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M. H. M. Rashid, T. Prasad (Mu'tah Univ., Al-Karak, Jordan)

STABILITY OF VERSIONS OF THE WEYL-TYPE THEOREMS UNDER TENSOR PRODUCT

СТАБІЛЬНІСТЬ РІЗНИХ ВЕРСІЙ ТЕОРЕМ ТИПУ ВЕЙЛЯ ДЛЯ ТЕНЗОРНОГО ДОБУТКУ

We study the transformation versions of the Weyl-type theorems from operators T and S for their tensor product $T \otimes S$ in the infinite-dimensional space setting.

Вивчаються трансформовані версії теорем типу Вейля для операторів T і S та їх тензорного добутку $T \otimes S$ у нескінченновимірній постановці.

1. Introduction. Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X} \otimes \mathcal{Y}$ denote the completion (in some reasonable uniform cross norm) of the tensor product of $\mathcal{X}$ and $\mathcal{Y}$. For Banach space operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, let $A \otimes B \in \mathcal{B}(\mathcal{X} \otimes \mathcal{Y})$ denote the *tensor product* of A and B. Recall that for an operator S, the *Browder spectrum* $\sigma_b(S)$ and the *Weyl spectrum* $\sigma_w(S)$ of S are the sets

$$\sigma_b(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \quad \text{is not Fredholm or} \quad \operatorname{asc}(S - \lambda) \neq \operatorname{dsc}(S - \lambda)\},$$

$$\sigma_w(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \quad \text{is not Fredholm or} \quad \operatorname{ind}(S - \lambda) \neq 0\}.$$

In the case in which $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, Kubrusly and Duggal [13] proved that

$$\text{if} \quad \sigma_b(A) = \sigma_w(A) \quad \text{and} \quad \sigma_b(B) = \sigma_w(B), \quad \text{then} \quad \sigma_b(A \otimes B) = \sigma_w(A \otimes B)$$

$$\text{if and only if} \quad \sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B).$$

In other words, if A and B satisfy Browder's theorem, then their tensor product satisfies Browder's theorem if and only if the Weyl spectrum identity holds true. The same proof still holds in a Banach space setting. Recently, Rashid and Prasad studied property (Sw): a Banach space operator T, $T \in \mathcal{B}(\mathcal{X})$, satisfies property (Sw) if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$, where σ denote the usual spectrum, $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} \colon T - \lambda \text{ is not an upper } B\text{-Fredholm or } \operatorname{ind}(T - \lambda) > 0\}$ denotes the *upper B-Weyl spectrum* and $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(T) \colon 0 < \alpha(T - \lambda) < \infty\}$ is the set of finite multiplicity isolated eigenvalues of T and that $T \in \mathcal{B}(\mathcal{X})$ obeys property (Sb) if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \pi^0(T)$, where $\pi^0(T)$ is the set of all poles of finite rank. This paper intents to discuss the stability of property (Sb) and property (Sw) under tensor product $T \otimes S$ of Banach space operators T and S.

2. Notation and complementary results. For a bounded linear operator $S \in \mathcal{B}(\mathcal{X})$, let $\sigma(S)$, $\sigma_p(S)$ and $\sigma_a(S)$ denote, respectively, the *spectrum*, the *point spectrum* and the *approximate point spectrum* of S and if $G \subseteq \mathbb{C}$, then $\operatorname{iso} G$ denote the *isolated points* of G. Let $\alpha(S)$ and $\beta(S)$ denote the *nullity* and the *deficiency* of S, defined by $\alpha(S) = \dim\ker(S)$ and $\beta(S) = \operatorname{codim}\Re(S)$. If the range $\Re(S)$ of S is closed and $\alpha(S) < \infty$ (resp. $\beta(S) < \infty$), then S is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. If $S \in \mathcal{B}(\mathcal{X})$ is either upper or lower semi-Fredholm, then S is called a *semi-Fredholm* operator, and $\operatorname{ind}(S)$, the index of S, is then defined by $\operatorname{ind}(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then S is a *Fredholm* operator. The *ascent*, denoted $\mathrm{asc}\,(S)$, and the *descent*, denoted $\mathrm{dsc}\,(S)$, of S are given by $\mathrm{asc}\,(S) = \inf\{n \in \mathbb{N} \colon \ker(S^n) = \ker(S^{n+1})\}$, $\mathrm{dsc}\,(S) = \inf\{n \in \mathbb{N} \colon \Re(S^n) = \Re(S^{n+1})\}$ (where the infimum is taken over the set of nonnegative integers; if no such integer n exists, then $\mathrm{asc}\,(S) = \infty$, respectively $\mathrm{dsc}\,(S) = \infty$).

According to Coburn [7], *Weyl's theorem* holds for S if $\Delta(S) = \sigma(S) \setminus \sigma_w(S) = E^0(S)$, and that *Browder's theorem* holds for $S(S \in \mathcal{B})$ if $\Delta(S) = \sigma(S) \setminus \sigma_w(S) = \pi^0(S)$, or equivalently $\sigma_b(S) = \sigma_w(T)$.

