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**THE CLASSICAL M. A. BUHL PROBLEM, ITS PFEIFFER – SATO SOLUTIONS
 AND THE CLASSICAL LAGRANGE – D’ALAMBERT PRINCIPLE
 FOR THE INTEGRABLE HEAVENLY TYPE NONLINEAR EQUATIONS**

**КЛАСИЧНА ЗАДАЧА М. А. БУЛЯ, ЇЇ РОЗВ’ЯЗКИ ПФАЙФЕРА – САТО
 І КЛАСИЧНИЙ ПРИНЦИП ЛАГРАНЖА–ДАЛАМБЕРА ДЛЯ ІНТЕГРОВНИХ
 НЕЛІНІЙНИХ РІВНЯНЬ НЕБЕСНОГО ТИПУ**

The survey is devoted to old and recent investigations of the classical M. A. Buhl problem of description of the compatible linear vector field equations and their general M. G. Pfeiffer and modern Lax – Sato-type special solutions. In particular, we analyze the related Lie-algebraic structures and the properties of integrability for a very interesting class of nonlinear dynamical systems called the dispersion-free heavenly type equations, which were introduced by Plebański and later analyzed in a series of articles. The AKS-algebraic and related \mathcal{R} -structure schemes are used to study the orbits of the corresponding coadjoint actions, which are intimately connected with the classical Lie – Poisson structures on them. It is shown that their compatibility condition coincides with the corresponding heavenly type equations under consideration. It is also demonstrated that all these equations are originated in this way and can be represented as a Lax compatibility condition for specially constructed loop vector fields on the torus. The infinite hierarchy of conservations laws related to the heavenly equations is described and its analytic structure connected with the Casimir invariants is indicated. In addition, we present typical examples of equations of this kind demonstrating in detail their integrability via the scheme proposed in the paper. The relationship between a very interesting Lagrange – d’Alembert-type mechanical interpretation of the devised integrability scheme and the Lax – Sato equations is also discussed.

Запропонований огляд присвячено старим та новим дослідженням класичної проблеми М. А. Буля про опис узгоджених лінійних рівнянь для функцій векторного поля та їх загальних розв’язків М. Г. Пфайфера та нових частинних розв’язків типу Лакса і Сато. Зокрема, проаналізовано відповідні алгебраїчні структури Лі та властивості інтегровності для дуже цікавого класу нелінійних димамічних систем, що називаються бездисперсійними рівняннями небесного типу, які були введені Плебанським і пізніше проаналізовані в серії статей. АКС-алгебраїчні та споріднені \mathcal{R} -структурні схеми використовуються для вивчення орбіт відповідних співспряжених операторів, тісно пов’язаних з класичними структурами Лі – Пуасона на них. Показано, що умова узгодженості для них співпадає з відповідними рівняннями небесного типу, що аналізуються. Також показано, що всі ці рівняння з’являються таким чином і їх можна представити у вигляді умови узгодженості Лакса для спеціально побудованих петельних (петльових ?) полів на торі. Описана нескінченна ієрархія законів збереження, пов’язаних з небесними рівняннями та вказана її аналітична структура, що пов’язана з інваріантами Казіміра. Крім того, наведено типові приклади таких рівнянь, що детально демонструють їх інтегровність в рамках запропонованої схеми. Також обговорюється співвідношення між дуже цікавою механічною інтерпретацією Лагранж – Даламбера запропонованої схеми інтегровності та рівняннями Лакса і Сато.

1. Introduction. In the classical works [12 – 14] still in 1928 the French mathematician M. A. Buhl posed the problem of classifying all infinitesimal symmetries of a given linear vector field equation

$$A\psi = 0, \quad (1.1)$$

where function $\psi \in C^2(\mathbb{R}^n; \mathbb{R})$, and

$$A := \sum_{j=1, n} a_j(x) \frac{\partial}{\partial x_j} \quad (1.2)$$

is a vector field operator on \mathbb{R}^n with coefficients $a_j \in C^1(\mathbb{R}^n; \mathbb{R})$, $j = \overline{1, n}$. It is easy to show that the problem under regard is reduced [49] to describing all possible vector fields

$$A^{(k)} := \sum_{j=\overline{1, n}} a_j^{(k)}(x) \frac{\partial}{\partial x_j} \tag{1.3}$$

with coefficients $a_j^{(k)} \in C^1(\mathbb{R}^n; \mathbb{R})$, $j, k = \overline{1, n}$, satisfying the Lax type commutator condition

$$[A, A^{(k)}] = 0 \tag{1.4}$$

for all $x \in \mathbb{R}^n$ and $k = \overline{1, n}$. The M. A. Buhl problem above was completely solved in 1931 by the Ukrainian mathematician G. Pfeiffer in the works [46–51], where he has constructed explicitly the searched set of independent vector fields (1.3), having made use effectively of the full set of invariants for the vector field (1.2) and the related solution set structure of the Jacobi–Mayer system of equations, naturally following from (1.4). Some results, yet not complete, were also obtained by C. Popovici in [53].

Some years ago the M. A. Buhl type equivalent problem was independently reanalyzed once more by Japanese mathematicians K. Takasaki and T. Takebe [67, 68] and later by L. V. Bogdanov, V. S. Dryuma and S. V. Manakov [9] for a very special case when the vector field operator (1.2) depends analytically on a “spectral” parameter $\lambda \in \mathbb{C}$:

$$\tilde{A} := \frac{\partial}{\partial t} + \sum_{j=\overline{1, n}} a_j(t, x; \lambda) \frac{\partial}{\partial x_j} + a_0(t, x; \lambda) \frac{\partial}{\partial \lambda}. \tag{1.5}$$

Based on the before developed Sato theory [58, 59], the authors mentioned above have shown for some special kinds of vector fields (1.5) that there exists an infinite hierarchy of the symmetry vector fields

$$\tilde{A}^{(k)} := \frac{\partial}{\partial \tau_k} + \sum_{j=\overline{1, n}} a_j^{(k)}(\tau, x; \lambda) \frac{\partial}{\partial x_j} + a_0^{(k)}(\tau, x; \lambda) \frac{\partial}{\partial \lambda}, \tag{1.6}$$

where $\tau = (t; \tau_1, \tau_2, \dots) \in \mathbb{R}^{\mathbb{Z}_+}$, $k \in \mathbb{Z}_+$, satisfying the Lax–Sato type compatible commutator conditions

$$[\tilde{A}, \tilde{A}^{(k)}] = 0 = [\tilde{A}^{(j)}, \tilde{A}^{(k)}] \tag{1.7}$$

for all $k, j \in \mathbb{Z}_+$. Moreover, in the cases under regard, the compatibility conditions (1.7) proved to be equivalent to some very important for applications heavenly type dispersionless equations in partial derivatives.

In the present work we investigate the Lax–Sato compatible systems, the related Lie-algebraic structures and complete integrability properties of an interesting class of nonlinear dynamical systems called the heavenly type equations, which were introduced by Plebański [52] and analyzed in a series of articles [9, 32, 41–43, 60, 60, 67, 68]. In our work, having employed the AKS-algebraic and related \mathcal{R} -structure schemes [5–7, 56, 57, 69], applied to the holomorphic loop Lie algebra $\tilde{\mathcal{G}} := \widehat{\text{diff}}(\mathbb{T}^n)$ of vector fields on torus \mathbb{T}^n , $n \in \mathbb{Z}_+$, the orbits of the corresponding coadjoint actions on $\tilde{\mathcal{G}}^*$,

closely related to the classical Lie–Poisson type structures, were reanalyzed and studied in detail. By constructing two commuting flows on the coadjoint space $\tilde{\mathcal{G}}^*$, generated by a chosen root element $\tilde{l} \in \tilde{\mathcal{G}}^*$ and some Casimir invariants, we have successively demonstrated that their compatibility condition coincides exactly with the corresponding heavenly equations under consideration.

As a by-product of the construction, devised recently in works [25, 55], we prove that all the heavenly equations have a similar origin and can be represented as a Lax compatibility condition for special loop vector fields on the torus \mathbb{T}^n . We analyze the structure of the infinite hierarchy of conservation laws, related to the heavenly equations, and demonstrate their analytical structure connected with the Casimir invariants is generated by the Lie–Poisson structure on $\tilde{\mathcal{G}}^*$. Moreover, we have extended the initial Lie-algebraic structure for the case when the basic Lie algebra $\tilde{\mathcal{G}} = \widehat{\text{diff}}(\mathbb{T}^n)$ is replaced by the adjacent holomorphic Lie algebra $\bar{\mathcal{G}} := \text{diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n) \subset \text{diff}(\mathbb{C} \times \mathbb{T}^n)$ of vector fields on $\mathbb{C} \times \mathbb{T}^n$. Typical examples are presented for all cases of the heavenly equations and it is shown in detail and their integrability is demonstrated using the scheme devised here. This scheme makes it possible to construct a very natural derivation of well known Lax–Sato representation for an infinite hierarchy of heavenly equations, related to the canonical Lie–Poisson structure on the adjoint space $\bar{\mathcal{G}}^*$. We also briefly discuss the Lagrangian representation of these equations following from their Hamiltonicity with respect to both intimately related commuting evolutionary flows, and the related bi-Hamiltonian structure as well as the Bäcklund transformations. As a matter of fact, there are only a few examples of multidimensional integrable systems for which such a detailed description of their mathematical structure has been given. As was aptly mentioned in [66], the heavenly equations comprise an important class of such integrable systems. This is due in part to the fact that some of them are obtained by a reduction of the Einstein equations with Euclidean (and neutral) signature for (anti-) self-dual gravity, which includes the theory of gravitational instantons. This and other cases of important applications of multidimensional integrable equations strongly motivated us to study this class of equations and the related mathematical structures. As a very interesting aspect of our approach to describing integrability of the heavenly dynamical systems, there is a very interesting Lagrange–d’Alembert type mechanical interpretation. We need to underline here that the main motivating idea behind this work was based both on the paper by P. P. Kulish [29], devoted to studying the superconformal Korteweg–de Vries equation as an integrable Hamiltonian flow on the adjoint space to the holomorphic loop Lie superalgebra of superconformal vector fields on the circle, and on the insightful investigation by V. G. Mikhalev [37], which studied Hamiltonian structures on the adjoint space to the holomorphic loop Lie algebra of smooth vector fields on the circle. We were also impressed by deep technical results [67, 68] of Takasaki and Takebe, who fully realized the vector field scheme of the Lax–Sato theory. Additionally, we were strongly influenced both by the works of M. V. Pavlov, L. V. Bogdanov, V. S. Dryuma, B. G. Konopelchenko and S. V. Manakov [9–11, 26], as well as by the work of E. V. Ferapontov and J. Moss [22], in which they devised new effective differential-geometric and analytical methods for studying an integrable degenerate multidimensional dispersionless heavenly type hierarchy of equations, the mathematical importance of which is still far from being properly appreciated. Concerning other Lie-algebraic approaches to constructing integrable heavenly equations, we mention

work by B. Szablikowski and A. Sergyeyev [62, 63], V. Ovsienko [41, 42] and by B. Kruglikov and O. Morozov [28].

We present interesting examples of the Lie-algebraic description of typical integrable heavenly equations amongst which the Mikhalev–Pavlov equation [37], the first and second reduced Shabat type [2] and Hirota [19] heavenly equations, the Liouville type [10] equations and some other.

We also generalized the Lie-algebraic scheme of [25, 55] subject to the loop Lie algebra $\widetilde{\text{diff}}(\mathbb{S}^{1|N})$ of superconformal vector fields on $\mathbb{S}^{1|N}$, being a Lie algebra of the Lie group of superconformal diffeomorphisms of the $1|N$ -dimensional supertorus $\mathbb{S}^{1|N} \simeq \mathbb{S}^1 \times \Lambda_1^N$, where $\Lambda := \Lambda_0 \oplus \Lambda_1$ is an infinite-dimensional Grassmann algebra over \mathbb{C} , $\Lambda_0 \supset \mathbb{C}$. It is applied to constructing the Lax–Sato integrable superanalogs of the Mikhalev–Pavlov heavenly type superequation for every $N \in \mathbb{N} \setminus \{4, 5\}$. As a result of suitably chosen superconformal mappings in the space of variables $(z; \theta_1, \dots, \theta_N) \in \mathbb{S}^{1|N}$ the superanalogs of Liouville type equations are obtained by means of using the loop Lie superalgebra $\widetilde{\text{diff}}(\mathbb{S}^{1|N})$. Some results are also presented for a special Lie-algebraic integrability scheme based on a metrized loop Lie algebra, generated by a semisimple sum of the holomorphic Lie algebra $\bar{\mathcal{G}} = \text{diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$ and its coadjoint space $\bar{\mathcal{G}}^*$.

2. The Lax–Sato type compatible systems of linear vector field equations. *2.1. A vector field on the torus and its invariants.* Consider a simple vector field $X : \mathbb{R} \times \mathbb{T}^n \rightarrow T(\mathbb{R} \times \mathbb{T}^n)$ on the $(n + 1)$ -dimensional toroidal manifold $\mathbb{R} \times \mathbb{T}^n$ for arbitrary $n \in \mathbb{Z}_+$, which we will write in the slightly special form

$$A = \frac{\partial}{\partial t} + \left\langle a(t, x), \frac{\partial}{\partial x} \right\rangle, \tag{2.1}$$

where $(t, x) \in \mathbb{R} \times \mathbb{T}^n$, $a(t, x) \in \mathbb{E}^n$, $\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)^\top$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product on the Euclidean space \mathbb{E}^n . With the vector field (2.1), one can associate the linear equation

$$A\psi = 0 \tag{2.2}$$

for some function $\psi \in C^2(\mathbb{R} \times \mathbb{T}^n; \mathbb{R})$, which we will call an “invariant” of the vector field.

