-\infty, +\infty)$, this implies that $\dfrac{d\varpi(s)}{ds}$ has at most one zero point for $s \in (-\infty, +\infty)$. If $\dfrac{d\varpi(s)}{ds}$ has no zero point in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in $(-\infty, +\infty)$. If $\dfrac{d\varpi(s)}{ds}$ has exactly one zero point s_0 in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in both $(-\infty, s_0]$ and $[s_0, +\infty)$. Therefore, we see that $\varpi(s)$ is monotonic when s tends to infinity. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty}\varpi(s)$ and $\lim_{s\to+\infty}\varpi(s)$ exist and (3.17) holds. From the monotonicity of $\dfrac{d\varpi(s)}{ds}$, we have $\dfrac{d\varpi(s)}{ds} \equiv 0$ and $\varpi(s) = \text{constant}$. Combining $\varpi = |\lambda^n - K|^{-1/n}$ and (3.1), we conclude that λ and μ are constant, that is, M^n is isoparametric. From the classical result of Cartan [3] (see also [4, p. 238]), we know that M^n is isometric to the Riemannian product $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, where $a^2 = 1/(1 + K^{\frac{2}{n-2}})$.

Case (ii). If $t \leq |K|$, we consider two subcases $|K| < t_0$ and $|K| \geq t_0$, where $t_0 = (-K)^{\frac{n}{n-2}}$.

If $|K| < t_0$, since $t \leq |K|$, we have $t < t_0$. From $K < 0$ and Lemma 3.2, it follows that $P_K(t) < P_K(t_0) = 0$. From (3.23), we have $\dfrac{d^2\varpi}{ds^2} > 0$. Thus $\dfrac{d\varpi(s)}{ds}$ is a strictly monotone increasing function of s. By the same arguments as in case (i), we conclude that $|K| < t_0$ does not occur, then $|K| \geq t_0$.

If $|K| \geq t_0$, since $t \leq |K|$, from Lemma 3.2 and (3.16), we have that $|A|^2 \leq (n-1)t_0^{2/n} + t_0^{-2/n}$ holds if and only if $t \geq t_0$ if and only if $P_K(t) \geq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \leq 0$. Also $|A|^2 \geq$ $\geq (n-1)t_0^{2/n} + t_0^{-2/n}$ holds if and only if $t \leq t_0$ if and only if $P_K(t) \leq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \geq 0$. By the same arguments as in the proof of case (i), we conclude.

(2) When $c = -1$, we also consider two cases $t \geq |K|$ and $t \leq |K|$.

Case (i). If $t \geq |K|$, we consider two subcases $|K| > t_0$ and $|K| \leq t_0$, where $t_0 = K^{\frac{n}{n-2}}$.

If $|K| > t_0$, since $t \geq |K|$, we have $t > t_0$. Since we assume that $K > 1$, by Lemma 3.2, we infer that $P_K(t) < P_K(t_0) = 0$. From (3.16), we have $\dfrac{d^2\varpi}{ds^2} > 0$, this implies that $\dfrac{d\varpi(s)}{ds}$ is a strictly monotone increasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. By use of the same method as in the proof of case (i) in (1), we know that the case $|K| > t_0$ does not occur. It follows that $|K| \leq t_0$.

If $|K| \leq t_0$, since $t \geq |K|$, from Lemma 3.2 and (3.16), we see that

$$|A|^2 \leq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}} = (n-1)t_0^{2/n} + t_0^{-2/n},$$

holds if and only if $t \leq t_0$ if and only if $P_K(t) \geq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \leq 0$. Also

$$|A|^2 \geq (n-1)K^{\frac{2}{n-2}} + K^{-\frac{2}{n-2}} = (n-1)t_0^{2/n} + t_0^{-2/n},$$

holds if and only if $t \geq t_0$ if and only if $P_K(t) \leq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \geq 0$. Thus $\dfrac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By use of the same method as in the proof of case (i) in (1), we conclude that λ and μ are constant, that is, M^n is isoparametric. From the classical result of Cartan [2] (see also [8] or [4, p. 238]), we know that M^n is isometric to the Riemannian product $S^{n-1}(a) \times H^1(-\sqrt{1+a^2})$, $a^2 = 1/(K^{\frac{2}{n-2}} - 1)$.

Case (ii). If $t \leq |K|$, we consider two subcases $|K| < t_0$ and $|K| \geq t_0$, where $t_0 = K^{\frac{n}{n-2}}$.

If $|K| < t_0$, since $t \leq |K|$, we have $t < t_0$. From $K > 1$ and Lemma 3.2, it follows that $P_K(t) > P_K(t_0) = 0$. From (3.16), we have $\dfrac{d^2\varpi}{ds^2} < 0$. Thus $\dfrac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s. By the same arguments as in case (i) of (1), we conclude that $|K| < t_0$ does not occur, then $|K| \geq t_0$.

If $|K| \geq t_0$, since $t \leq |K|$, from Lemma 3.2 and (3.16), we have that $|A|^2 \leq (n-1)t_0^{2/n} + t_0^{-2/n}$ holds if and only if $t \geq t_0$ if and only if $P_K(t) \leq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \geq 0$. Also $|A|^2 \geq$ $\geq (n-1)t_0^{2/n} + t_0^{-2/n}$ holds if and only if $t \leq t_0$ if and only if $P_K(t) \geq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \leq 0$. By the same arguments as in the proof of case (i) of (2), we conclude.

Theorem 1.1 is proved.

***Proof of Theorem* 1.2.** (1) When $c = 1$, since we assume that $K < 0$ and $t = \lambda^n (> 0)$, we see that $t > K$. From $t_0 > K$, where $t_0 = (-K)^{\frac{n}{n-2}}$, Lemma 3.2 and (3.16), we see that

$$H \leq \frac{1}{n}\{(n-1)|K|^{\frac{1}{n-2}} - |K|^{-\frac{1}{n-2}}\} = \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\},$$

holds if and only if $t \leq t_0$ if and only if $P_K(t) \leq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \geq 0$. Also

$$H \geq \frac{1}{n}\{(n-1)|K|^{\frac{1}{n-2}} - |K|^{-\frac{1}{n-2}}\} = \frac{1}{n}\{(n-1)t_0^{1/n} - t_0^{-1/n}\},$$

holds if and only if $t \geq t_0$ if and only if $P_K(t) \geq 0$ and if and only if $\dfrac{d^2\varpi}{ds^2} \leq 0$. Thus $\dfrac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. By use of the same method as in the proof of (1) in Theorem 1.1, we know that M^n is isometric to the Riemannian product $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, where $a^2 = 1/(1+K^{\frac{2}{n-2}})$.

(2) When $c = -1$, since we assume that $K > 1$, we consider two cases $t \geq K$ and $t \leq K$. By Lemma 3.2, we see that the function ϖ is bounded. It suffices to use the same method as in the proof of (2) in Theorem 1.1.

Theorem 1.2 is proved.

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