For $S \in \mathcal{B}(\mathcal{X})$ and a nonnegative integer n define $S_{[n]}$ to be the restriction of S to $\Re(S^n)$ viewed as a map from $\Re(S^n)$ into $\Re(S^n)$ (in particular, $S_{[0]} = S$). If for some integer n the range space $\Re(S^n)$ is closed and $S_{[n]}$ is an upper (a lower) semi-Fredholm operator, then S is called an *upper* (a *lower*) *semi-B-Fredholm* operator. In this case the index of S is defined as the index of the semi-B-Fredholm operator $S_{[n]}$, see [4]. Moreover, if $S_{[n]}$ is a Fredholm operator, then S is called a *B-Fredholm* operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator S is said to be a *B-Weyl operator* [3] (Definition 1.1) if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(S)$ of S is defined by $\sigma_{BW}(S) = \{\lambda \in \mathbb{C} \colon S - \lambda I \text{ is not a B} - \text{Weyl operator}\}$. An operator $S \in \mathcal{B}(\mathcal{X})$ is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(S)$ of an operator S is defined by $\sigma_D(S) = \{\lambda \in \mathbb{C} \colon S - \lambda I \text{ is not Drazin invertible}\}$. Define also the set $LD(\mathcal{X})$ by $LD(\mathcal{X}) = \{S \in \mathcal{B}(\mathcal{X}) \colon a(S) < \infty \text{ and } \Re(S^{a(S)+1}) \text{ is closed}\}$ and

$$\sigma_{LD}(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \notin LD(\mathcal{X})\}.$$

Following [2], an operator $S \in \mathcal{B}(\mathcal{X})$ is said to be left Drazin invertible if $S \in LD(\mathcal{X})$. We say that $\lambda \in \sigma_a(S)$ is a left pole of S if $S - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(S)$ is a left pole of S of finite rank if λ is a left pole of S and $\alpha(S - \lambda I) < \infty$. Let $\pi_a(S)$ denotes the set of all left poles of S and let $\pi_a^0(S)$ denotes the set of all left poles of S of finite rank. From [2] (Theorem 2.8) it follows that if $S \in \mathcal{B}(\mathcal{X})$ is left Drazin invertible, then S is an upper semi-B-Fredholm operator of index less than or equal to 0. Note that $\pi_a(S) = \sigma(S) \setminus \sigma_{LD}(S)$ and hence $\lambda \in \pi_a(S)$ if and only if $\lambda \notin \sigma_{LD}(S)$.

According to [17], $T \in \mathcal{B}(\mathcal{X})$ satisfies property (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. We say that T satisfies property (Bb) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)$ [18]. Property (Bw) implies Weyl's theorem but converse is not true also property (Bw) implies property (Bb) but converse is not true [18]. Let $\mathcal{SBF}_+^-(\mathcal{X})$ denote the class of all upper B-Fredholm operators such that $\mathrm{ind}(T) \le 0$. The upper B-Weyl spectrum $\sigma_{SBF_+^-}(T)$ of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} \colon T - \lambda \notin \mathcal{SBF}_+^-(\mathcal{X})\}$.

Rashid and Prasad [20] introduced and studied new versions of the Weyl-type theorems property (Sw) and property (Sb).

Definition 2.1. *A bounded linear operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy*

(i) *property* (Sw) *if* $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$ [20],

(ii) *property* (Sb) *if* $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \pi^0(T)$ [20],

(iii) *property*(Bgw) *if* $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$ [18].

The operator $T \in \mathcal{B}(\mathcal{X})$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}$ centred at λ_0, the only analytic function $f \colon \mathbb{D} \to \mathcal{X}$ which satisfies the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Obviously, every $T \in \mathcal{B}(\mathcal{X})$ has SVEP at the points of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function, it easily follows that $T \in \mathcal{B}(\mathcal{X})$, as well as its dual T^*, has SVEP at every point of the boundary $\partial\sigma(T) = \partial\sigma(T^*)$ of the spectrum $\sigma(T)$. In particular, both T and T^* have SVEP at every isolated point of the spectrum, see [1]. Let $T \in \mathcal{B}(\mathcal{X})$ and let $s \in \mathbb{N}$ then T has uniform descent for $n \geq s$ if $\Re(T) + \ker(T^n) = \Re(T) + \ker(T^s)$ for all $n \geq s$. If in addition $\Re(T) + \ker(T^s)$ is closed, then T is said to have topological descent for $n \geq s$ [10]. Let

$$\mathcal{SF}_+(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \quad \text{is upper semi-Fredholm}\},$$

$$\mathcal{F}(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \quad \text{is Fredholm}\},$$

$$\sigma_{SF_+}(S) = \{\lambda \in \sigma_a(S) \colon \lambda \notin \mathcal{SF}_+(S)\},$$

$$\sigma_{SF_+^-}(S) = \{\lambda \in \sigma_a(S) \colon \lambda \in \sigma_{SF_+}(S) \text{ or } \operatorname{ind}(S - \lambda) > 0\},$$

$$\sigma_{ub}(S) = \{\lambda \in \sigma_a(S) \colon \lambda \in \sigma_{SF_+}(S) \text{ or } \operatorname{asc}(S - \lambda) = \infty\},$$

$$\pi_a^0(S) = \{\lambda \in \operatorname{iso}\sigma_a(S) \colon \lambda \in \mathcal{SF}_+(S),\ \operatorname{asc}(S - \lambda) < \infty\},$$

$$E_a^0(S) = \{\lambda \in \operatorname{iso}\sigma_a(S) \colon 0 < \alpha(S - \lambda) < \infty\},$$

$$\mathcal{SBF}_+(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \text{ is upper semi-}B\text{-Fredholm}\},$$