Next, we study the existence and number of such functionally-independent invariants to the equation (2.2). For this let us pose the following Cauchy problem for equation (2.2): find a function $\psi \in C^2(\mathbb{R} \times \mathbb{T}^n; \mathbb{R})$, which at point $t^{(0)} \in \mathbb{R}$ satisfies the condition $\psi(t, x)|_{t=t^{(0)}} = \bar{\psi}(x)$, $x \in \mathbb{R}^n$, for a given function $\bar{\psi} \in C^2(\mathbb{T}^n; \mathbb{R})$. For the equation (2.2) there is naturally related parametric vector field on the torus \mathbb{T}^n in the form of the ordinary vector differential equation

$$dx/dt = a(t, x), \tag{2.3}$$

to which there corresponds the following Cauchy problem: find a function $x : \mathbb{R} \rightarrow \mathbb{T}^n$ satisfying

$$x(t)|_{t=t^{(0)}} = z \tag{2.4}$$

for an arbitrary constant vector $z \in \mathbb{T}^n$. Assuming that the vector-function $a \in C^1(\mathbb{R} \times \mathbb{T}^n; \mathbb{R}^n)$, it follows from the classical Cauchy theorem [15] on the existence and unicity of the solution to (2.3) and (2.4), that we can obtain a unique solution to the vector equation (2.3) as some function $\Phi \in C^1(\mathbb{R} \times \mathbb{T}^n; \mathbb{T}^n)$, $x = \Phi(t, z)$, such that the matrix $\partial\Phi(t, z)/\partial z$ is nondegenerate for all $t \in \mathbb{R}$ sufficiently close to $t^{(0)} \in \mathbb{R}$. Hence, the Implicit Function theorem [15, 16] implies that there exists

a mapping $\Psi : \mathbb{R} \times \mathbb{T}^n \rightarrow \mathbb{T}^n$, such that

$$\Psi(t, x) = z \quad (2.5)$$

for every $z \in \mathbb{T}^n$ and all $t \in \mathbb{R}$ sufficiently enough to $t^{(0)} \in \mathbb{R}$. Supposing now that the functional vector $\Psi(t, x) = (\psi^{(1)}(t, x), \psi^{(2)}(t, x), \dots, \psi^{(n)}(t, x))^T$, $(t, x) \in \mathbb{R} \times \mathbb{T}^n$, is constructed, from the arbitrariness of the parameter $z \in \mathbb{T}^n$ one can deduce that all functions $\psi^{(j)} : \mathbb{R} \times \mathbb{T}^n \rightarrow \mathbb{R}$, $j = \overline{1, n}$, are functionally independent invariants of the vector field equation (2.2), that is $A\psi^{(j)} = 0$, $j = \overline{1, n}$. Thus, the vector field equation (2.2) has exactly $n \in \mathbb{Z}_+$ functionally independent invariants, which make it possible, in particular, to solve the Cauchy problem posed above. Namely, let a mapping $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ be chosen such that $\alpha(\Psi(t, x))|_{t=t^{(0)}} = \bar{\psi}(x)$ for all $x \in \mathbb{R}^n$ and a fixed $t^{(0)} \in \mathbb{R}$. Inasmuch as the superposition of functions $\alpha \circ \Psi : \mathbb{R} \times \mathbb{T}^n \rightarrow \mathbb{R}$ is, evidently, also an invariant for the equation (2.2), it provides the solution to this Cauchy problem, which we can formulate as the following result.

Proposition 2.1. *The linear equation (2.2), generated by the vector field (2.3) on the toroidal manifold $\mathbb{R} \times \mathbb{T}^n$, has exactly $n \in \mathbb{Z}_+$ functionally independent invariants.*

Consider now a differential form $\chi^{(n)} \in \Lambda^n(\mathbb{T}^n)$, generated by the vector of independent invariants (2.5), additionally depending parametrically on the vector evolution parameter $t \in \mathbb{R}^n$:

$$\chi^{(n)} := d\psi^{(1)} \wedge d\psi^{(2)} \wedge \dots \wedge d\psi^{(n)}, \quad (2.6)$$

where, by definition, for any $\psi \in C^2(\mathbb{R}^n \times \mathbb{T}^n; \mathbb{R})$ the differential

$$d\psi := \left\langle \frac{\partial \psi}{\partial x}, dx \right\rangle. \quad (2.7)$$

As follows from the Frobenius theorem [15, 24, 26], the Plucker type form (2.6) is for $t \in \mathbb{R}^n$ nonzero on the torus \mathbb{T}^n owing to the functional independence of the invariants. It is easy to see that the following [51] Jacobi – Mayer type relationship:

$$\left| \frac{\partial \Psi}{\partial x} \right|^{-1} d\psi^{(1)} \wedge d\psi^{(2)} \wedge \dots \wedge d\psi^{(n)} = dx_1 \wedge dx_2 \wedge \dots \wedge dx_n, \quad (2.8)$$

holds on the manifold \mathbb{T}^n , where on the right-hand side one has the volume measure on the torus \mathbb{T}^n , which is naturally dependent on $t \in \mathbb{R}^n$ owing to the vector field relationships (2.3). Taking into account that for all invariants $\psi^{(k)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$, $k = \overline{1, n}$, there hold the differential expressions

$$d\psi^{(k)} = - \sum_{s=\overline{1, n}} \frac{\partial \psi^{(k)}}{\partial t_s} dt_s,$$

their substitution into (2.8) gives rise, owing to the independence of the differentials dt_s , $s = \overline{1, n}$, to the following set of the compatible vector field relationships:

$$\frac{\partial \Psi}{\partial t_s} + \sum_{j,k=\overline{1, n}} \left[\left(\frac{\partial \Psi}{\partial x} \right)^{-1}_{jk} \frac{\partial \psi^{(k)}}{\partial t_s} \right] \frac{\partial \Psi}{\partial x_j} = 0$$

for any $s = \overline{1, n}$. The latter property, as it was demonstrated by M. G. Pfeiffer in [51], makes it possible to solve effectively the M. A. Buhl problem and has interesting applications [9, 26] in the

theory of completely integrable dynamical systems of heavenly type, which are considered in the next section.

2.2. Vector field hierarchies on the torus with “spectral” parameter and the Lax–Sato integrable heavenly dynamical systems. Consider some naturally ordered infinite set of parametric vector fields (2.1) on the infinite dimensional toroidal manifold $\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n$ in the form

$$A^{(k)} = \frac{\partial}{\partial t_k} + \left\langle a^{(k)}(t, x; \lambda), \frac{\partial}{\partial x} \right\rangle + a_0^{(k)}(t, x; \lambda) \frac{\partial}{\partial \lambda} := \frac{\partial}{\partial t_k} + A^{(k)}, \tag{2.9}$$

where $t_k \in \mathbb{R}$, $k \in \mathbb{Z}_+$, $(t, x; \lambda) \in (\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n) \times \mathbb{C}$ are the evolution parameters, and the dependence of smooth vectors $(a_0^{(k)}, a^{(k)})^\top \in \mathbb{E} \times \mathbb{E}^n$, $k \in \mathbb{Z}_+$, on the “spectral” parameter $\lambda \in \mathbb{C}$ is assumed to be holomorphic. Suppose now that the infinite hierarchy of linear equations

$$A^{(k)}\psi = 0 \tag{2.10}$$

for $k \in \mathbb{Z}_+$ has exactly $n + 1 \in \mathbb{Z}_+$ common functionally independent invariants $\psi^{(j)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C})$, $j = \overline{0, n}$, on the torus \mathbb{T}^n , suitably depending on the parameter $\lambda \in \mathbb{C}$. Then, owing to the existence theory [15, 16] for ordinary differential equations depending on the “spectral” parameter $\lambda \in \mathbb{C}$, these invariants may be assumed to be such that allow analytical continuation in the parameter $\lambda \in \mathbb{C}$ both inside $\mathbb{S}_+^1 \subset \mathbb{C}$ of some circle $\mathbb{S}^1 \subset \mathbb{C}$ and subject to the parameter $\lambda^{-1} \in \mathbb{C}$, $|\lambda| \rightarrow \infty$, outside $\mathbb{S}_-^1 \subset \mathbb{C}$ of this circle $\mathbb{S}^1 \subset \mathbb{C}$. This means that as $|\lambda| \rightarrow \infty$ we have the following expansions:

$$\begin{aligned} \psi^{(0)}(\lambda) &\sim \lambda + \sum_{k=0}^{\infty} \psi_k^{(0)}(t, x) \lambda^{-k}, \\ \psi^{(1)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau_k^{(1)}(t, x) \psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(1)}(t, x) \psi_0(\lambda)^{-k}, \\ \psi^{(2)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau_k^{(2)}(t, x) \psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(2)}(t, x) \psi_0(\lambda)^{-k}, \\ &\dots\dots\dots \\ \psi^{(n)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau_k^{(n)}(t, x) \psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(n)}(t, x) \psi_0(\lambda)^{-k}, \end{aligned}$$

where we took into account that $\psi^{(0)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C})$, $\lambda \in \mathbb{C}$, is the basic invariant solution to the equations (2.10), the functions $\tau_l^{(s)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ for all $s = \overline{1, n}$, $l \in \mathbb{Z}_+$, are assumed to be independent and $\psi_k^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ for all $k \in \mathbb{N}$, $j = \overline{0, n}$, are arbitrary. Write down now the condition (2.8) on the manifold $\mathbb{C} \times \mathbb{T}^n$ in the form $\lambda \in \mathbb{C}$

$$\left| \frac{\partial \Psi}{\partial \mathbf{x}} \right|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \wedge \dots \wedge d\psi^{(n)} = d\lambda \wedge dx_1 \wedge dx_2 \wedge \dots \wedge dx_n, \tag{2.11}$$

where $\mathbf{x} := (\lambda, x) \in \mathbb{C} \times \mathbb{T}^n$, $\left| \frac{\partial \Psi}{\partial \mathbf{x}} \right|$ is the Jacobi determinant of the mapping $\Psi := (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n)})^\top \in C^2(\mathbb{C} \times (\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n); \mathbb{C}^{n+1})$ on the manifold $\mathbb{C} \times \mathbb{T}^n$. Inasmuch this mapping subject

to the parameter $\lambda \in \mathbb{C}$ has analytical continuation [16] inside $\mathbb{S}_+^1 \subset \mathbb{C}$ of the circle $\mathbb{S}^1 \subset \mathbb{C}$ and subject to the parameter $\lambda^{-1} \in \mathbb{C}$ as $|\lambda| \rightarrow \infty$ outside $\mathbb{S}_-^1 \subset \mathbb{C}$ of this circle $\mathbb{S}^1 \subset \mathbb{C}$, one can easily obtain from the vanishing differential expressions

$$d\psi^{(j)} = d\psi^{(j)} + \sum_{k=0}^{\infty} \frac{\partial \psi^{(j)}}{\partial \tau_k^{(j)}} d\tau_k^{(j)} = 0 \tag{2.12}$$

for all $j = \overline{1, n}$ and the relationship (2.11) on the manifold $\mathbb{C} \times \mathbb{T}^n$ of the independent variables $x \in \mathbb{C} \times \mathbb{T}^n$, evolving analytically with respect to the parameters $\tau_k^{(j)} \in \mathbb{R}$, $j = \overline{1, n}$, $k \in \mathbb{Z}_+$, the following Lax – Sato criterion:

$$\left(\left| \frac{\partial \Psi}{\partial x} \right|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \wedge \dots \wedge d\psi^{(n)} \right)_- = 0, \tag{2.13}$$

where $(\dots)_-$ means the asymptotic part of an expression in the bracket, depending on the parameter $\lambda^{-1} \in \mathbb{C}$ as $|\lambda| \rightarrow \infty$. The substitution of expressions (2.12) into (2.13) easily yields

$$\frac{\partial \Psi}{\partial \tau_k^{(j)}} = \left[\left(\frac{\partial \Psi}{\partial x} \right)_{0j}^{-1} \psi^{(0)}(\lambda)^k \right]_+ \frac{\partial \Psi}{\partial \lambda} + \sum_{s=1}^n \left[\left(\frac{\partial \Psi}{\partial x} \right)_{sj}^{-1} \psi^{(0)}(\lambda)^k \right]_+ \frac{\partial \Psi}{\partial x_s} \tag{2.14}$$

for all $k \in \mathbb{Z}_+$, $j = \overline{1, n}$. These relationships (2.14) comprise an infinite hierarchy of Lax – Sato compatible [67, 68] linear equations, where $(\dots)_+$ denotes the asymptotic part of an expression in the bracket, depending on nonnegative powers of the complex parameter $\lambda \in \mathbb{C}$. As for the independent functional parameters $\tau_k^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ for all $k \in \mathbb{Z}_+$, $j = \overline{1, n}$, one can state their functional independence by taking into account their *a priori* linear dependence on the independent evolution parameters $t_k \in \mathbb{R}$, $k \in \mathbb{Z}_+$. On the other hand, taking into account the explicit form of the hierarchy of equations (2.14), following [9], it is not hard to show that the corresponding vector fields

$$A_k^{(j)} := \left[\left(\frac{\partial \Psi}{\partial x} \right)_{0j}^{-1} \psi^{(0)}(\lambda)^k \right]_+ \frac{\partial}{\partial \lambda} + \sum_{s=1}^n \left[\left(\frac{\partial \Psi}{\partial x} \right)_{sj}^{-1} \psi^{(0)}(\lambda)^k \right]_+ \frac{\partial}{\partial x_s} \tag{2.15}$$

on the manifold $\mathbb{C} \times \mathbb{T}^n$ satisfy for all $k, m \in \mathbb{Z}_+$, $j, l = \overline{1, n}$, the Lax – Sato compatibility conditions

$$\frac{\partial A_m^{(l)}}{\partial \tau_k^{(j)}} - \frac{\partial A_k^{(j)}}{\partial \tau_m^{(l)}} = [A_k^{(j)}, A_m^{(l)}], \tag{2.16}$$

which are equivalent to the independence of the all functional parameters $\tau_k^{(j)} \in C^1(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$, $k \in \mathbb{Z}_+$, $j = \overline{1, n}$. As a corollary of the analysis above, one can show that the infinite hierarchy of vector fields (2.9) is a linear combination of the basic vector fields (2.15) and also satisfies the Lax type compatibility condition (2.16). Inasmuch the coefficients of vector fields (2.15) are suitably smooth functions on the manifold $\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n$, the compatibility conditions (2.16) yield the corresponding sets of differential-algebraic relationships on their coefficients, which have the common infinite set of invariants, thereby comprising an infinite hierarchy of completely integrable so called *heavenly* nonlinear dynamical systems on the corresponding multidimensional functional manifolds. That is, all of the above can be considered as an introduction to a recently devised

[9, 11, 25, 67, 68] constructive algorithm for generating infinite hierarchies of completely integrable nonlinear dynamical systems of heavenly type on functional manifolds of arbitrary dimension. It is worthwhile to stress here that the above constructive algorithm for generating completely integrable nonlinear multidimensional dynamical systems still does not make it possible to directly show they are Hamiltonian and construct other related mathematical structures. This important problem is solved by employing other mathematical theories; for example, the analytical properties of the related loop diffeomorphisms groups generated by the hierarchy of vector fields (2.9).

Remark 2.1. The compatibility condition (2.16) allows an alternative differential-geometric description based on the Lie-algebraic properties of the basic vector fields (2.15). Namely, consider the manifold $\mathbb{R}^{n \times \mathbb{Z}_+}$, as the base manifold of the vector bundle $E(\mathbb{R}^{n \times \mathbb{Z}_+}, G)$, $E = \cup_{\tau \in \mathbb{R}^{n \times \mathbb{Z}_+}} \{(G^* \otimes \otimes \tau) / \rho\}$, $G^* := \{\varphi^* : \varphi^* \beta^{(1)} := \alpha^{(1)} \circ \varphi, \beta^{(1)} \in \tilde{\Lambda}^{(1)}(\mathbb{C} \times \mathbb{T}^n; \mathbb{C}), \varphi \in G\}$ for an equivalence relation ρ and the (holomorphic in $\lambda \in \mathbb{S}_+^1 \cup \mathbb{S}_-^1 \subset \mathbb{C}$) structure group $G = \text{Diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$, naturally acting on the vector space E . The structure group can be endowed with a connection Υ by means of a mapping $d_h : \Gamma(E) \rightarrow \Gamma(T^*(\mathbb{R}^{n \times \mathbb{Z}_+}) \otimes E) \cong \Gamma(\text{Hom}(T(\mathbb{R}^{n \times \mathbb{Z}_+}); E))$, where

$$d_h \varphi_\tau^* := \sum_{j \in \mathbb{Z}_+} d\tau_j^{(k)} \otimes \frac{\partial}{\partial \tau_j^{(k)}} \circ \varphi_\tau^* + \varphi_\tau^* \circ \left\langle \alpha^{(1)}, \frac{\partial}{\partial \mathbf{x}} \right\rangle, \tag{2.17}$$

$\alpha^{(1)} := \sum_{j \in \mathbb{Z}_+} a_j^{(k)} d\tau_j^{(k)} \in \Lambda(\mathbb{R}^{n \times \mathbb{Z}_+}) \otimes \Gamma(E)$, which is defined for any cotangent diffeomorphism $\varphi_\tau^* \in E$, $\tau \in \mathbb{R}^{n \times \mathbb{Z}_+}$, generated by the set of parametric vector fields (2.15), and naturally acting on any mapping $\psi \in C^2(\mathbb{R}^{n \times \mathbb{Z}_+} \times (\mathbb{C} \times \mathbb{T}^n); \mathbb{C})$ as $\varphi_\tau^* \circ \psi(\tau, \mathbf{x}) := \psi(\tau, \varphi_\tau(\mathbf{x}))$, $(\tau, \mathbf{x}) \in \mathbb{R}^{n \times \mathbb{Z}_+} \times \mathbb{T}^n$. It is easy now to see that the corresponding to (2.17) zero curvature condition $d_h^2 = 0$ is equivalent to the set of compatibility equations (2.16). Moreover, the parallel transport equation

$$d_h \varphi_\tau^* \circ \psi = 0 \tag{2.18}$$

coincides exactly with the infinite hierarchy of linear vector field equations (2.14), where $\psi \in C^2(\mathbb{R}^{n \times \mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ is their invariant. Conversely, the Cartan integrable ideal of differential forms $h(\alpha) \in \Lambda(\mathbb{R}^{n \times \mathbb{Z}_+} \times \mathbb{T}^n) \otimes \Gamma(T^*(\mathbb{R}^{n \times \mathbb{Z}_+}))$, which is equivalent to the zero curvature condition $d_h^2 = 0$, makes it possible to retrieve [5, 54] the corresponding connection Υ by constructing a mapping $d_h : \Gamma(E) \rightarrow \Gamma(T^*(\mathbb{R}^{n \times \mathbb{Z}_+}) \otimes E) \cong \Gamma(\text{Hom}(T(\mathbb{R}^{n \times \mathbb{Z}_+}); E))$ in the form (2.17). These and other interesting related aspects of the integrable heavenly dynamical systems shall be investigated separately elsewhere.

2.3. Example: the vector field representation for the Mikhalev–Pavlov equation. As an example, let us consider the connection Υ

$$d_h = dt \otimes \frac{\partial}{\partial t} + dy \otimes \frac{\partial}{\partial y} + [(\lambda^2 + \lambda u_x - u_y)dt + (\lambda + u_x)dy] \otimes \frac{\partial}{\partial x},$$

where $u \in C^2(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R})$, $(t, y) \in \mathbb{R}^2$ are the base manifold parameters, $x \in \mathbb{T}^1$ and the holomorphic in $\lambda \in \mathbb{C}$ differential 1-form $(\lambda^2 + \lambda u_x - u_y) dt + (\lambda + u_x) dy \in \tilde{\Lambda}^1(\mathbb{R}^2)$. It is easy to check that the curvature 2-form $\Omega^{(2)} := d_h^2$ vanishes on the three-dimensional integral submanifold of the integrable ideal $h(\alpha) \in \Lambda(\mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{R}^6) \otimes \Gamma(T^*(\mathbb{R}))$, generated by the following set $\alpha \subset \Lambda^2(\mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{R}^6)$ of differential 2-forms:

$$\begin{aligned} \alpha_1^{(2)} &= du \wedge dt - vdx \wedge dt - wdy \wedge dt, \\ \alpha_2^{(2)} &= dv \wedge dy - dw \wedge dt - wdv \wedge dt - vdw \wedge dt, \\ \alpha_3^{(2)} &= dv \wedge dx - qdy \wedge dx + zdt \wedge dx + wdv \wedge dt - \\ &\quad - wqdy \wedge dt - vdv \wedge dt + vqdy \wedge dt = 0, \\ \alpha_4^{(2)} &= dv \wedge dt - rdx \wedge dt - qdy \wedge dt = 0, \\ \alpha_5^{(2)} &= dw \wedge dt - zdy \wedge dt - qdx \wedge dt = 0 \end{aligned}$$

on a fibered manifold with local coordinates $(t, y; x; u, v, w, q, r, z) \in \mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{R}^6$, that is $d_h^2 = 0$ if the following equation in partial derivatives:

$$u_{xt} + u_{yy} = u_y u_{xx} - u_x u_{xy} \tag{2.19}$$

holds for all $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$, where $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R})$. As is well known, the dispersionless equation in partial derivatives (2.19) was first constructed in [37] and is called the the Mikhalev – Pavlov equation. It belongs [43] to the mentioned above integrable heavenly type class.

Assume now [9] that the two functions

$$\psi^{(0)} = \lambda, \quad \psi^{(1)} \sim \sum_{k=3}^{\infty} \lambda^k \tau_k - \lambda^2 t + \lambda y + x + \sum_{j=1}^{\infty} \psi_j^{(1)}(t, y, \tau; x) \lambda^{-j}, \tag{2.20}$$

where $\psi_1^{(1)}(t, y, \tau; x) = u$, $(t, y, \tau; x) \in \mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1$, are invariants of the set of vector field equations (2.10) on the torus \mathbb{T}^1 for an infinite set of constant parameters $\tau_k \in \mathbb{R}$, $k = \overline{3, \infty}$, as the complex parameter $\lambda \rightarrow \infty$. By applying to the invariants (2.20) the criterion (2.13), (2.13) in the form

$$\left(\left(\frac{\partial \psi^{(1)}}{\partial x} \right)^{-1} d\psi^{(1)} \right)_- = 0,$$

one can easily obtain the following compatible linear vector field equations:

$$\begin{aligned} \frac{\partial \psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y) \frac{\partial \psi}{\partial x} &= 0, \\ \frac{\partial \psi}{\partial y} + (\lambda + u_x) \frac{\partial \psi}{\partial x} &= 0, \\ \dots\dots\dots \\ \frac{\partial \psi}{\partial \tau_k} + P_k(u; \lambda) \frac{\partial \psi}{\partial x} &= 0, \end{aligned} \tag{2.21}$$

where $P_k(u; \lambda)$, $k = \overline{3, \infty}$, are independent differential-algebraic polynomials in the variable $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions (2.14). Moreover, as one can check, the compatibility condition for the first two vector field equations of (2.21) yields exactly the Mikhalev – Pavlov equation (2.19).

The description of the Lax – Sato equations presented above, especially their alternative differential-geometric interpretation (2.17) and (2.18), makes it possible to realize that the structure group $\text{Diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$ should play an important role in unveiling the hidden Lie-algebraic nature of the integrable heavenly dynamical systems. This is actually the case, and a detailed analysis is presented in the sequel.

2.4. Example: The Dunajski metric nonlinear equation. The equations for the Dunajski metric [18] are

$$\begin{aligned} u_{x_1 t} + u_{y x_2} + u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2} - v &= 0, \\ v_{x_1 t} + v_{x_2 y} + u_{x_1 x_1} v_{x_2 x_2} - 2u_{x_1 x_2} v_{x_1 x_2} &= 0, \end{aligned} \tag{2.22}$$

where $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2)$, $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$. One can construct now, by definition, the following asymptotic expansions:

$$\begin{aligned} \psi^{(0)} &\sim \lambda + \sum_{j=1}^{\infty} \psi_j^{(0)}(t, y; x) \lambda^{-j}, \\ \psi^{(1)} &\sim \sum_{k=2}^{\infty} (\psi^{(0)})^k \tau_k^{(1)} - \psi^{(0)} y + x_1 + \sum_{j=1}^{\infty} \psi_j^{(1)}(t, y; x) (\psi^{(0)})^{-j}, \\ \psi^{(2)} &\sim \sum_{k=2}^{\infty} (\psi^{(0)})^k \tau_k^{(2)} + \psi^{(0)} t + x_2 + \sum_{j=1}^{\infty} \psi_j^{(2)}(t, y; x) (\psi^{(0)})^{-j}, \end{aligned}$$

where $\partial u / \partial x_1 := \psi_1^{(2)}$, $\partial u / \partial x_2 := \psi_1^{(1)}$, $v := \psi_1^{(0)}$ and $\tau_k^{(s)} \in \mathbb{R}$, $s = \overline{1, 2}$, $k = \overline{2, \infty}$, are constant parameters. Then the Lax – Sato conditions (2.13), (2.14)

$$\left(\left| \frac{\partial(\psi^{(0)}, \psi^{(1)}, \psi^{(2)})}{\partial(\lambda, x_1, x_2)} \right|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \right)_- = 0 \tag{2.23}$$

yield a compatible hierarchy of the following linear vector field equations:

$$\begin{aligned} cX^{(t_0)}\psi &:= \frac{\partial\psi}{\partial t} + X^{(t_0)}\psi = 0, & X^{(t_0)} &:= u_{x_2 x_2} \frac{\partial}{\partial x_1} - (\lambda + u_{x_1 x_2}) \frac{\partial}{\partial x_2} + v_{x_2} \frac{\partial}{\partial \lambda} = 0, \\ X^{(t_1)}\psi &:= \frac{\partial\psi}{\partial y} + X^{(t_1)}\psi = 0, & X^{(t_1)} &:= (\lambda - u_{x_1 x_2}) \frac{\partial}{\partial x_1} + u_{x_1 x_1} \frac{\partial}{\partial x_2} - v_{x_1} \frac{\partial}{\partial \lambda} = 0, \\ X^{(t_k)}\psi &:= \frac{\partial\psi}{\partial \tau_k^s} + \left\langle P_k^s(u; \lambda), \frac{\partial\psi}{\partial x} \right\rangle = 0, \end{aligned}$$

where $P_k^s(u, v; \lambda)$, $s = \overline{1, 2}$, $k = \overline{2, \infty}$, are independent analytical in $\lambda \in \mathbb{C}$ and differential-algebraic vector polynomials [11] in the variables $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^2; \mathbb{R}^2)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions (2.14). In particular, the compatibility condition (2.16) for the first two equations of (2.23) is equivalent to the Dunajski metric nonlinear equations (2.22).

The description of the Lax – Sato equations presented above, especially their alternative differential-geometric interpretation (2.17) and (2.18), makes it possible to realize that the structure group $\text{Diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$ should play an important role in unveiling the hidden Lie-algebraic nature of the integrable heavenly dynamical systems. This is actually the case, and a detailed analysis is presented in the sequel.

3. Heavenly type equations: the Lie-algebraic integrability scheme. Let $\tilde{G}_{\pm} := \widetilde{\text{Diff}}_{\pm}(\mathbb{T}^n)$, $n \in \mathbb{Z}_+$, be subgroups of the loop diffeomorphisms group $\widetilde{\text{Diff}}(\mathbb{T}^n) := \{\mathbb{C} \supset \mathbb{S}^1 \rightarrow \text{Diff}(\mathbb{T}^n)\}$, holomorphically extended in the interior $\mathbb{S}_+^1 \subset \mathbb{C}$ and in the exterior $\mathbb{S}_-^1 \subset \mathbb{C}$ regions of the unit circle $\mathbb{S}^1 \subset \mathbb{C}^1$, such that for any $g(\lambda) \in \tilde{G}_{\pm}$, $\lambda \in \mathbb{S}_{\pm}^1$, $g(\infty) = 1 \in \text{Diff}(\mathbb{T}^n)$. The corresponding Lie subalgebras $\tilde{\mathcal{G}}_{\pm} := \widetilde{\text{diff}}_{\pm}(\mathbb{T}^n)$ of the loop subgroups \tilde{G}_{\pm} are vector fields on \mathbb{T}^n holomorphic, respectively, on $\mathbb{S}_{\pm}^1 \subset \mathbb{C}^1$, where one can single out two cases: the first one, if for any $\tilde{a}(\lambda) \in \tilde{\mathcal{G}}_+$ the value $\tilde{a}(0) = 0$, and the second one, if for any $\tilde{a}(\lambda) \in \tilde{\mathcal{G}}_-$ the value $\tilde{a}(\infty) = 0$. The split loop Lie algebra $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ can be naturally identified with a dense subspace of the dual space $\tilde{\mathcal{G}}^*$ through the pairing

$$(\tilde{l}, \tilde{a}) := \text{res}_{\lambda \in \mathbb{C}} (\lambda^{-p} l(x, \lambda), a(x, \lambda))_{H^q}, \tag{3.1}$$

for some fixed $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$. We took above, by definition [15, 55], a loop vector field $\tilde{a} \in \Gamma(\tilde{T}(\mathbb{T}^n))$ and a loop differential 1-form $\tilde{l} \in \tilde{\Lambda}^1(\mathbb{T}^n)$ given as

$$\tilde{a} = \sum_{j=1}^n a^{(j)}(x, \lambda) \frac{\partial}{\partial x_j} := \left\langle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle,$$

$$\tilde{l} = \sum_{j=1}^n l_j(x, \lambda) dx_j := \langle l(x; \lambda), dx \rangle,$$

introduced for brevity the gradient operator $\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)^T$ in the Euclidean space \mathbb{E}^n and chose the Sobolev type metric $(\cdot, \cdot)_{H^q}$ on the space $C^\infty(\mathbb{T}^n; \mathbb{R}^n) \subset H^q(\mathbb{T}^n; \mathbb{R}^n)$ for some $q \in \mathbb{Z}_+$ as

$$(l(x; \lambda), a(x; \lambda))_{H^q} := \sum_{j=1}^n \sum_{|\alpha|=0}^q \int dx \left(\frac{\partial^{|\alpha|} l_j(x; \lambda)}{\partial x^\alpha} \frac{\partial^{|\alpha|} a^{(j)}(x; \lambda)}{\partial x^\alpha} \right),$$

where $\partial x^\alpha := \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}$, $|\alpha| = \sum_{j=1}^n \alpha_j$ for $\alpha \in \mathbb{Z}_+^n$, generalizing the metric used before in [38]. The Lie commutator of vector fields $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$ is calculated the standard way and equals

$$\begin{aligned} [\tilde{a}, \tilde{b}] &= \tilde{a}\tilde{b} - \tilde{b}\tilde{a} = \left\langle \left\langle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle b(x; \lambda), \frac{\partial}{\partial x} \right\rangle - \\ &\quad - \left\langle \left\langle b(x; \lambda), \frac{\partial}{\partial x} \right\rangle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle. \end{aligned}$$

The Lie algebra $\tilde{\mathcal{G}}$ naturally splits into the direct sum of two Lie subalgebras

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-,$$

for which one can identify the dual spaces

$$\tilde{\mathcal{G}}_+^* \simeq \lambda^{p-1} \tilde{\mathcal{G}}_-, \quad \tilde{\mathcal{G}}_-^* \simeq \lambda^{p-1} \tilde{\mathcal{G}}_+,$$

where for any $l(\lambda) \in \tilde{\mathcal{G}}_-^*$ one has the constraint $\tilde{l}(0) = 0$. Having defined now the projections

$$P_{\pm} \tilde{\mathcal{G}} := \tilde{\mathcal{G}}_{\pm} \subset \tilde{\mathcal{G}},$$

one can construct a classical \mathcal{R} -structure [57, 65, 69] on the Lie algebra $\tilde{\mathcal{G}}$ as the endomorphism $\mathcal{R} : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}$, where

$$\mathcal{R} := (P_+ - P_-)/2,$$

which allows to determine on the vector space $\tilde{\mathcal{G}}$ the new Lie algebra structure

$$[\tilde{a}, \tilde{b}]_{\mathcal{R}} := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}]$$

for any $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$, satisfying the standard Jacobi identity.