$$\mathcal{SBF}(S) = \{\lambda \in \mathbb{C} \colon S - \lambda \text{ is } B\text{-Fredholm}\},$$

$$\sigma_{SBF_+}(S) = \{\lambda \in \sigma_a(S) \colon \lambda \notin \mathcal{SBF}_+(S)\},$$

$$\sigma_{SBF_+^-}(S) = \{\lambda \in \sigma_a(S) \colon \lambda \in \sigma_{SBF_+}(S) \text{ or } \operatorname{ind}(S - \lambda) > 0\},$$

$$H_0(S) = \{x \in \mathcal{X} \colon \lim_{n\to\infty} \|S^n x\|^{1/n} = 0\},$$

$$\Delta^g(S) = \{\lambda \in \mathbb{C} \colon \lambda \in \sigma(S) \setminus \sigma_{BW}(S)\},$$

$$\Delta_a^g(S) = \{\lambda \in \mathbb{C} \colon \lambda \in \sigma_a(S) \setminus \sigma_{SBF_+^-}(S)\}.$$

Recall that $\sigma_{SF_+^-}(S)$ is the Weyl approximate point spectrum of S, $\sigma_{ub}(S)$ is the Browder approximate point spectrum of S, and $H_0(S)$ is the quasinilpotent of S [1].

We say that $S \in \mathcal{B}(\mathcal{X})$ satisfies a-Browder's theorem ($S \in a\mathcal{B}$) if $\sigma_{SF_+^-}(S) = \sigma_{ub}(S)$ or equivalently, $\Delta_a(S) = \sigma_a(S) \setminus \sigma_{SF_+^-}(S) = \pi_a^0(S)$ and that $S \in \mathcal{B}(\mathcal{X})$ satisfies a-Weyl's theorem ($S \in a\mathcal{W}$) if $\Delta_a(S) = E_0^a(S)$ [21].

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$. Then*

(i) $\sigma_x(T \otimes S) = \sigma_x(T)\sigma_x(S)$, *where $\sigma_x = \sigma$ or σ_a* [5, 22],

(ii) $\sigma_{SF_+}(T \otimes S) = \sigma_{SF_+}(T)\sigma_a(S) \cup \sigma_a(T)\sigma_{SF_+}(S)$ [8].

Recall that an operator T is said to be isoloid if $\lambda \in \operatorname{iso}\sigma(T)$ implies $\lambda \in \sigma_p(T)$ and that $T \in \mathcal{B}(\mathcal{X})$ is said to be a-isoloid if $\lambda \in \operatorname{iso}\sigma_a(T)$ implies $\lambda \in \sigma_p(T)$. It is well-known that if T is a-isoloid, then T is isoloid but not conversely.

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$. If T and S are isoloid, then*

(i) *$T \otimes S$ is isoloid* [11],

(ii) $E^0(T \otimes S) \subseteq E^0(T)E^0(S)$ [14].

Lemma 2.3 ([9], Theorem 3)**.** *If T and S satisfy Browder's theorem, then the following conditions are equivalent:*

(i) $T \otimes S \in \mathcal{B}$,

(ii) $\sigma_w(T \otimes S) = \sigma(T)\sigma_w(S) \cup \sigma_w(T)\sigma(S)$,

(iii) T *has SVEP at points* $\mu \in \mathcal{F}(T)$ *and* S *has SVEP at points* $\nu \in \mathcal{F}(S)$ *such that* $(0 \neq)\lambda = \mu\nu \notin \sigma_w(T \otimes S)$.

3. Property (Sw) and tensor product. We first give some useful lemmas.

Lemma 3.1. *Let* $T \in \mathcal{B}(\mathcal{X})$. *If* T *obeys property* (Sb) *or satisfy any one of the following two conditions:*

(i) $\sigma_{SBF_+^-}(T) = \sigma_b(T)$,

(ii) $\sigma_{SBF_+^-}(T) \cup E^0(T) = \sigma(T)$.

Then the following statements are equivalent:

(i) T *obeys property* (Sw),

(ii) $\sigma_{SBF_+^-}(T) \cap E^0(T) = \varnothing$,

(iii) $E_0(T) = \pi^0(T)$.

Let $H_0(T) = \{x \in \mathcal{X} \colon \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$ and $K(T)$=$\{x \in \mathcal{X} \colon$ there exists a sequence $\{x_n\} \subset \mathcal{X}$ and $\delta > 0$ for which $x = x_0$, $T(x_{n+1}) = x_n$ and $\|x_n\| \leq \delta^n \|x\|$ for all $n = 1, 2, \ldots\}$ denotes the quasinilpotent part and the analytic core of $T \in \mathcal{B}(\mathcal{X})$. It is well known that $H_0(T)$ and $K(T)$ are nonclosed hyperinvariant subspace of $\mathcal{X}$ such that $T^{-q}(0) \subseteq H_0(T)$ for all $q = 0, 1, 2, \ldots$ and $TK(T) = K(T)$ [15].

Lemma 3.2. *Let* $T \in \mathcal{B}(\mathcal{X})$ *and* $S \in \mathcal{B}(\mathcal{Y})$ *obey property* (Sb). *Then* $T \otimes S$ *obeys property* (Sb) *if and only if* $\sigma_{SBF_+^-}(T \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma_{SBF_+^-}(S)\sigma(T)$.