Let $D(\tilde{\mathcal{G}}^*)$ denote the space of smooth functions on $\tilde{\mathcal{G}}^*$. Then for any $f, g \in D(\tilde{\mathcal{G}}^*)$ one can write the canonical [5, 54, 57, 69] Lie–Poisson bracket

$$\{f, g\} := (\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]), \tag{3.2}$$

where $\tilde{l} \in \tilde{\mathcal{G}}^*$ is a seed element and $\nabla f, \nabla g \in \tilde{\mathcal{G}}$ are the standard functional gradients at $\tilde{l} \in \tilde{\mathcal{G}}^*$ with respect to the metric (3.1). The related to (3.2) space $I(\tilde{\mathcal{G}}^*)$ of Casimir invariants is defined as the set $I(\tilde{\mathcal{G}}^*) \subset D(\tilde{\mathcal{G}}^*)$ of smooth independent functions $h^{(j)} \in D(\tilde{\mathcal{G}}^*)$, $j = \overline{1, n}$, for which

$$ad_{\nabla h^{(j)}(\tilde{l})}^* \tilde{l} = 0, \tag{3.3}$$

where for any seed element

$$\tilde{l} = \langle l, dx \rangle$$

the gradients

$$\nabla h^{(j)}(\tilde{l}) := \left\langle \nabla h^{(j)}(l), \frac{\partial}{\partial x} \right\rangle$$

and the coadjoint action (3.3) can be equivalently rewritten, for instance, in the case $q = 0$, as

$$\left\langle \frac{\partial}{\partial x}, \nabla h^{(j)}(l) \right\rangle l + \left\langle l, \left(\frac{\partial}{\partial x} \nabla h^{(j)}(l) \right) \right\rangle = 0 \tag{3.4}$$

for any $j = \overline{1, n}$. If one takes two smooth functions $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*) \subset D(\tilde{\mathcal{G}}^*)$, their second Poisson bracket

$$\{h^{(1)}, h^{(2)}\}_{\mathcal{R}} := \left(\tilde{l}, [\nabla h^{(1)}, \nabla h^{(2)}]_{\mathcal{R}} \right)$$

on the space $\tilde{\mathcal{G}}^*$ vanishes, that is

$$\{h^{(1)}, h^{(2)}\}_{\mathcal{R}} = 0$$

at any seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$. Since the functions $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$, the following coadjoint action relationships hold:

$$ad_{\nabla h^{(1)}(\tilde{l})}^* \tilde{l} = 0, \quad ad_{\nabla h^{(2)}(\tilde{l})}^* \tilde{l} = 0,$$

which can be equivalently rewritten (as above in the case $q = 0 = p$) as

$$\begin{aligned} & \left\langle \frac{\partial}{\partial x}, \nabla h^{(1)}(l) \right\rangle l + \left\langle l, \left(\frac{\partial}{\partial x} \nabla h^{(1)}(l) \right) \right\rangle = \\ & = \left\langle \nabla h^{(1)}(l), \frac{\partial}{\partial x} \right\rangle l + \left\langle \left(\frac{\partial}{\partial x}, \nabla h^{(1)}(l) \right) \right\rangle l + \left\langle l, \left(\frac{\partial}{\partial x} \nabla h^{(1)}(l) \right) \right\rangle := \\ & = \left(\nabla h^{(1)}(\tilde{l}) + B_{\nabla h^{(1)}(\tilde{l})} \right) l \end{aligned} \tag{3.5}$$

and similarly

$$\left\langle \frac{\partial}{\partial x}, \nabla h^{(2)}(l) \right\rangle l + \left\langle l, \left(\frac{\partial}{\partial x}, \nabla h^{(2)}(l) \right) \right\rangle := \left(\nabla h^{(2)}(\tilde{l}) + B_{\nabla h^{(2)}(\tilde{l})} \right) l,$$

where the expressions

$$\nabla h^{(1)}(\tilde{l}) := \left\langle \nabla h^{(1)}(l), \frac{d}{dx} \right\rangle, \quad \nabla h^{(2)}(\tilde{l}) := \left\langle \nabla h^{(2)}(l), \frac{\partial}{\partial x} \right\rangle$$

are true vector fields on \mathbb{T}^n , yet the expressions

$$\begin{aligned} B_{\nabla h^{(1)}(\tilde{l})} & := \left\langle \left(\frac{\partial}{\partial x}, \nabla h^{(1)}(l) \right) \right\rangle + \left(\frac{\partial}{\partial x} \nabla h^{(1)}(l) \right), \\ B_{\nabla h^{(2)}(\tilde{l})} & := \left\langle \left(\frac{\partial}{\partial x}, \nabla h^{(2)}(l) \right) \right\rangle + \left(\frac{\partial}{\partial x} \nabla h^{(2)}(l) \right), \end{aligned}$$

are the usual matrix homomorphisms of the Euclidean space \mathbb{E}^n .

Consider now the following Hamiltonian flows on the space $\tilde{\mathcal{G}}^*$:

$$\begin{aligned} \partial \tilde{l} / \partial y & := \{h^{(1)}, \tilde{l}\}_{\mathcal{R}} = -ad_{\nabla h^{(1)}(\tilde{l})_+}^* \tilde{l}, \\ \partial \tilde{l} / \partial t & := \{h^{(2)}, \tilde{l}\}_{\mathcal{R}} = -ad_{\nabla h^{(2)}(\tilde{l})_+}^* \tilde{l}, \end{aligned} \tag{3.6}$$

where $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ and $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ are Casimirs, the flows (3.6) commute. Thus, taking into account the representations (3.5), one can recast the flows (3.6) as

$$\partial l / \partial t = -\left(\nabla h^{(y)}(\tilde{l})_+ + B_{\nabla h^{(1)}(\tilde{l})_+} \right) l, \quad \partial l / \partial y = -\left(\nabla h^{(y)}(\tilde{l}) + B_{\nabla h^{(2)}(\tilde{l})_+} \right) l, \tag{3.7}$$

where

$$\nabla h^{(y)}(\tilde{l}) := \nabla h^{(1)}(\tilde{l})_+, \quad \nabla h^{(t)}(\tilde{l}) := \nabla h^{(2)}(\tilde{l})_+. \tag{3.8}$$

Lemma 3.1. *The compatibility of commuting flows (3.7) is equivalent to the Lax–Sato type vector fields relationship*

$$\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}) - \frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}) + [\nabla h^{(y)}(\tilde{l}), \nabla h^{(t)}(\tilde{l})] = 0, \tag{3.9}$$

which holds for all $y, t \in \mathbb{R}$ and arbitrary $\lambda \in \mathbb{C}$.

Proof. The compatibility of commuting flows (3.7) implies that $\partial^2 \tilde{l} / \partial t \partial y - \partial^2 \tilde{l} / \partial y \partial t = 0$ for all $y, t \in \mathbb{R}$ and arbitrary $\lambda \in \mathbb{C}$. Taking into account the expressions (3.6), one has for any vector field $\tilde{Z} = \left\langle Z, \frac{\partial}{\partial x} \right\rangle \in \tilde{\mathcal{G}}$

$$\begin{aligned}
 0 &= \left(\frac{\partial^2 \tilde{l}}{\partial t \partial y} - \frac{\partial^2 \tilde{l}}{\partial y \partial t}, \tilde{Z} \right) = \\
 &= - \left(\frac{\partial}{\partial t} \text{ad}_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}, \tilde{Z} \right) + \left(\frac{\partial}{\partial y} \text{ad}_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l}, \tilde{Z} \right) = \\
 &= - \frac{\partial}{\partial t} (\tilde{l}, [\nabla h^{(y)}(\tilde{l}), \tilde{Z}]) + \frac{\partial}{\partial y} (\tilde{l}, [\nabla h^{(t)}(\tilde{l}), \tilde{Z}]) = \\
 &= - \left(\frac{\partial}{\partial t} \tilde{l}, [\nabla h^{(y)}(\tilde{l}), \tilde{Z}] \right) - \left(\tilde{l}, \left[\frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}), \tilde{Z} \right] \right) + \\
 &+ \left(\frac{\partial}{\partial y} \tilde{l}, [\nabla h^{(t)}(\tilde{l}), \tilde{Z}] \right) + \left(\tilde{l}, \left[\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}), \tilde{Z} \right] \right) = \\
 &= (\text{ad}_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l}, [\nabla h^{(y)}(\tilde{l}), \tilde{Z}]) - \left(\tilde{l}, \left[\frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}), \tilde{Z} \right] \right) - \\
 &- (\text{ad}_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}, [\nabla h^{(t)}(\tilde{l}), \tilde{Z}]) + \left(\tilde{l}, \left[\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}), \tilde{Z} \right] \right) = \\
 &= (\tilde{l}, [\nabla h^{(t)}(\tilde{l}), [\nabla h^{(y)}(\tilde{l}), \tilde{Z}]] - \left(\tilde{l}, \left[\frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}), \tilde{Z} \right] \right) - \\
 &- (\tilde{l}, [\nabla h^{(y)}(\tilde{l}), [\nabla h^{(t)}(\tilde{l}), \tilde{Z}]]) + \left(\tilde{l}, \left[\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}), \tilde{Z} \right] \right) = \\
 &= (\tilde{l}, [\nabla h^{(t)}(\tilde{l}), [\nabla h^{(y)}(\tilde{l}), \tilde{Z}]] - [\nabla h^{(y)}(\tilde{l}), [\nabla h^{(t)}(\tilde{l}), \tilde{Z}]] + \\
 &+ \left(\tilde{l}, \left[\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}) - \frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}), \tilde{Z} \right] \right) = \\
 &= \left(\tilde{l}, [\nabla h^{(y)}(\tilde{l}), \nabla h^{(t)}(\tilde{l})] + \frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}) - \frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}), \tilde{Z} \right) = \\
 &= \left(\tilde{l}, \left[\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}) - \frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}) + [\nabla h^{(y)}(\tilde{l}), \nabla h^{(t)}(\tilde{l})], \tilde{Z} \right] \right) = \\
 &= \left(\tilde{l}, [Q^{(t,y)}(\tilde{l}), \tilde{Z}] \right), \tag{3.10}
 \end{aligned}$$

where

$$Q^{(t,y)}(\tilde{l}) := \frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l}) - \frac{\partial}{\partial t} \nabla h^{(y)}(\tilde{l}) + [\nabla h^{(y)}(\tilde{l}), \nabla h^{(t)}(\tilde{l})]. \tag{3.11}$$

From (3.10) we obtain that $ad_{Q^{(t,y)}(\tilde{l})}^* \tilde{l} = 0$ for all $y, t \in \mathbb{R}$ and arbitrary $\lambda \in \mathbb{C}$; that is the vector field (3.11) is an analytical Casimir type vector field $Q^{(t,y)}(\tilde{l}) \in \tilde{\mathcal{G}}_+$. Now based on the analyticity of the vector field expression (3.11), one easily shows [9] that $Q^{(t,y)}(\tilde{l}) = 0$.

Lemma 3.1 is proved.

For the exact representatives of the functions $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$, it is necessary to solve the determining equation (3.4), taking into account that if the chosen element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \rightarrow \infty$, the related expansion

$$\nabla h^{(p)}(l) \simeq \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla h_j(l) \lambda^{-j}, \tag{3.12}$$

where the degree $p \in \mathbb{Z}_+$ can be taken as arbitrary. Upon substituting (3.12) into (3.4) one can find recurrently all the coefficients $\nabla h_j(l)$, $j \in \mathbb{Z}_+$, and then construct functional gradients of the Casimir functions $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ projected on $\tilde{\mathcal{G}}_+$ as

$$\nabla h^{(t)}(l) := (\lambda^{p_t} \nabla h(l))_+, \quad \nabla h^{(y)}(l) := (\lambda^{p_y} \nabla h(l))_+ \tag{3.13}$$

for some positive integers $p_y, p_t \in \mathbb{Z}_+$.

Remark 3.1. As mentioned above, the expansion (3.12) is effective if a chosen seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \rightarrow \infty$. In the case when it is singular as $|\lambda| \rightarrow 0$, the expression (3.12) should be replaced by the expansion

$$\nabla h^{(p)}(l) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla h_j(l) \lambda^j$$

for an arbitrary $p \in \mathbb{Z}_+$, and the projected Casimir function gradients then are given by the expressions

$$\nabla h^{(y)}(l) := \nabla h^{(p_y)}(l)_-, \quad \nabla h^{(t)}(l) := \nabla h^{(p_t)}(l)_-$$

for some positive integers $p_y, p_t \in \mathbb{Z}_+$. Then the corresponding flows are, respectively, written as

$$\partial \tilde{l} / \partial t = -ad_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l}, \quad \partial \tilde{l} / \partial y = -ad_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}.$$

The above results, owing to lemma 3.1, can be formulated as the following main proposition.

Proposition 3.1. Let a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ and let $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ be Casimir functions subject to the loop Lie algebra $\tilde{\mathcal{G}}$ and the natural coadjoint action on the loop co-algebra $\tilde{\mathcal{G}}^*$. Then the dynamical systems

$$\partial \tilde{l} / \partial y = -ad_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}, \quad \partial \tilde{l} / \partial t = -ad_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l} \tag{3.14}$$

are commuting Hamiltonian flows for all $y, t \in \mathbb{R}$. Moreover, the compatibility condition of these flows is equivalent to the vector fields representation

$$\left(\partial / \partial t + \nabla h^{(t)}(\tilde{l}) \right) \psi = 0, \quad \left(\partial \partial y + \nabla h^{(y)}(\tilde{l}) \right) \psi = 0,$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^n; \mathbb{C})$ and the vector fields $\nabla h^{(y)}(\tilde{l}), \nabla h^{(t)}(\tilde{l}) \in \tilde{\mathcal{G}}_+$, given by the expressions (3.8) and (3.13), satisfy the Lax–Sato type relationship (3.9).

The proposition above makes it possible to describe in a very effective way the Bäcklund transformations between two solution sets to the dispersionless heavenly type equations resulting from the Lax compatibility condition (3.9). Namely, let a diffeomorphism $\xi \in \widetilde{\text{Diff}}(\mathbb{T}^n)$, depending parametrically on $\lambda, \mu \in \mathbb{C}$ and evolution variables $(y, t) \in \mathbb{R}^2$, be such that a seed loop differential form $\tilde{l}(x; \lambda, \mu) \in \tilde{\mathcal{G}}^* \simeq \tilde{\Lambda}^1(\mathbb{T}^n)$ satisfies the invariance condition

$$\tilde{l}(\xi(x; \lambda, \mu); \lambda) = k\tilde{l}(x; \mu) \tag{3.15}$$

for some non-zero constant $k \in \mathbb{C} \setminus \{0\}$, any $x \in \mathbb{T}^n$ and arbitrarily chosen $\lambda \in \mathbb{C}$. As the seed element $\tilde{l}(\xi(x; \lambda, \mu); \lambda) \in \tilde{\Lambda}^1(\mathbb{T}^n)$, by the construction, simultaneously satisfies the system of compatible equations following from (3.14), the loop diffeomorphism $\xi \in \widetilde{\text{Diff}}(\mathbb{T}^n)$, found analytically from the invariance condition (3.15), should satisfying the relationships

$$\frac{\partial}{\partial y}\xi = \nabla h^{(y)}(l), \quad \frac{\partial}{\partial t}\xi = \nabla h^{(t)}(l), \tag{3.16}$$

giving rise exactly to the Bäcklund type relationships for coefficients of the seed loop differential form $\tilde{l} \in \tilde{\mathcal{G}}^* \simeq \tilde{\Lambda}^1(\mathbb{T}^n)$.

4. Integrable heavenly type equations: Examples. *4.1. The Mikhalev–Pavlov heavenly type equation.* This equation [37, 43] is

$$u_{xt} + u_{yy} = u_y u_{xx} - u_x u_{xy}, \tag{4.1}$$

where $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1)$ and $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. Set $\tilde{\mathcal{G}}^* := \widetilde{\text{diff}}^*(\mathbb{T}^1)$ and take the corresponding seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ as

$$\tilde{l} = (\lambda - 2u_x)dx. \tag{4.2}$$

It generates a Casimir invariant $h \in I(\tilde{\mathcal{G}}^*)$ for which the expansion (3.12) as $|\lambda| \rightarrow \infty$ is given by the asymptotic series

$$\nabla h(l) \sim 1 + u_x/\lambda - u_y/\lambda^2 + O(1/\lambda^3)$$

and so on. If further one defines

$$\nabla h^{(t)}(l) := (\lambda^2 \nabla h)_+ = \lambda^2 + \lambda u_x - u_y,$$

$$\nabla h^{(y)}(l) := (\lambda^1 \nabla h)_+ = \lambda + u_x,$$

it is easy to verify that

$$\begin{aligned} \nabla h^{(t)}(\tilde{l}) &:= \left\langle \nabla h^{(t)}(l)_+, \frac{\partial}{\partial x} \right\rangle = (\lambda^2 + \lambda u_x - u_y) \frac{\partial}{\partial x}, \\ \nabla h^{(y)}(\tilde{l}) &:= \left\langle \nabla h^{(y)}(l)_+, \frac{\partial}{\partial x} \right\rangle = (\lambda + u_x) \frac{\partial}{\partial x}. \end{aligned} \tag{4.3}$$

As a result of (4.3) and the commuting flows (3.14) on $\tilde{\mathcal{G}}^*$ we retrieve (the equivalent to the Mikhalev–Pavlov [43]) equation (4.1) vector field compatibility relationships

$$\frac{\partial \psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y) \frac{\partial \psi}{\partial x} = 0 = \frac{\partial \psi}{\partial y} + (\lambda + u_x) \frac{\partial \psi}{\partial x}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C})$, any $(y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and all $\lambda \in \mathbb{C}$.

We now study the Bäcklund transformation for two special solutions $u, \tilde{u} \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R})$ to the Mikhalev–Pavlov equation (4.1). Let us consider a loop diffeomorphism $\xi \in \text{Diff}(\mathbb{T}^1)$ that is the mapping $\mathbb{T}^1 \ni x \rightarrow \tilde{x} = \xi(x; y, t, \lambda) \in \mathbb{T}^1$, which parametrically depends on $\lambda \in \mathbb{C}^1$ and the evolution variables $(y, t) \in \mathbb{R}^2$, satisfying the invariance condition (3.15) for the seed loop differential form (4.2):

$$[\lambda - 2\tilde{u}_{\tilde{x}}(\tilde{x}; t, y)]d\tilde{x} = [\lambda - 2u_x(x; t, y)]dx, \quad (4.4)$$

where for simplicity, we define $\mu = \lambda \in \mathbb{C}$ and the constant parameter $k = 1$. From (4.4) one easily finds that

$$\lambda \xi_x(x; t, y) = 2[\tilde{u}(\tilde{x}; t, y) - u(x; t, y)]_x + \lambda,$$

or, equivalently,

$$\xi(x; \lambda) = x + 2(\tilde{u} - u)/\lambda + \alpha(y, t; \lambda), \quad (4.5)$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\alpha \in C^2(\mathbb{R}^2; \mathbb{R})$ is some mapping. As the loop diffeomorphism (4.5) should simultaneously satisfy the vector field equations (3.16), giving rise (at $\alpha(y, t; \lambda) = 0$) to the following Bäcklund type differential relationships:

$$2(\tilde{u}_y - u_y)/\lambda = \lambda + \tilde{u}_{\tilde{x}}, \quad 2(\tilde{u}_t - u_t)/\lambda = \lambda^2 + \lambda\tilde{u}_{\tilde{x}} - \tilde{u}_y,$$

which hold for any $\lambda \in \mathbb{C}$ and two special solutions $u, \tilde{u} \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R})$ to the Mikhalev–Pavlov equation (4.1).

4.2. Example: the Witham heavenly type equation. Consider the following [21, 27, 35, 45] heavenly type equation:

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \quad (4.6)$$

where $u \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R})$ and $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. To prove the Lax–Sato type integrability of (4.6), let us consider a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$, defined as

$$\tilde{l} = (u_y^{-2}\lambda^{-1} + 2u_x + \lambda)dx, \quad (4.7)$$

where $\lambda \in \mathbb{C}$ is a complex parameter. The following asymptotic expressions are gradients of the Casimir functionals $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$, related with the holomorphic loop Lie algebra $\tilde{\mathcal{G}} = \widetilde{\text{diff}}(\mathbb{T}^1)$:

$$\nabla h^{(1)}(l) \sim [(u_x \lambda^{-1} - 1) + O(1/\lambda^2)], \quad (4.8)$$

as $\lambda \rightarrow \infty$, and

$$\nabla h^{(2)}(l) \sim u_y + O(\lambda), \quad (4.9)$$

as $\lambda \rightarrow 0$. Based on the expressions (4.8) and (4.9), one can construct [55] the following commuting to each other Hamiltonian flows:

$$\frac{\partial}{\partial y} \tilde{l} = -ad_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}, \quad \frac{\partial}{\partial t} \tilde{l} = -ad_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l} \quad (4.10)$$

with respect to the evolution parameters $y, t \in \mathbb{R}$, where

$$\nabla h^{(y)}(l) := (\lambda \nabla h^{(1)}(l))|_+, \quad \nabla h^{(t)}(l) := (\lambda^{-1} \nabla h^{(2)}(l))|_-. \quad (4.11)$$

Now one ensues easily from (4.10) and (4.11) the system of equations

$$\begin{aligned} u_{yt} &= u_x u_{xy} - u_y u_{xx}, \\ u_t &= -u_y^{-2}/2 + 3u_x^2/2, \\ u_{yy} &= -u_y^3[(u_x u_y)_x + u_x u_{xy}], \end{aligned} \tag{4.12}$$

where the projected gradients $\nabla h^{(y)}(\tilde{l})$ and $\nabla h^{(t)}(\tilde{l}) \in \tilde{\mathcal{G}}$, are equal to the loop vector fields

$$\nabla h^{(t)}(\tilde{l}) = (u_x - \lambda) \frac{\partial}{\partial x}, \quad \nabla h^{(y)}(\tilde{l}) = \frac{u_y}{\lambda} \frac{\partial}{\partial x}, \tag{4.13}$$

satisfying for evolution parameters $y, t \in \mathbb{R}^2$ the Lax–Sato vector field compatibility condition:

$$\frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l}) - \frac{\partial}{\partial t} \nabla h^{(t)}(\tilde{l}) + [\nabla h^{(t)}(\tilde{l}), \nabla h^{(y)}(\tilde{l})] = 0. \tag{4.14}$$

As a simple consequence of the condition one finds exactly the first equation of the (4.12), coinciding with the heavenly type equation (4.6). Thereby, we have stated that this equation is a completely integrable heavenly type dynamical system with respect to both evolution parameters.

Remark 4.1. It is worth to observe that the third equation of (4.12) entails the interesting relationship

$$\frac{\partial}{\partial y} (1/u_y) = \frac{\partial}{\partial x} (u_x u_y^2),$$

whose compatibility makes it possible to introduce a new function $v \in C^2(\mathbb{S}^1; \mathbb{R})$, satisfying the next differential expressions:

$$v_x = 1/u_y, \quad v_y = u_x u_y^2, \tag{4.15}$$

which hold for all $(x, y) \in \mathbb{S}^1 \times \mathbb{R}$. Based on (4.15) the seed element (4.7) is rewritten as

$$\tilde{l} = (v_x^2 \lambda^{-1} + 2u_x + \lambda) dx,$$

and the vector fields (4.13) are rewritten as

$$\nabla h^{(t)}(\tilde{l}) = (u_x - \lambda) \frac{\partial}{\partial x}, \quad \nabla h^{(y)}(\tilde{l}) = \frac{1}{v_x \lambda} \frac{\partial}{\partial x},$$

whose compatibility condition (4.14) gives rise to the following system of heavenly type nonlinear integrable flows:

$$\begin{aligned} v_y &= u_x v_x^{-2}, & v_{xt} &= u_x v_{xy} + u_{xx} v_x, \\ u_y &= 1/v_x, & u_t &= -v_x^2/2 + 3u_x^2/2, \end{aligned}$$

compatible for arbitrary evolution parameters $y, t \in \mathbb{R}$.

4.3. Plebański heavenly equation. This equation [52] is

$$u_{tx_1} - u_{yx_2} + u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2 = 0 \tag{4.16}$$

for a function $u \in C^\infty(\mathbb{R}^2; \mathbb{T}^2)$, where $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$. We set $\tilde{\mathcal{G}}^* := \widetilde{\text{diff}}^*(\mathbb{T}^2)$ and take the corresponding seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ as

$$\tilde{l} = (\lambda - u_{x_1x_2} + u_{x_1x_1}) dx_1 + (\lambda - u_{x_2x_2} + u_{x_1x_2}) dx_2.$$

This generates two independent Casimir functionals $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradient expansions (3.12) as $|\lambda| \rightarrow \infty$ are given by the expressions

$$\begin{aligned} \nabla h^{(1)}(l) &\sim (0, 1)^\top + (u_{x_2x_2}, -u_{x_1x_2})^\top \lambda^{-1} + O(\lambda^{-2}), \\ \nabla h^{(2)}(l) &\sim (1, 0)^\top + (u_{x_1x_2}, -u_{x_1x_2})^\top \lambda^{-1} + O(\lambda^{-2}), \end{aligned}$$

and so on. Now, by defining

$$\begin{aligned} \nabla h^{(y)}(l) &:= (\lambda \nabla h^{(1)}(l))_+ = (u_{x_2x_2}, \lambda - u_{x_1x_2})^\top, \\ \nabla h^{(t)}(l) &:= (\lambda \nabla h^{(2)}(l))_+ = (\lambda + u_{x_1x_2}, -u_{x_1x_1})^\top, \end{aligned}$$

where we put $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ with $\tilde{\mathcal{G}}_-|_{\lambda=\infty} = 0$, one obtains for the heavenly equation (4.16) the following [52] vector field representation

$$\begin{aligned} \frac{\partial \psi}{\partial t} + u_{x_1x_1} \frac{\partial \psi}{\partial x_1} + (\lambda - u_{x_1x_2}) \frac{\partial \psi}{\partial x_2} &= 0, \\ \frac{\partial \psi}{\partial z} + (\lambda + u_{x_1x_2}) \frac{\partial \psi}{\partial x_1} - u_{x_1x_1} \frac{\partial \psi}{\partial x_2} &= 0, \end{aligned}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C})$, any $(t, y; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$ and all $\lambda \in \mathbb{C}$.

4.4. General heavenly equation. This equation was first suggested and analyzed by W. K. Schief in [60, 61], where it was shown to be equivalent to the first Plebański heavenly equation, and later studied by B. Doubrov and E. V. Ferapontov [17]; it has the form

$$\alpha u_{yt} u_{x_1x_2} + \beta u_{tx_2} u_{yx_1} + \gamma u_{tx_1} u_{yx_2} = 0, \tag{4.17}$$

where α, β and $\gamma \in \mathbb{R}$ are arbitrary constants, satisfying the constraint

$$\alpha + \beta + \gamma = 0,$$

and $t, y \in \mathbb{R}, (x_1, x_2) \in \mathbb{T}^2$. To demonstrate the Lax integrability of the equation (4.17) we choose now a seed vector field $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{\text{diff}}^*(\mathbb{T}^2)$ in the following rational form:

$$\begin{aligned} \tilde{l} &= \left(\frac{\mu u_{x_1x_2}^2}{\gamma(\mu + \beta)} + \frac{u_{x_1x_2}^2}{\alpha} - \frac{\mu u_{x_1x_2}^2}{\beta(\mu - \gamma)} \right) dx_1 + \\ &+ \left(\frac{\mu u_{x_1x_2} u_{x_2x_2}}{\gamma(\mu + \beta)} + \frac{u_{x_1x_2} u_{x_2x_2}}{\alpha} - \frac{\mu u_{x_1x_2} u_{x_2x_2}}{\beta(\mu - \gamma)} \right) dx_2, \end{aligned}$$

where $a_j, b_j \in C^\infty(\mathbb{T}^2; \mathbb{R}), j = \overline{0, 1}$, are smooth functions and $\mu \in \mathbb{C}$ is a complex parameter. The corresponding equations for independent Casimir invariants $h^{(j)} \in I(\tilde{\mathcal{G}}^*), j = \overline{1, 2}$, are given with respect to the standard bilinear form (\cdot, \cdot) by the following asymptotic expansions:

$$\nabla h^{(1)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(1)}(l) \lambda^j, \tag{4.18}$$

as $\mu + \beta = \lambda \rightarrow 0$, and

$$\nabla h^{(2)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(2)}(l) \lambda^j, \tag{4.19}$$

as $\mu - \gamma = \lambda \rightarrow 0$. For the first case (4.18) one obtains that

$$\nabla h^{(1)}(l) \sim \left(-\frac{\beta u_{tx_2}}{u_{x_1x_2}} + \frac{u_{tx_2}}{u_{x_1x_2}} \lambda, \frac{\beta u_{x_1x_1}}{u_{x_1x_2}} \right)^\top + O(\lambda^2)$$

and for the second one (4.19) one finds that

$$\nabla h^{(2)}(l) \sim \left(\frac{\gamma u_{yx_2}}{u_{x_1x_2}} + \frac{u_{yx_2}}{u_{x_1x_2}} \lambda, -\frac{\gamma u_{x_1x_1}}{u_{x_1x_2}} \right)^\top + O(\lambda^2).$$

Here we took into account that $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, where $\tilde{\mathcal{G}}_+|_{\lambda=0} = 0$, and the following two Hamiltonian flows on $\tilde{\mathcal{G}}^*$

$$\partial \tilde{l} / \partial y = ad_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}, \quad \partial \tilde{l} / \partial t = ad_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$ hold for the corresponding conservation laws gradients:

$$\begin{aligned} \nabla h^{(t)}(l) &:= (\lambda^{-1} \nabla h^{(1)}(l))_- \Big|_{\lambda=\mu+\beta} = \left(\frac{\mu u_{tx_2}}{u_{x_1x_2}(\mu + \beta)}, \frac{\beta u_{tx_1}}{u_{x_1x_2}(\mu + \beta)} \right)^\top, \\ \nabla h^{(y)}(l) &:= (\lambda^{-1} \nabla h^{(2)}(l))_- \Big|_{\lambda=\mu-\gamma} = \left(\frac{\mu u_{yx_2}}{u_{x_1x_2}(\mu - \gamma)}, -\frac{\gamma u_{yx_1}}{u_{x_1x_2}(\mu - \gamma)} \right)^\top. \end{aligned} \tag{4.20}$$

Owing to the compatibility condition of two commuting flows (4.20), one can easily rewrite it as the Lax relationship

$$\partial \nabla h^{(y)}(\tilde{l}) / \partial t - \partial \nabla h^{(t)}(\tilde{l}) / \partial y = [\nabla h^{(y)}(\tilde{l}), \nabla h^{(t)}(\tilde{l})], \tag{4.21}$$

where

$$\begin{aligned} \nabla h^{(t)}(\tilde{l}) &:= \left\langle \nabla h^{(t)}(l), \frac{\partial}{\partial x} \right\rangle = \frac{\mu u_{tx_2}}{u_{x_1x_2}(\mu + \beta)} \frac{\partial}{\partial x_1} + \frac{\beta u_{tx_1}}{u_{x_1x_2}(\mu + \beta)} \frac{\partial}{\partial x_2}, \\ \nabla h^{(y)}(\tilde{l}) &:= \left\langle \nabla h^{(y)}(l), \frac{\partial}{\partial x} \right\rangle = \frac{\mu u_{yx_2}}{u_{x_1x_2}(\mu - \gamma)} \frac{\partial}{\partial x_1} - \frac{\gamma u_{yx_1}}{u_{x_1x_2}(\mu - \gamma)} \frac{\partial}{\partial x_2}. \end{aligned}$$

An easy calculation shows that the general heavenly equation (4.17) follows from the compatibility condition (4.21), whose equivalent vector field representation is given as

$$\begin{aligned} \frac{\mu u_{tx_2}}{u_{x_1x_2}(\mu + \beta)} \frac{\partial \psi}{\partial x_1} + \frac{\beta u_{tx_1}}{u_{x_1x_2}(\mu + \beta)} \frac{\partial \psi}{\partial x_2} + \frac{\partial \psi}{\partial t} &= 0, \\ \frac{\mu u_{yx_2}}{u_{x_1x_2}(\mu - \gamma)} \frac{\partial \psi}{\partial x_1} - \frac{\gamma u_{yx_1}}{u_{x_1x_2}(\mu - \gamma)} \frac{\partial \psi}{\partial x_2} + \frac{\partial \psi}{\partial y} &= 0 \end{aligned}$$

for a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C})$ for all $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$.