Proof. First, we have to show that $\sigma_{SBF_+^-}(T \otimes S) \subseteq \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma_{SBF_+^-}(S)\sigma(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma_{SBF_+^-}(S)\sigma(T)$. For every factorization $\lambda = \mu\nu$ such that $\mu \in \sigma(T)$ and $\nu \in \sigma(S)$ we have that $\mu \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ and $\mu \in \sigma(S) \setminus \sigma_{SBF_+^-}(S)$. That is, $T - \mu I$ and $S - \nu I$ are upper semi-*B*- Fredholm operators. In particular $\lambda \notin \sigma_{SBF_+}(T \otimes S)$. Now we obtain to prove that $\operatorname{ind}(T \otimes S - \lambda) \leq 0$. If $\operatorname{ind}(T \otimes S - \lambda) > 0$, then it follows that $(T \otimes S - \lambda) \leq \infty$ have finite indices and so $(T \otimes S - \lambda) \in \mathcal{F}$. Let $E = \{(\mu_i, \nu_i) \in \sigma(T)\sigma(S) \colon 1 \leq i \leq p, \mu_i\nu_i = \lambda\}$. Then from [12] (Theorem 3.5) $\operatorname{ind}(T \otimes S - \lambda) = \sum_{j=n+1}^{p} \operatorname{ind}(T - \mu_j) \dim H_0(S - \nu_j) + \sum_{j=1}^{n} \operatorname{ind}(S - \nu_j) \dim H_0(T - \nu_j)$. Since $\operatorname{ind}(T - \mu_i) < 0$ and $\operatorname{ind}(S - \nu_i) < 0$, we get a contradiction. Consequently, $\lambda \notin \sigma_{SBF_+^-}(T \otimes S)$. Since the inclusion $\sigma_w(T)\sigma(S) \cup \sigma_w(S)\sigma(T) \subseteq \sigma_b(T)\sigma(S) \cup \sigma_b(S)\sigma(T) = \sigma_b(T \otimes S)$ is true and since $\sigma_{SBF_+^-}(T) \subseteq \sigma_w(T)$ and $\sigma_{SBF_+^-}(S) \subseteq \sigma_w(S)$, we have $\sigma_{SBF_+^-}(T \otimes S) \subseteq \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma_{SBF_+^-}(S)\sigma(T) \subseteq \sigma_w(T)\sigma(S) \cup \sigma_w(S)\sigma(T) \subseteq \sigma_b(T)\sigma(S) \cup \sigma_b(S)\sigma(T) = \sigma_b(T \otimes S)$. Then the equality $\sigma_{SBF_+^-}(T \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma_{SBF_+^-}(S)\sigma(T)$ follows from Lemma 3.1. Conversely, suppose the equality $\sigma_{SBF_+^-}(T \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma_{SBF_+^-}(S)\sigma(T)$ holds. Since T and S satisfy property (Sb), it follows that

$$\sigma_{SBF_+^-}(T \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S) \cup \sigma_{SBF_+^-}(S)\sigma(T) =$$

$$= \sigma_b(T)\sigma(S) \cup \sigma_b(S)\sigma(T) = \sigma_b(T \otimes S).$$

That is, $T \otimes S$ obeys property (Sb).

Lemma 3.2 is proved.

In [14], Kubrusly and Duggal studied the stability of Weyl's theorem under tensor product in the infinite dimensional space setting. Rashid [19] studied the stability of generalized Weyl's theorem under tensor product in the infinite dimensional Banach space. The following main theorem shows if isoloid operators T and S satisfies property (Sw) and the equality $\sigma_{SBF_+^-}(T\otimes S) = \sigma_{SBF_+^-}(T)\sigma(S)\cup \cup\sigma_{SBF_+^-}(S)\sigma(T)$ holds, then $T\otimes S$ satisfies property (Sw) in the infinite dimensional space setting. Let $\sigma_{PF}(T) = \{\lambda \in \sigma_p(T) \colon \alpha(T-\lambda) < \infty\} = \{\lambda \in \mathbb{C} \colon 0 < \alpha(T-\lambda) < \infty\}$.

Theorem 3.1. *If isoloid operators T and S satisfies property (Sw) and the equality $\sigma_{SBF_+^-}(T\otimes \otimes S) = \sigma_{SBF_+^-}(T)\sigma(S)\cup\sigma_{SBF_+^-}(S)\sigma(T)$ holds, then $T\otimes S$ satisfies property (Sw).*

Proof. Since T and S satisfies property (Sw), T and S satisfies property (Sb) by [20] (Theorem 2.7). Then by the equality hypothesis $\sigma_{SBF_+^-}(T\otimes S) = \sigma_{SBF_+^-}(T)\sigma(S)\cup\sigma_{SBF_+^-}(S)\sigma(T)$, $T\otimes S$ satisfies property (Sb) (see Lemma 3.2). Suppose $T\otimes S$ does not satisfies property (Sw). Then we have the result $\sigma_{SBF_+^-}(T\otimes S)\cap E^0(T\otimes S)\neq\varnothing$.

Since $\sigma_{SBF_+^-}(T\otimes S) = \sigma_{SBF_+^-}(T)\sigma(S)\cup\sigma_{SBF_+^-}(S)\sigma(T)$, we get $\lambda = \mu\upsilon \in \sigma_{SBF_+^-}(T\otimes S)$ if and only if $(\mu,\upsilon)\in\sigma_{SBF_+^-}(T)\sigma(S)$ or $(\mu,\upsilon)\in\sigma_{SBF_+^-}(S)\sigma(T)$. If $\lambda\in E^0(T\otimes S)$, then by applying [14] (Lemma 3), $\lambda\in E^0(T)E^0(S)$. Thus if, $\lambda=\mu\upsilon\in\sigma_{SBF_+^-}(T\otimes S)\cap E^0(T\otimes S)$, then it follows that $0\neq\lambda=\mu\nu=\mu'\nu'$ with $\mu=\dfrac{\lambda}{\nu}\in\sigma_{SBF_+^-}(T)$, $\mu'=\dfrac{\lambda}{\nu'}\in E^0(S)$, $\nu=\dfrac{\lambda}{\mu}\in\sigma_{SBF_+^-}(S)$, $\nu'=\dfrac{\lambda}{\nu'}\in E^0(T)$. Thus, $E^0(T)\neq\varnothing$ and $E^0(S)\neq\varnothing$. Since $\lambda=\mu\nu\in E^0(T\otimes S)$, it follows by [14] (Lemma 5) that $\mu\in\sigma_{\rm iso}(T)$ and $\nu\in\sigma_{\rm iso}(S)$. Since T and S are isoloid, and since $\lambda=\mu\nu\in E^0(T\otimes S)$, it follows that $\mu\in\sigma_{PF}(T)$ and $\nu\in\sigma_{PF}(S)$. Since T and S satisfies property (Sw), $\mu\in\sigma_{SBF_+^-}(T)\cap\sigma_{\rm iso}(T)\cap\sigma_{PF}(T)$ and $\nu\in\sigma_{SBF_+^-}(S)\cap\sigma_{\rm iso}(S)\cap\sigma_{PF}(S)$ which implies that both $\sigma_{SBF_+^-}(T)\cap E^0(T)$ and $\sigma_{SBF_+^-}(S)\cap E^0(S)$ are nonempty. This contradicts the fact that T and S satisfies property (Sw) (see Lemma 3.1).

Theorem 3.1 is proved.

4. Perturbations. Let $[T,S]=TS-ST$ denote the commutator of the operators T and S. If $Q_1\in\mathcal{B}(\mathcal{X})$ and $Q_2\in\mathcal{B}(\mathcal{Y})$ are quasinilpotent operators such that $[Q_1,T]=[Q_2,S]=0$ for some operators $T\in\mathcal{B}(\mathcal{X})$ and $S\in\mathcal{B}(\mathcal{Y})$, then

$$(T+Q_1)\otimes(S+Q_2)=(T\otimes S)+Q,$$

where $Q=Q_1\otimes S+T\otimes Q_2+Q_1\otimes Q_2\in\mathcal{B}(\mathcal{X}\otimes\mathcal{Y})$ is quasinilpotent operator.

Recall that $T\in\mathcal{B}(\mathcal{X})$ is finitely isoloid if $\lambda\in\operatorname{iso}\sigma(T)$ implies $\lambda\in E^0(T)$.

Theorem 4.1. *Let $T\in\mathcal{B}(\mathcal{X})$ and $S\in\mathcal{B}(\mathcal{Y})$ having SVEP and let $Q_1\in\mathcal{B}(\mathcal{X})$ and $Q_2\in\mathcal{B}(\mathcal{Y})$ be quasinilpotent operators such that $[Q_1,T]=[Q_2,S]=0$. If $T\otimes S$ is finitely isoloid, then $T\otimes S$ satisfies property (Sw) implies $(T+Q_1)\otimes(S+Q_2)$ satisfies property (Sw).*

Proof. Recall that $\sigma((T+Q_1)\otimes(S+Q_2))=\sigma(T\otimes S)$, $\sigma_a((T+Q_1)\otimes(S+Q_2))=\sigma_a(T\otimes S)$, $\sigma_{SBF_+^-}((T+Q_1)\otimes(S+Q_2))=\sigma_{SBF_+^-}(T\otimes S)$ and that the perturbation of an operator by a commuting quasinilpotent has SVEP if and only if the operator has SVEP. If $T\otimes S$ satisfies property (Sw), then

$$E^0(T\otimes S)=\sigma(T\otimes S)\setminus\sigma_{SBF_+^-}(T\otimes S)=$$

$$=\sigma((T+Q_1)\otimes(S+Q_2))\setminus\sigma_{SBF_+^-}((T+Q_1)\otimes(S+Q_2)).$$

We prove $E^0(T\otimes S)=E^0((T+Q_1)\otimes(S+Q_2))$. Observe that if $\lambda\in\operatorname{iso}\sigma(T\otimes S)$, then $T^*\otimes S^*$ has SVEP at λ; equivalently, $(T^*+Q_1^*)\otimes(S^*+Q_2^*)$ has SVEP at λ. Let $\lambda\in E^0(T\otimes S)$. Then

$\lambda \in \sigma((T+Q_1) \otimes (S+Q_2)) \setminus \sigma_{SBF_+^-}((T+Q_1) \otimes (S+Q_2))$. Since $(T+Q_1)^* \otimes (S+Q_2)^*$ has SVEP at λ, it follows that $\lambda \notin \sigma_w((T+Q_1) \otimes (S+Q_2))$ and $\lambda \in \operatorname{iso}((T+Q_1) \otimes (S+Q_2))$. Thus $\lambda \in E^0((T+Q_1) \otimes (S+Q_2))$. Hence $E^0(T \otimes S) \subseteq E^0((T+Q_1) \otimes (S+Q_2))$. Conversely, if $\lambda \in E^0((T+Q_1) \otimes (S+Q_2))$, then $\lambda \in \operatorname{iso}(T \otimes S)$, and this, since $T \otimes S$ is finitely isoloid, implies that $\lambda \in E^0(T \otimes S)$. Hence $E^0((T+Q_1) \otimes (S+Q_2)) \subseteq E^0(T \otimes S)$.