4.5. The Alonso–Shabat heavenly type equation. This equation [2] has the form

$$u_{yx_2} - u_t u_{yx_1} + u_y u_{tx_1} = 0, \tag{4.22}$$

where $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R})$, $(y, t) \in \mathbb{R}^2$ and $(x_1, x_2) \in \mathbb{T}^2$. To prove its Lax integrability, we define a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{\text{diff}}^*(\mathbb{T}^2)$ of the form

$$\tilde{l} = z_{x_1}^2(\lambda + 1) dx_1 + z_{x_1} z_{x_2}(\lambda + 1) dx_2,$$

for a fixed function $z \in C^\infty(\mathbb{T}^2; \mathbb{R})$. Then one easily obtains asymptotic expansions as $|\lambda| \rightarrow \infty$ for coefficients of the two independent Casimir functionals $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ gradients:

$$\nabla h^{(1)}(l) \sim (1/z_{x_1} + kz_{x_2}/z_{x_1}, -k)^\top + O(1/\lambda^2), \tag{4.23}$$

$$\nabla h^{(2)}(l) \sim (z_{x_2}/z_{x_1}, -1)^\top + O(1/\lambda^2),$$

where $k \neq 1$ is a constant and $\alpha_1, \alpha_2 \in C^\infty(\mathbb{T}^1; \mathbb{R})$ are different functions. Using the Casimir functionals (4.23) in the case $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ with $\tilde{\mathcal{G}}_-|_{\lambda=\infty} = 0$, one can construct the simplest two commuting flows

$$\partial \tilde{l} / \partial y = -ad_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}, \quad \partial \tilde{l} / \partial t = -ad_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l} \tag{4.24}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$, where

$$\nabla h^{(y)}(l) := (\lambda \nabla h^{(1)}(l))_+ = (\lambda/z_{x_1} + \lambda k z_{x_2}/z_{x_1}, -\lambda k)^\top := (\lambda u_y, -\lambda k)^\top, \tag{4.25}$$

$$\nabla h^{(t)}(l) := (\lambda \nabla h^{(2)}(l))_+ = (\lambda z_{x_2}/z_{x_1}, -\lambda)^\top := (\lambda u_t, -\lambda)^\top$$

for some function $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R})$. From relationships (4.25), as a result of the commutativity of the flows (4.24), one derives the equivalent Lax type relationship (3.9) for the vector fields, namely

$$\nabla h^{(y)}(\tilde{l}) := \langle \nabla h^{(y)}(l), \partial / \partial x \rangle = \lambda u_y \partial / \partial x_1 - k \lambda \partial / \partial x_2,$$

$$\nabla h^{(t)}(\tilde{l}) := \langle \nabla h^{(t)}(l), \partial / \partial x \rangle = \lambda u_t \partial / \partial x_1 - \lambda \partial / \partial x_2,$$

which can be rewritten as the compatibility condition for the following vector field equations:

$$\frac{\partial \psi}{\partial t} + \lambda u_t \frac{\partial \psi}{\partial x_1} - \lambda \frac{\partial \psi}{\partial x_2} = 0, \quad \frac{\partial \psi}{\partial y} + \lambda u_y \frac{\partial \psi}{\partial x_1} - k \lambda \frac{\partial \psi}{\partial x_2} = 0,$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C})$, any $(t, y; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$ and all $\lambda \in \mathbb{C}$. The resulting equation is then

$$u_{yx_2} - u_t u_{yx_1} + u_y u_{tx_1} + k u_{tx_2} = 0,$$

which reduces at $k = 0$ to the Alonso – Shabat heavenly equation (4.22).

Remark 4.2. It is interesting to observe that the seed elements $\tilde{l} \in \tilde{\mathcal{G}}^*$ of the examples presented above have the differential geometric structure

$$\tilde{l} = \eta d\rho,$$

where η and $\rho \in C^\infty(\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{T}^2); \mathbb{C})$ are some smooth functions. For instance,

$$\tilde{l} = d(\lambda x - 2u) \quad - \text{Mikhalev – Pavlov heavenly type equation,}$$

$$\tilde{l} = d(\lambda x_1 + \lambda x_2 - u_{x_2} + u_{x_1}) \quad - \text{Plebański heavenly equation,}$$

$$\tilde{l} = u_{x_1 x_2} \xi du_{x_2}, \xi := (\mu[\gamma(\mu + \beta)]^{-1} + \alpha^{-1} - \mu[\beta(\mu - \gamma)]^{-1}) \quad - \text{general heavenly equation,}$$

$$\tilde{l} = (\lambda + 1) z_{x_1} dz \quad - \text{Alonso – Shabat heavenly type equation.}$$

5. The generalized Lie-algebraic structures and related heavenly type quasi-Hamiltonian systems. *5.1. The metrized Lie algebra of holomorphic vector fields on torus.* It worth mention now that, following Ovsienko’s scheme [41, 42], one can consider a virtually wider class of integrable heavenly equations, realized as compatible Hamiltonian flows on the semidirect product of the holomorphic loop Lie algebra $\tilde{\mathcal{G}}$ of vector fields on the torus \mathbb{T}^n and its regular co-adjoint space $\tilde{\mathcal{G}}^*$, supplemented with naturally related cocycles.

Below we proceed to describing the Lie algebraic structure and integrability properties of a generalized hierarchy of the Lax–Sato type compatible systems of vector field equations and related nonlinear dynamical systems, based on the AKS-algebraic and related \mathcal{R} -structure schemes [5–7, 57, 69], recently devised in B. Szablikowski and A. Sergyeyev [62, 63], V. Ovsienko [41, 42] and also in [25, 55]. The devised there technique we apply to the holomorphic metrized loop Lie algebra $\tilde{\mathcal{G}} := \text{diff}(\mathbb{T}^n) \ltimes \text{diff}(\mathbb{T}^n)^*$, the semidirect sum of the loop Lie algebra $\text{diff}(\mathbb{T}^n) := \widetilde{\text{Vect}}(\mathbb{T}^n)$ of vector fields on the torus \mathbb{T}^n , $n \in \mathbb{Z}_+$, and its dual space $\text{diff}(\mathbb{T}^n)^*$, studying the orbits of the coadjoint actions on the dual space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$, related with the classical Lie–Poisson type structures on them. This construction, being centrally extended, makes it possible to construct commuting to each other Hamiltonian flows on the suitably extended adjoint space $\tilde{\mathcal{G}}^*$, generated by a suitably chosen seed element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ and the related Casimir invariants. We successively demonstrate that their compatibility condition, being equivalent to that for a system of specially constructed three linear Lax–Sato type vector field equations, gives rise to an infinite hierarchy of new heavenly type Hamiltonian flows, interesting for applications.

Let $\widetilde{\text{Diff}}_{\pm}(\mathbb{T}^n)$, $n \in \mathbb{Z}_+$, be subgroups of the loop diffeomorphisms group $\widetilde{\text{Diff}}(\mathbb{T}^n) := \{\mathbb{C} \supset \supset \mathbb{S}^1 \rightarrow \text{Diff}(\mathbb{T}^n)\}$, holomorphically extended, respectively, on the interior $\mathbb{S}^1_+ \subset \mathbb{C}$ and on the exterior $\mathbb{S}^1_- \subset \mathbb{C}$ regions of the unit circle $\mathbb{S}^1 \subset \mathbb{C}^1$, such that for any $g(\lambda) \in \widetilde{\text{Diff}}_{\pm}(\mathbb{T}^n)$, $\lambda \in \mathbb{S}^1_{\pm}$, either $g(\infty) = 1 \in \text{Diff}(\mathbb{T}^n)$ or $g(0) = 1 \in \text{Diff}(\mathbb{T}^n)$. The corresponding Lie subalgebras $\widetilde{\text{diff}}_{\pm}(\mathbb{T}^n) \simeq \widetilde{\text{Vect}}_{\pm}(\mathbb{T}^n)$ of the loop subgroups $\widetilde{\text{Diff}}_{\pm}(\mathbb{T}^n)$ are vector fields on \mathbb{T}^n holomorphic, respectively, on $\mathbb{S}^1_{\pm} \subset \mathbb{C}^1$, where either for any $\tilde{a}(\lambda) \in \widetilde{\text{diff}}_-(\mathbb{T}^n)$ the value $\tilde{a}(\infty) = 0$ or for any $\tilde{a}(\lambda) \in \widetilde{\text{diff}}_+(\mathbb{T}^n)$ the value $\tilde{a}(0) = 0$. The split loop Lie algebra $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ can be naturally identified with a dense subspace of the dual space $\text{diff}(\mathbb{T}^n)^*$ through the pairing

$$(\tilde{l}, \tilde{a})_1 := \text{res}_{\lambda \in \mathbb{C}} (l(x; \lambda), a(x; \lambda))_{H^0} \tag{5.1}$$

for any elements $\tilde{l} \in \widetilde{\text{diff}}(\mathbb{T}^n)^*$ and $\tilde{a} \in \widetilde{\text{diff}}(\mathbb{T}^n)$. We took above, by definition [15, 55], that a loop vector field $\tilde{a} \in \widetilde{\text{diff}}(\mathbb{T}^n) \simeq \widetilde{\text{Vect}}_{\pm}(\mathbb{T}^n) = \Gamma(\tilde{T}(\mathbb{T}^n))$ and a loop differential 1-form $\tilde{l} \in \tilde{\Lambda}^1(\mathbb{T}^n)$ are presented in the form

$$\tilde{a} = \sum_{j=1}^n a^{(j)}(x; \lambda) \frac{\partial}{\partial x_j} := \left\langle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle,$$

$$\tilde{l} = \sum_{j=1}^n l_j(x; \lambda) dx_j := \langle l(x; \lambda), dx \rangle,$$

introduced for brevity the gradient operator $\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)^T$ in the Euclidean space \mathbb{E}^n and chose the metric $(\cdot, \cdot)_{H^0}$ on the space $C^\infty(\mathbb{T}^n; \mathbb{R}^n) \subset H^0(\mathbb{T}^n; \mathbb{R}^n)$ as

$$(l(x; \lambda), a(x; \lambda))_{H^0} := \sum_{j=1}^n \int_{\mathbb{T}^n} l_j(x; \lambda) a^{(j)}(x; \lambda) dx.$$

The Lie commutator of vector fields $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$ is calculated the standard way and equals

$$\begin{aligned} [\tilde{a}, \tilde{b}] &= \tilde{a}\tilde{b} - \tilde{b}\tilde{a} = \left\langle \left\langle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle b(x; \lambda), \frac{\partial}{\partial x} \right\rangle - \\ &\quad - \left\langle \left\langle b(x; \lambda), \frac{\partial}{\partial x} \right\rangle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle. \end{aligned}$$

Construct now the Lie algebra $\tilde{\mathcal{G}} := \widetilde{\text{diff}}(\mathbb{T}^n) \times \widetilde{\text{diff}}(\mathbb{T}^n)^*$ as the semidirect sum of the Lie algebra $\widetilde{\text{diff}}(\mathbb{T}^n)$ and its dual space $\widetilde{\text{diff}}(\mathbb{T}^n)^*$, whose Lie structure is given by the following expression:

$$[\tilde{a}_1 \times \tilde{l}_1, \tilde{a}_2 \times \tilde{l}_2] := [\tilde{a}_1, \tilde{a}_2] \times (ad_{\tilde{a}_1}^* \tilde{l}_2 - ad_{\tilde{a}_2}^* \tilde{l}_1) \tag{5.2}$$

for any pair of elements $(\tilde{a}_1 \times \tilde{l}_1), (\tilde{a}_2 \times \tilde{l}_2) \in \tilde{\mathcal{G}}$, where $ad_{\widetilde{\text{diff}}(\mathbb{T}^n)}^* : \widetilde{\text{diff}}(\mathbb{T}^n)^* \rightarrow \widetilde{\text{diff}}(\mathbb{T}^n)^*$ is the standard coadjoint mapping of the Lie algebra $\widetilde{\text{diff}}(\mathbb{T}^n)$ on its adjoint space $\widetilde{\text{diff}}(\mathbb{T}^n)^*$ with respect to the pairing (5.1). The Lie algebra $\tilde{\mathcal{G}}$ proves to be metrizable, as it can be endowed with the nondegenerate ad-invariant bilinear symmetric product:

$$(\tilde{a}_1 \times \tilde{l}_1, \tilde{a}_2 \times \tilde{l}_2) := (\tilde{l}_2, \tilde{a}_1)_1 + (\tilde{l}_1, \tilde{a}_2)_1, \tag{5.3}$$

where $\tilde{a}_1 \times \tilde{l}_1, \tilde{a}_2 \times \tilde{l}_2 \in \tilde{\mathcal{G}}$ are arbitrary elements. Owing to the holomorphic structure of the Lie algebra $\widetilde{\text{diff}}(\mathbb{T}^n)$, the ad-invariant product (5.3) makes it possible to identify the Lie algebra $\tilde{\mathcal{G}}$ with its dual $\tilde{\mathcal{G}}^*$, that is $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$. Moreover, the Lie algebra $\tilde{\mathcal{G}}$ can be naturally split [41, 42, 62] with respect to the the pairing (5.1) and the Lie bracket (5.2) into two subalgebras $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, where, by definition,

$$\tilde{\mathcal{G}}_+ := \widetilde{\text{diff}}(\mathbb{T}^n)_+ \times \widetilde{\text{diff}}(\mathbb{T}^n)_-^*, \quad \tilde{\mathcal{G}}_- := \widetilde{\text{diff}}(\mathbb{T}^n)_- \times \widetilde{\text{diff}}(\mathbb{T}^n)_+^*.$$

The latter allows to define on the Lie algebra $\tilde{\mathcal{G}}$ a new Lie bracket

$$[\tilde{w}_1, \tilde{w}_2]_{\mathcal{R}} := [\mathcal{R}\tilde{w}_1, \tilde{w}_2] + [\tilde{w}_1, \mathcal{R}\tilde{w}_2]$$

for any elements $\tilde{w}_1, \tilde{w}_2 \in \tilde{\mathcal{G}}$, where $\mathcal{R} := (P_+ - P_-)/2$ is the standard \mathcal{R} -matrix homomorphism [5, 6, 69] on $\tilde{\mathcal{G}}$ and, by definition, $P_{\pm} : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}_{\pm} \subset \tilde{\mathcal{G}}$ are projectors. The construction above makes it possible to apply to the Lie algebra $\tilde{\mathcal{G}}$ the classical AKS-scheme and, respectively, to generate a wide class of completely integrable Hamiltonian systems as the commuting flows on the adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$, generated by the corresponding hierarchies of the Casimir invariants subject to the basic Lie bracket (5.3).