Theorem 4.1 is proved.

From [6], we recall that an operator $R \in \mathcal{B}(\mathcal{X})$ is said to be Riesz if $R - \lambda I$ is Fredholm for every non-zero complex number λ.

For a bounded operator T on $\mathcal{X}$, we denote by $E_{0f}(T)$ the set of isolated points λ of $\sigma(T)$ such that $\ker(T - \lambda I)$ is finite-dimensional. Evidently, $E_0(T) \subseteq E_{0f}(T)$.

Lemma 4.1. *Let T be a bounded operator on $\mathcal{X}$. If R is a Riesz operator that commutes with T, then*

$$E^0(T+R) \cap \sigma(T) \subseteq \operatorname{iso} \sigma(T).$$

Proof. Clearly,

$$E^0(T+R) \cap \sigma(T) \subseteq E_{0f}(T+R) \cap \sigma(T)$$

and by Lemma 2.3 of [16] the last set contained in $\operatorname{iso} \sigma(T)$.

Lemma 4.1 is proved.

Now we consider the perturbations by commuting Riesz operators. Let $T, R \in \mathcal{B}(\mathcal{X})$ be such that R is Riesz and $[T, R] = 0$. The tensor product $T \otimes R$ is not a Riesz operator (the Fredholm spectrum $\sigma_F(T \otimes R) = \sigma(T)\sigma_F(R) \cup \sigma_F(T)\sigma(R) = \sigma_F(T)\sigma(R) = \{0\}$ for a particular choice of T only). However, σ_w (also, σ_b) is stable under perturbation by commuting Riesz operators [23], and so T satisfies Browder's theorem if and only if $T + R$ satisfies Browder's theorem. Thus, if $\sigma(T) = \sigma(T+R)$ for a certain choice of operators $T, R \in \mathcal{B}(\mathcal{X})$ (such that R is Riesz and $[T, R] = 0$), then

$$\pi^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T+R) \setminus \sigma_w(T+R) = \pi^0(T+R),$$

where $\pi^0(T)$ is the set of $\lambda \in \operatorname{iso} \sigma(T)$ which are finite rank poles of the resolvent of T. If we now suppose additionally that T satisfies property (Sw), then

$$E^0(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T+R) \setminus \sigma_w(T+R), \tag{4.1}$$

and a necessary and sufficient condition for $T+R$ to satisfy property (Sw) is that $E_a^0(T+R) = E_a^0(T)$. One such condition, namely T is finitely isoloid.

Proposition 4.1. *Let $T, R \in \mathcal{B}(\mathcal{X})$, where R is Riesz, $[T, R] = 0$ and T is finitely isoloid. Then T satisfies property (Sw) implies $T + R$ satisfies property (Sw).*

Proof. Observe that if T obeys property (Sw), then identity (4.1) holds. Let $\lambda \in E^0(T)$. Then it follows from Lemma 4.1 that $\lambda \in E^0(T) \cap \sigma(T) = E^0(T + R - R) \subseteq \operatorname{iso} \sigma(T+R)$ and so $T^* + R^*$ has SVEP at λ. Since $\lambda \in \sigma(T+R) \setminus \sigma_w(T+R)$, $T^* + R^*$ has SVEP at λ implies $T + R - \lambda$ is Fredholm of index 0 and so $\lambda \in E^0(T+R)$. Thus $E^0(T) \subseteq E^0(T+R)$. Now let $\lambda \in E^0(T+R)$. Then $\lambda \in E^0(T+R) \cap \sigma(T+R) = E^0(T+R) \cap \sigma(T) \subseteq \operatorname{iso} \sigma(T)$, which by the finite isoloid property of T implies $\lambda \in E^0(T)$. Hence $E^0(T+R) \subseteq E^0(T)$.

Proposition 4.1 is proved.

Theorem 4.2. *Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{X})$ be finitely isoloid operators which satisfy property (Sw). If $R_1 \in \mathcal{B}(\mathcal{X})$ and $R_2 \in \mathcal{B}(\mathcal{Y})$ are Riesz operators such that $[T, R_1] = [S, R_2] = 0$, $\sigma(T + +R_1) = \sigma(T)$ and $\sigma(S+R_2) = \sigma(S)$, then $T \otimes S$ satisfies property (Sw) implies $(T+R_1)\otimes(S+R_2)$ satisfies property (Sw) if and only if Browder's theorem transforms from $T + R_1$ and $S + R_2$ to their tensor product.*