To describe this scheme in more details, we need to find the corresponding Casimir functionals $h \in I(\tilde{\mathcal{G}}^*)$, satisfying, by definition, the following relationship:

$$[\nabla h(\tilde{a}, \tilde{l}), \tilde{a} \times \tilde{l}] = 0, \tag{5.4}$$

where, by definition, the gradient $\nabla h(\tilde{a}, \tilde{l}) := \nabla h_{\tilde{l}} \times \nabla h_{\tilde{a}} \in \tilde{\mathcal{G}}$ satisfies the following from (5.4) differential-algebraic equations:

$$[\nabla h_{\tilde{l}}, \tilde{a}] = 0, \quad ad_{\nabla h_{\tilde{l}}}^* \tilde{l} - ad_{\tilde{a}}^* \nabla h_{\tilde{a}} = 0 \tag{5.5}$$

for arbitrarily chosen element $\tilde{a} \times \tilde{l} \in \tilde{\mathcal{G}} \simeq \tilde{\mathcal{G}}^*$. The equations (5.5) can be rewritten [55] in details as

$$\begin{aligned} \left\langle \nabla h_l, \frac{\partial}{\partial x} \right\rangle a - \left\langle a, \frac{\partial}{\partial x} \right\rangle \nabla h_l &= 0, \\ \left\langle \frac{\partial}{\partial x}, \nabla h_l \right\rangle l + \left\langle l, \left(\frac{\partial}{\partial x} \nabla h_l \right) \right\rangle - \left\langle \frac{\partial}{\partial x}, a \right\rangle \nabla h_a - \left\langle \nabla h_a, \left(\frac{\partial}{\partial x} a \right) \right\rangle &= 0, \end{aligned} \tag{5.6}$$

where we put, by definition, that

$$\nabla h_{\tilde{l}} := \left\langle \nabla h_l, \frac{\partial}{\partial x} \right\rangle, \quad \tilde{a} := \left\langle a, \frac{\partial}{\partial x} \right\rangle,$$

$$\tilde{l} := \langle l, dx \rangle, \quad \nabla h_{\tilde{a}} := \langle \nabla h_a, dx \rangle.$$

The system of linear equation (5.6) for a given element $\tilde{a} \times \tilde{l} \in \tilde{\mathcal{G}} \simeq \tilde{\mathcal{G}}^*$, singular as $\lambda \rightarrow \infty$, can be, in general, resolved by means of the asymptotical expressions

$$\nabla h_l \sim \sum_{j \in \mathbb{Z}_+} \nabla h_l^{(j)} \lambda^{-j}, \quad \nabla h_a \sim \sum_{j \in \mathbb{Z}_+} \nabla h_a^{(j)} \lambda^{-j}$$

giving rise to an infinite hierarchy of gradients $\nabla h^{(p)}(\tilde{a}, \tilde{l}) := \lambda^p \nabla h(\tilde{a}, \tilde{l}) \in \tilde{\mathcal{G}}, p \in \mathbb{Z}_+$, for the corresponding Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*), p \in \mathbb{Z}_+$. Similarly, if a given element $\tilde{a} \times \tilde{l} \in \tilde{\mathcal{G}} \simeq \tilde{\mathcal{G}}^*$ is chosen to be singular as $\lambda \rightarrow 0$, the system of linear equations (5.6) can be resolved by means of the asymptotical expressions

$$\nabla h_l \sim \sum_{j \in \mathbb{Z}_+} \nabla h_l^{(j)} \lambda^j, \quad \nabla h_a \sim \sum_{j \in \mathbb{Z}_+} \nabla h_a^{(j)} \lambda^j$$

also generating an infinite hierarchy of gradients $\nabla h^{(p)}(\tilde{a}, \tilde{l}) = \lambda^{-p} \nabla h(\tilde{a}, \tilde{l}) \in \tilde{\mathcal{G}}, p \in \mathbb{Z}_+$, for the corresponding Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*), p \in \mathbb{Z}_+$.

Let us now assume that we have already found the projected gradients $\nabla h^{(y)}(\tilde{a}, \tilde{l}) := (\lambda^{p_y} \nabla h^{(1)}(\tilde{a}, \tilde{l}))_+, \nabla h^{(t)}(\tilde{a}, \tilde{l}) := (\lambda^{p_t} \nabla h^{(2)}(\tilde{a}, \tilde{l}))_+ \in \tilde{\mathcal{G}}_+$, related to two chosen Casimir invariants $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ (not necessary different) for some integers $p_y, p_t \in \mathbb{Z}$, satisfying the determining equations (5.6). Then, owing to the classical AKS-scheme [5, 6, 65, 69], one can construct two commuting to each other flows with respect to the evolution parameters $y, t \in \mathbb{R}$ on the adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$

$$\frac{\partial}{\partial y} \tilde{a} = -[\nabla h_l^{(y)}(\tilde{a}, \tilde{l}), \tilde{a}], \quad \frac{\partial}{\partial t} \tilde{a} = -[\nabla h_l^{(t)}(\tilde{a}, \tilde{l}), \tilde{a}], \tag{5.7}$$

and

$$\frac{\partial}{\partial y} \tilde{l} = ad_{\nabla h_l^{(y)}(\tilde{a}, \tilde{l})}^* \tilde{l} - ad_{\tilde{a}}^*(\nabla h_a^{(y)}(\tilde{a}, \tilde{l})), \quad \frac{\partial}{\partial t} \tilde{l} = ad_{\nabla h_l^{(t)}(\tilde{a}, \tilde{l})}^* \tilde{l} - ad_{\tilde{a}}^*(\nabla h_a^{(t)}(\tilde{a}, \tilde{l})). \tag{5.8}$$

The flows (5.7) and (5.8) are, by construction, Hamiltonian, as they result from the expressions

$$\frac{\partial}{\partial y}(\tilde{a} \times \tilde{l}) = \{h^{(1)}, \tilde{a} \times \tilde{l}\}_{\mathcal{R}}, \quad \frac{\partial}{\partial t}(\tilde{a} \times \tilde{l}) = \{h^{(2)}, \tilde{a} \times \tilde{l}\}_{\mathcal{R}} \tag{5.9}$$

for a chosen element $\tilde{a} \times \tilde{l} \in \tilde{\mathcal{G}} \simeq \tilde{\mathcal{G}}^*$, stemming from the \mathcal{R} -deformed Lie – Poisson [5, 6, 57, 65, 69] bracket

$$\{h, f\}_{\mathcal{R}} := (\tilde{a} \times \tilde{l}, [\nabla h(\tilde{l}, \tilde{a}), \nabla f(\tilde{l}, \tilde{a})]_{\mathcal{R}}) \tag{5.10}$$

on the space $\tilde{\mathcal{G}}^*$, defined for any smooth functionals $h, f \in D(\tilde{\mathcal{G}}^*)$. From the reasonings above one can derive by means of easy yet cumbersome calculations the following important proposition.

Proposition 5.1. *The Hamiltonian flows (5.9) generate the separately commuting evolution flows (5.7) and (5.8), giving rise to the following unique Lax – Sato type compatibility condition:*

$$[\nabla h_l^{(y)}(\tilde{a}, \tilde{l}), \nabla h_l^{(t)}(\tilde{a}, \tilde{l})] - \frac{\partial}{\partial t} \nabla h_l^{(y)}(\tilde{a}, \tilde{l}) + \frac{\partial}{\partial y} \nabla h_l^{(t)}(\tilde{a}, \tilde{l}) = 0,$$

being equivalent to some system of nonlinear heavenly type equations in partial derivatives.

The presented above construction of Hamiltonian flows on the adjoint space $\tilde{\mathcal{G}}^*$ still allows the next important generalization. Namely, let us endow the point product $\tilde{\mathcal{G}} := \prod_{z \in \mathbb{S}^1} \tilde{\mathcal{G}}$ of loop Lie algebra $\tilde{\mathcal{G}}$ with the central extension generated by a two-cocycle $\omega_2 : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \rightarrow \mathbb{C}$, where

$$\omega_2(\tilde{a}_1 \times \tilde{l}_1, \tilde{a}_2 \times \tilde{l}_2) := \int_{\mathbb{S}^1} dz [(l_1, \partial \tilde{a}_2 / \partial z)_1 - (l_2, \partial \tilde{a}_1 / \partial z)_1]$$

for any pair of elements $\tilde{a}_1 \times \tilde{l}_1, \tilde{a}_2 \times \tilde{l}_2 \in \tilde{\mathcal{G}}$. The resulting \mathcal{R} -deformed Lie – Poisson bracket (5.10) for any smooth functionals $h, f \in D(\tilde{\mathcal{G}}^*)$ on the adjoint space $\tilde{\mathcal{G}}^*$ becomes equal to

$$\begin{aligned} \{h, f\}_{\mathcal{R}} := & \int_{\mathbb{S}^1} dz (\tilde{a} \times \tilde{l}, [\nabla h(\tilde{a}, \tilde{l}), \nabla f(\tilde{a}, \tilde{l})]_{\mathcal{R}}) + \\ & + \omega_2(\mathcal{R} \nabla h(\tilde{a}, \tilde{l}), \nabla f(\tilde{a}, \tilde{l})) + \omega_2(\nabla h(\tilde{a}, \tilde{l}), \mathcal{R} \nabla f(\tilde{a}, \tilde{l})). \end{aligned} \tag{5.11}$$

The corresponding Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*)$, $p \in \mathbb{Z}_+$, are defined with respect to the standard Lie – Poisson bracket as

$$\{h^{(p)}, f\} := \int_{\mathbb{S}^1} dz (\tilde{a} \times \tilde{l}, [\nabla h^{(p)}(\tilde{a}, \tilde{l}), \nabla f(\tilde{a}, \tilde{l})]) + \omega_2(\nabla h^{(p)}(\tilde{a}, \tilde{l}), \nabla f(\tilde{a}, \tilde{l})) = 0 \tag{5.12}$$

for all smooth functionals $f \in D(\tilde{\mathcal{G}}^*)$. Based on the equality (5.12) one easily finds that the gradients $\nabla h^{(p)} \in \tilde{\mathcal{G}}$ of the Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*)$, $p \in \mathbb{Z}_+$, satisfy the following equations:

$$\begin{aligned} -[\nabla h_l^{(p)}(\tilde{a}, \tilde{l}), \tilde{a}] + \frac{\partial}{\partial z} \nabla h_l^{(p)}(\tilde{a}, \tilde{l}) &= 0, \\ ad_{\nabla h_l^{(p)}(\tilde{a}, \tilde{l})}^* \tilde{l} - ad_{\tilde{a}}^* \nabla h_a^{(p)}(\tilde{a}, \tilde{l}) - \frac{\partial}{\partial z} \nabla h_a^{(p)}(\tilde{a}, \tilde{l}) &= 0 \end{aligned}$$

for a chosen element $\tilde{a} \times \tilde{l} \in \tilde{\mathcal{G}}^*$. Making use of the suitable Casimir functionals $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$, one can construct from (5.11) the following commuting Hamiltonian flows on the adjoint space $\tilde{\mathcal{G}}^*$:

$$\frac{\partial}{\partial y}(\tilde{a} \times \tilde{l}) = \{h^{(1)}, \tilde{a} \times \tilde{l}\}_{\mathcal{R}}, \quad \frac{\partial}{\partial t}(\tilde{a} \times \tilde{l}) = \{h^{(2)}, \tilde{a} \times \tilde{l}\}_{\mathcal{R}}, \quad (5.13)$$

which are equivalent to the evolution equations

$$\begin{aligned} \frac{\partial}{\partial y} \tilde{a} &= -[\nabla h_l^{(y)}(\tilde{a}, \tilde{l}), \tilde{a}] + \frac{\partial}{\partial z} \nabla h_l^{(y)}(\tilde{a}, \tilde{l}), \\ \frac{\partial}{\partial t} \tilde{a} &= -[\nabla h_l^{(t)}(\tilde{a}, \tilde{l}), \tilde{a}] + \frac{\partial}{\partial z} \nabla h_l^{(t)}(\tilde{a}, \tilde{l}) \end{aligned} \quad (5.14)$$

and

$$\begin{aligned} \frac{\partial}{\partial y} \tilde{l} &= ad_{\nabla h_l^{(y)}(\tilde{a}, \tilde{l})}^* \tilde{l} - ad_{\tilde{a}}^*(\nabla h_a^{(y)}(\tilde{a}, \tilde{l})) - \frac{\partial}{\partial z} \nabla h_a^{(y)}(\tilde{a}, \tilde{l}), \\ \frac{\partial}{\partial t} \tilde{l} &= ad_{\nabla h_l^{(t)}(\tilde{a}, \tilde{l})}^* \tilde{l} - ad_{\tilde{a}}^*(\nabla h_a^{(t)}(\tilde{a}, \tilde{l})) - \frac{\partial}{\partial z} \nabla h_a^{(t)}(\tilde{a}, \tilde{l}), \end{aligned} \quad (5.15)$$

where we put, by definition,

$$\begin{aligned} \nabla h_l^{(y)}(\tilde{a}, \tilde{l}) &:= \nabla h_l^{(1)}(\tilde{a}, \tilde{l})_+, & \nabla h_l^{(t)}(\tilde{a}, \tilde{l}) &:= \nabla h_l^{(2)}(\tilde{a}, \tilde{l})_+, \\ \nabla h_a^{(y)}(\tilde{a}, \tilde{l}) &:= \nabla h_a^{(1)}(\tilde{a}, \tilde{l})_+, & \nabla h_a^{(t)}(\tilde{a}, \tilde{l}) &:= \nabla h_a^{(2)}(\tilde{a}, \tilde{l})_+ \end{aligned}$$

the corresponding projections on the subalgebra $\check{\mathcal{G}}_+$. As a consequence of the obtained above results one can formulate a next proposition.