Proof. The hypotheses imply (by Proposition 4.1) that both $T + R_1$ and $S + R_2$ satisfy property (Sw). Suppose that $T \otimes B$ satisfies property (Sw). Then $\sigma(T \otimes B) \setminus \sigma_{SBF_+^-}(T \otimes S) = E^0(T \otimes S)$. Evidently $T \otimes B$ satisfies Browder's theorem, and so the hypothesis T and B satisfy property (Sw) implies that Browder's theorem transfers from T and S to $T \otimes S$. Furthermore, since, $\sigma(T + R_1) = = \sigma(T)$, $\sigma(S + R_2) = \sigma(S)$, and σ_w is stable under perturbations by commuting Riesz operators,

$$\sigma_{SBF_+^-}(T \otimes S) = \sigma_w(T \otimes S) = \sigma(T)\sigma_w(S) \cup \sigma_w(T)\sigma(S) =$$
$$= \sigma(T + R_1)\sigma_w(S + R_2) \cup \sigma_w(T + R_1)\sigma(S + R_2) =$$
$$= \sigma(T + R_1)\sigma_{SBF_+^-}(S + R_2) \cup \sigma_{SBF_+^-}(T + R_1)\sigma(S + R_2).$$

Suppose now that Browder's theorem transfers from $T + R_1$ and $S + R_2$ to $(T + R_1) \otimes (S + R_2)$. Then

$$\sigma_w(T \otimes S) = \sigma_w((T + R_1) \otimes (S + R_2))$$

and

$$E^0(T \otimes S) = \sigma((T + R_1) \otimes (S + R_2)) \setminus \sigma_w((T + R_1) \otimes (S + R_2)).$$

Let $\lambda \in E^0(T \otimes S)$. Then $\lambda \neq 0$, and hence there exist $\mu \in \sigma(T + R_1) \setminus \sigma_w(T + R_1)$ and $\nu \in \sigma(S + R_2) \setminus \sigma_w(S + R_2)$ such that $\lambda = \mu\nu$. As observed above, both $T + R_1$ and $S + R_2$ satisfy property (Sw); hence $\mu \in E^0(S + R_1)$ and $\nu \in E^0(S + R_2)$. This, since $\lambda \in \sigma(T \otimes S) = \sigma((T + +R_1)\otimes(S+R_2))$, implies $\lambda \in E^0((T+R_1)\otimes(S+R_2))$. Conversely, if $\lambda \in E^0((T+R_1)\otimes(S+R_2))$, then $\lambda \neq 0$ and there exist $\mu \in E^0(T + R_1) \subseteq \operatorname{iso}\sigma(T)$ and $\nu \in E^0(S + R_2) \subseteq \operatorname{iso}\sigma(S)$ such that $\lambda = \mu\nu$. Recall that $E^0((T + R_1) \otimes (S + R_2)) \subseteq E^0(T + R_1)E^0(S + R_2)$. Since T and S are finite isoloid, $\mu \in E^0(T)$ and $\nu \in E^0(S)$. Hence, since $\sigma((T + R_1) \otimes (S + R_2)) = \sigma(T \otimes S)$, $\lambda = \mu\nu \in E^0(T \otimes S)$. To complete the proof, we observe that if the implication of the statement of the theorem holds, then (necessarily) $(T + R_1) \otimes (S + R_2)$ satisfies Browder's theorem. This, since $T + R_1$ and $S + R_2$ satisfy Browder's theorem, implies Browder's theorem transfers from $T + R_1$ and $S + R_2$ to $(T + R_1) \otimes (S + R_2)$.

Theorem 4.2 is proved.

5. Property (Sw) for direct sum. Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional Hilbert spaces. In this section we show that if T and S are two operators on $\mathcal{H}$ and $\mathcal{K}$ respectively and at least one of them satisfies property (Sw) then their direct sum $T \oplus S$ obeys property (Sw). We also explore various conditions on T and S to ensure that $T \oplus S$ satisfies property (Sw).

Theorem 5.1. *Suppose that property (Sw) holds for $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$. If T and S are isoloid and $\sigma_{SBF_+^-}(T \oplus S) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)$, then property (Sw) holds for $T \oplus S$.*

Proof. We know that $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ for any pairs of operators. If T and S are isoloid, then

$$E^0(T \oplus S) = \left[E^0(T) \cap \rho(S)\right] \cup \left[\rho(T) \cap E^0(S)\right] \cup \left[E^0(T) \cap E^0(S)\right],$$

where $\rho(.) = \mathbb{C} \setminus \sigma(.)$.

If property (Sw) holds for T and S, then

$$[\sigma(T)\cup\sigma(S)]\setminus\left[\sigma_{SBF_+^-}(T)\cup\sigma_{SBF_+^-}(S)\right]=$$
$$=\left[E^0(T)\cap\rho(S)\right]\cup\left[\rho(T)\cap E^0(S)\right]\cup\left[E^0(T)\cap E^0(S)\right].$$

Thus, $E^0(T\oplus S)=[\sigma(T)\cup\sigma(S)]\setminus\left[\sigma_{SBF_+^-}(T)\cup\sigma_{SBF_+^-}(S)\right]$.

If $\sigma_{SBF_+^-}(T\oplus S)=\sigma_{SBF_+^-}(T)\cup\sigma_{SBF_+^-}(S)$, then

$$E^0(T\oplus S)=\sigma(T\oplus S)\setminus\sigma_{SBF_+^-}(T\oplus S).$$

Hence property (Sw) holds for $T\oplus S$.

Theorem 5.1 is proved.

Theorem 5.2. *Suppose that $T\in\mathcal{B}(\mathcal{H})$ such that $\operatorname{iso}\sigma(T)=\varnothing$ and $S\in\mathcal{B}(\mathcal{K})$ satisfies property (Sw). If $\sigma_{SBF_+^-}(T\oplus S)=\sigma(T)\cup\sigma_{SBF_+^-}(S)$, then property (Sw) holds for $T\oplus S$.*

Proof. We know that $\sigma(T\oplus S)=\sigma(T)\cup\sigma(S)$ for any pairs of operators. Then

$$\sigma(T\oplus S)\setminus\sigma_{SBF_+^-}(T\oplus S)=[\sigma(T)\cup\sigma(S)]\setminus\left[\sigma(T)\cup\sigma_{SBF_+^-}(S)\right]=$$
$$=\sigma(S)\setminus\left[\sigma(T)\cup\sigma_{SBF_+^-}(S)\right]=$$
$$=\left[\sigma(S)\setminus\sigma_{SBF_+^-}(S)\right]\setminus\sigma(T)=E^0(S)\cap\rho(T).$$

If $\operatorname{iso}\sigma(T)=\varnothing$ it implies that $\sigma(T)=\operatorname{acc}\sigma(T)$, where $\operatorname{acc}\sigma(T)=\sigma(T)\setminus\operatorname{iso}\sigma(T)$ is the set of all accumulation points of $\sigma(T)$. Thus we have

$$\operatorname{iso}\sigma(T\oplus S)=[\operatorname{iso}\sigma(T)\cup\operatorname{iso}\sigma(S)]\setminus[(\operatorname{iso}\sigma(T)\cap\operatorname{acc}\sigma(S))\cup(\operatorname{acc}\sigma(T)\cap\operatorname{iso}\sigma(S))]=$$
$$=[\operatorname{iso}\sigma(T)\setminus\operatorname{acc}\sigma(S)]\cup[\operatorname{iso}\sigma(S)\setminus\operatorname{acc}\sigma(T)]=\operatorname{iso}\sigma(S)\setminus\sigma(T)=\operatorname{iso}\sigma(S)\cap\rho(T).$$

We know that $\sigma_p(T\oplus S)=\sigma_p(T)\cup\sigma_p(S)$ and $\alpha(T\oplus S)=\alpha(T)+\alpha(S)$ for any pairs of operators T and S, so that

$$\sigma_{PF}(T\oplus S)=\{\lambda\in\sigma_{PF}(T)\cup\sigma_{PF}(S)\alpha(T-\lambda I)+\alpha(S-\lambda I)<\infty\}.$$

Therefore,

$$E^0(T\oplus S)=\operatorname{iso}\sigma(T\oplus S)\cap\sigma_{PF}(T\oplus S)=\operatorname{iso}\sigma(S)\cap\rho(T)\cap\sigma_{PF}(S)=E^0(S)\cap\rho(T).$$

Thus $\sigma(T\oplus S)\setminus\sigma_{SBF_+^-}(T\oplus S)=E^0(T\oplus S)$. Hence $T\oplus S$ satisfies property (Sw).

Theorem 5.2 is proved.

***Corollary* 5.1.** *Suppose that $T\in\mathcal{B}(\mathcal{H})$ is such that $\operatorname{iso}\sigma(T)=\varnothing$ and $S\in\mathcal{B}(\mathcal{H})$ satisfies property (Sw) with $\operatorname{iso}\sigma(S)\cap\sigma_p(S)=\varnothing$, and $\Delta_a^g(T\oplus S)=\varnothing$, then $T\oplus S$ satisfies property (Sw).*

Proof. Since S satisfies property (Sw), therefore given condition $\operatorname{iso}\sigma(S)\cap\sigma_p(S)=\varnothing$ implies that $\sigma(S)=\sigma_{SBF_+^-}(S)$. Now $\Delta_a^g(T\oplus S)=\varnothing$ gives that $\sigma_{SBF_+^-}(T\oplus S)=\sigma(T\oplus S)=\sigma(T)\cup$ $\cup\,\sigma_{SBF_+^-}(S)$. Thus from Theorem 5.2, we have that $T\oplus S$ satisfies property (Sw).

***Corollary* 5.2.** *Suppose that* $T \in \mathcal{B}(\mathcal{H})$ *is such that* $\mathrm{iso}\,\sigma(T) \cup \Delta_a^g(T) = \varnothing$ *and* $S \in \mathcal{B}(\mathcal{K})$ *satisfies property* (Sw). *If* $\sigma_{SBF_+^-}(T \oplus S) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)$, *then* $T \oplus S$ *satisfies property* (Sw).

Theorem 5.3. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be an isoloid operator that satisfies property* (Sw). *If* $S \in \mathcal{B}(\mathcal{K})$ *is a normal operator satisfies property* (Sw), *then property* (Sw) *holds for* $T \oplus S$.

Proof. If S is normal, then both S and S^* have SVEP, and $\mathrm{ind}\,(S - \lambda I) = 0$ for every λ such that $S - \lambda I$ is a B-Fredholm. Observe that $\lambda \notin \sigma_{SBF_+^-}(T \oplus S)$ if and only if $S - \lambda I \in SBF_+(K)$ and $T - \lambda I \in SBF_+(H)$ and $\mathrm{ind}\,(T - \lambda I) + \mathrm{ind}\,(S - \lambda I) = \mathrm{ind}\,(T - \lambda I) \leq 0$ if and only if $\lambda \notin \Delta_a^g(T) \cap \Delta_a^g(S)$. Hence $\sigma_{SBF_+^-}(T \oplus S) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)$. It is well known that the isolated points of the approximate point spectrum of a normal operator are simple poles of the resolvent of the operator implies that S is isoloid. So the result follows now from Theorem 5.1.

Theorem 5.3 is proved.

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