Proposition 5.2. *The Hamiltonian flows (5.13) on the adjoint space $\check{\mathcal{G}}^*$ generate the separately commuting evolution flows (5.14) and (5.15), giving rise to the following unique Lax–Sato type compatibility condition:*

$$[\nabla h_l^{(y)}(\tilde{a}, \tilde{l}), \nabla h_l^{(t)}(\tilde{a}, \tilde{l})] - \frac{\partial}{\partial t} \nabla h_l^{(y)}(\tilde{a}, \tilde{l}) + \frac{\partial}{\partial y} \nabla h_l^{(t)}(\tilde{a}, \tilde{l}) = 0,$$

being equivalent to some system of nonlinear heavenly type equations in partial derivatives. Moreover, the system of evolution flows (5.14) and (5.15) can be considered as the compatibility condition for the following set of linear vector equations:

$$\frac{\partial \psi}{\partial y} + \nabla h_l^{(y)}(\tilde{a}, \tilde{l})\psi = 0, \quad \frac{\partial \psi}{\partial z} + \tilde{a}\psi = 0, \quad \frac{\partial \psi}{\partial t} + \nabla h_l^{(t)}(\tilde{a}, \tilde{l})\psi = 0$$

for all $(y, t; \lambda, z, x) \in \mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1) \times \mathbb{T}^n$ and a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times (\mathbb{S}^1 \times \mathbb{T}^n); \mathbb{C})$.

5.2. The generalized Lie algebra of holomorphic vector fields on torus. It is well known that the loop Lie algebra $\check{\mathcal{G}} := \widetilde{\text{diff}}(\mathbb{T}^n)$ can be centrally extended as $\hat{\mathcal{G}} := (\widetilde{\text{diff}}(\mathbb{T}^n); \mathbb{R}^1)$ only [23] for the case $n = 1$, where for any two elements $(\tilde{a}; \alpha)$ and $(\tilde{b}; \beta) \in \hat{\mathcal{G}}$ the commutator

$$[(\tilde{a}; \alpha), (\tilde{b}; \beta)] = ([\tilde{a}, \tilde{b}]; \omega_2(\tilde{a}, \tilde{b})) \in \check{\mathcal{G}}$$

and the 2-cocycle $\omega_2 : \check{\mathcal{G}} \times \check{\mathcal{G}} \rightarrow \mathbb{R}^1$ satisfies the condition

$$\omega_2([\tilde{a}, \tilde{b}], \tilde{c}) + \omega_2([\tilde{b}, \tilde{c}], \tilde{a}) + \omega_2([\tilde{c}, \tilde{a}], \tilde{b}) = 0$$

for any \tilde{a}, \tilde{b} and $\tilde{c} \in \tilde{\mathcal{G}}$. For the case $n = 1$, the Gelfand–Fuchs 2-cocycle [23] on the loop Lie algebra $\tilde{\mathcal{G}}$ equals the expression

$$\omega_2(\tilde{a}, \tilde{b}) = \operatorname{res}_{\lambda \in \mathbb{C}} \left(\lambda^{-p} \frac{\partial^2 a(x; \lambda)}{\partial x^2}, \frac{\partial b(x; \lambda)}{\partial x} \right)_{H^0}$$

for any vector fields $\tilde{a} = a(x; \lambda) \frac{\partial}{\partial x}$, $\tilde{b} = b(x; \lambda) \frac{\partial}{\partial x} \in \tilde{\mathcal{G}}$ on \mathbb{T}^1 and a fixed integer $p \in \mathbb{Z}$.

The integrable dynamical systems related to this central extension were described in detail in [38]. Concerning a further generalization of the multidimensional case related to the loop group $\tilde{\mathcal{G}}$ for $n \in \mathbb{Z}_+$ one can proceed in the following natural way: as the Lie algebra $\tilde{\mathcal{G}} = \widetilde{\operatorname{diff}(\mathbb{T}^n)}$ consists of the elements formally depending additionally on the “spectral” variable $\lambda \in \mathbb{C}^1$, one can extend the basic Lie structure on $\mathcal{G} = \operatorname{diff}(\mathbb{T}^n)$ to that on the adjacent holomorphic in $\lambda \in \mathbb{S}_{\pm}^1$ Lie algebra $\bar{\mathcal{G}} := \operatorname{diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n) \subset \operatorname{diff}(\mathbb{C} \times \mathbb{T}^n)$ of vector fields on $\mathbb{C} \times \mathbb{T}^n$. This has elements representable as $\bar{a}(x; \lambda) := \left\langle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle = \sum_{j=1}^n a_j(x; \lambda) \frac{\partial}{\partial x_j} + a_0(x; \lambda) \frac{\partial}{\partial \lambda} \in \bar{\mathcal{G}}$ for some holomorphic in $\lambda \in \mathbb{S}_{\pm}^1$ vectors $a(x; \lambda) \in \mathbb{E} \times \mathbb{E}^n$ for all $x \in \mathbb{T}^n$, where $\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)^{\top}$ is the generalized Euclidean vector gradient with respect to the vector variable $x := (\lambda, x) \in \mathbb{C} \times \mathbb{T}^n$.

It is now important to mention that the Lie algebra $\bar{\mathcal{G}} \subset \operatorname{diff}(\mathbb{C} \times \mathbb{T}^n)$ also splits into the direct sum of two subalgebras:

$$\bar{\mathcal{G}} = \bar{\mathcal{G}}_+ \oplus \bar{\mathcal{G}}_-,$$

allowing to introduce on it the classical \mathcal{R} -structure:

$$[\bar{a}, \bar{b}]_{\mathcal{R}} := [\mathcal{R}\bar{a}, \bar{b}] + [\bar{a}, \mathcal{R}\bar{b}]$$

for any $\bar{a}, \bar{b} \in \bar{\mathcal{G}}$, where

$$\mathcal{R} := (P_+ - P_-)/2,$$

and

$$P_{\pm} \bar{\mathcal{G}} := \bar{\mathcal{G}}_{\pm} \subset \bar{\mathcal{G}}.$$

The space $\bar{\mathcal{G}}^* \simeq \Lambda^1(\mathbb{C} \times \mathbb{T}^n)$, adjoint to the Lie algebra $\bar{\mathcal{G}}$ of vector fields on $\mathbb{C} \times \mathbb{T}^n$, can be functionally identified with $\bar{\mathcal{G}}$ subject to the Sobolev type metric

$$(\bar{l}, \bar{a}) = \operatorname{res}_{\lambda \in \mathbb{C}} (l, a)_{H^q}, \tag{5.16}$$

where $q \in \mathbb{Z}_+$ and for arbitrary $\bar{l} := \langle l(x; \lambda), dx \rangle = \sum_{j=0, \dots, n} l_j(x; \lambda) dx_j \in \bar{\mathcal{G}}^*$, $\bar{a} = \sum_{j=0, \dots, n} a_j(x; \lambda) \frac{\partial}{\partial x_j} \in \bar{\mathcal{G}}$ one defines

$$(l, a)_{H^q} = \sum_{j=0}^n \sum_{|\alpha|=0}^q \int_{\mathbb{T}^n} dx \frac{\partial^{|\alpha|} l_j}{\partial x^{\alpha}} \frac{\partial^{|\alpha|} a_j}{\partial x^{\alpha}}.$$

In particular, for $q = 0$ one has

$$(\bar{l}, \bar{a}) = \operatorname{res}_{\lambda \in \mathbb{C}} \int_{\mathbb{T}^n} dx \left(\sum_{j=0}^n l_j a_j \right),$$

the case which will be mainly chosen later on. Then for arbitrary $f, g \in D(\bar{\mathcal{G}}^*)$, one can determine two Lie–Poisson brackets

$$\{f, g\} := (\bar{l}, [\nabla f(\bar{l}), \nabla g(\bar{l})])$$

and

$$\{f, g\}_{\mathcal{R}} := (\bar{l}, [\nabla f(\bar{l}), \nabla g(\bar{l})]_{\mathcal{R}}), \tag{5.17}$$

where at any seed element $\bar{l} \in \bar{\mathcal{G}}^*$ the gradient element $\nabla f(\bar{l})$ and $\nabla g(\bar{l}) \in \bar{\mathcal{G}}$ are calculated with respect to the metric (5.16).

Now let us assume that a smooth function $h \in I(\bar{\mathcal{G}}^*)$ is a Casimir invariant, that is

$$ad_{\nabla h(\bar{l})}^* \bar{l} = 0 \tag{5.18}$$

for a chosen seed element $\bar{l} \in \bar{\mathcal{G}}^*$. As the adjoint mapping $ad_{\nabla f(\bar{l})}^* \bar{l}$ for any $f \in D(\bar{\mathcal{G}}^*)$ can be rewritten in the reduced form as

$$ad_{\nabla f(\bar{l})}^* \bar{l} = \left\langle \frac{\partial}{\partial \mathbf{x}}, \nabla f(l) \right\rangle \bar{l} + \sum_{j=1}^n \left\langle \left\langle l, \frac{\partial}{\partial \mathbf{x}} \nabla f(l) \right\rangle, dx \right\rangle,$$

where $\nabla f(\bar{l}) := \left\langle \nabla f(l), \frac{\partial}{\partial \mathbf{x}} \right\rangle \in \bar{\mathcal{G}}$. For the Casimir function $h \in D(\bar{\mathcal{G}}^*)$, the condition (5.18) is then equivalent to the equation

$$\left\langle \frac{\partial}{\partial \mathbf{x}}, \nabla f(l) \right\rangle l + \left\langle l, \left(\frac{\partial}{\partial \mathbf{x}} \nabla h(l) \right) \right\rangle = 0, \tag{5.19}$$

which should be solved analytically. In the case when an element $\bar{l} \in \bar{\mathcal{G}}^*$ is singular as $|\lambda| \rightarrow \infty$, one can consider the general asymptotic expansion

$$\nabla h := \nabla h^{(p)} \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(p)} \lambda^{-j} \tag{5.20}$$

for some suitably chosen integers $p \in \mathbb{Z}_+$, and upon substituting (5.20) into the equation (5.19), one can solve it recurrently.

Now let $h^{(p_y)}, h^{(p_t)} \in I(\bar{\mathcal{G}}^*)$ be such Casimir functions for which the Hamiltonian vector field generators

$$\nabla h^{(y)}(l) := (\nabla h^{(p_y)}(l))_+, \quad \nabla h^{(t)}(l) := (\nabla h^{(p_t)}(l))_+ \tag{5.21}$$

are, respectively, defined for special integers $p_y, p_t \in \mathbb{Z}_+$. These invariants generate, owing to the Lie–Poisson bracket (5.17), for the case $q = 0 = p$ the following commuting flows:

$$\begin{aligned} \partial l / \partial t &= - \left\langle \frac{\partial}{\partial \mathbf{x}}, \nabla h^{(t)}(l) \right\rangle l - \left\langle l, \left(\frac{\partial}{\partial \mathbf{x}} \nabla h^{(t)}(l) \right) \right\rangle, \\ \partial l / \partial y &= - \left\langle \frac{\partial}{\partial \mathbf{x}}, \nabla h^{(y)}(l) \right\rangle l - \left\langle l, \left(\frac{\partial}{\partial \mathbf{x}} \nabla h^{(y)}(l) \right) \right\rangle, \end{aligned} \tag{5.22}$$

where $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since the invariants $h^{(p_y)}, h^{(p_t)} \in I(\bar{\mathcal{G}}^*)$ commute with respect to the Lie–Poisson bracket (5.17), the flows (5.22) also commute, implying that the corresponding Hamiltonian vector field generators

$$\nabla h^{(t)}(\bar{l}) := \left\langle \nabla h^{(t)}(l), \frac{\partial}{\partial \mathbf{x}} \right\rangle, \quad \nabla h^{(y)}(\bar{l}) := \left\langle \nabla h^{(y)}(l), \frac{\partial}{\partial \mathbf{x}} \right\rangle \tag{5.23}$$

satisfy the Lax – Sato compatibility condition

$$\frac{\partial}{\partial y} \nabla h^{(t)}(\bar{l}) - \frac{\partial}{\partial t} \nabla h^{(y)}(\bar{l}) = [\nabla h^{(t)}(\bar{l}), \nabla h^{(y)}(\bar{l})] \tag{5.24}$$

for all $y, t \in \mathbb{R}$. On the other hand, the condition (5.24) is equivalent to the compatibility condition of two linear equations

$$\left(\frac{\partial}{\partial t} + \nabla h^{(t)}(\bar{l}) \right) \psi = 0, \quad \left(\frac{\partial}{\partial y} + \nabla h^{(y)}(\bar{l}) \right) \psi = 0 \tag{5.25}$$

for a function $\psi \in C^2(\mathbb{R} \times \mathbb{T}^n; \mathbb{C})$ for all $y, t \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$.

The above can be formulated as the following key result:

Proposition 5.3. *Let a seed vector field be $\bar{l} \in \bar{\mathcal{G}}^*$ and $h^{(p_y)}, h^{(p_t)} \in I(\bar{\mathcal{G}}^*)$ be Casimir functions subject to the metric (\cdot, \cdot) on the loop Lie algebra $\bar{\mathcal{G}}$ and the natural coadjoint action on the loop co-algebra $\bar{\mathcal{G}}^*$. Then the dynamical systems*

$$\partial \bar{l} / \partial y = -ad_{\nabla h^{(y)}(\bar{l})}^* \bar{l}, \quad \partial \bar{l} / \partial t = -ad_{\nabla h^{(t)}(\bar{l})}^* \bar{l} \tag{5.26}$$

are commuting Hamiltonian flows for all $y, t \in \mathbb{R}$. Moreover, the compatibility condition of these flows is equivalent to the vector fields representation

$$\left(\partial / \partial t + \nabla h^{(t)}(\bar{l}) \right) \psi = 0, \quad \left(\partial / \partial y + \nabla h^{(y)}(\bar{l}) \right) \psi = 0,$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$ and the vector fields $\nabla h^{(t)}(\bar{l}), \nabla h^{(y)}(\bar{l}) \in \bar{\mathcal{G}}$ are given by the expressions (5.23) and (5.21).

Remark 5.1. As mentioned above, the expansion (5.20) is effective if a chosen seed element $\bar{l} \in \bar{\mathcal{G}}^*$ is singular as $|\lambda| \rightarrow \infty$. In the case when it is singular as $|\lambda| \rightarrow 0$, the expression (5.20) should be replaced by the expansion

$$\nabla h^{(p)}(l) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(p)}(l) \lambda^j$$

for suitably chosen integers $p \in \mathbb{Z}_+$, and the reduced Casimir function gradients then are given by the Hamiltonian vector field generators

$$\nabla h^{(y)}(l) := (\nabla h^{(p_y)}(l))_-, \quad \nabla h^{(t)}(l) := (\nabla h^{(p_t)}(l))_-$$

for suitably chosen positive integers $p_y, p_t \in \mathbb{Z}_+$ and the splitting $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ with either $\tilde{\mathcal{G}}_-|_{\lambda=\infty} = 0$ or $\tilde{\mathcal{G}}_+|_{\lambda=0} = 0$. Moreover, the corresponding Hamiltonian flows are, respectively, written as

$$\partial \bar{l} / \partial t = -ad_{\nabla h^{(t)}(\bar{l})}^* \bar{l}, \quad \partial \bar{l} / \partial y = -ad_{\nabla h^{(y)}(\bar{l})}^* \bar{l}, \tag{5.27}$$

where we need to mention that, owing to the analytical structure of the seed element $\bar{l} \in \bar{\mathcal{G}}^*$, the corresponding functional evolution equations on its coefficients become only quasi-Hamiltonian, being suitable reductions of the true Hamiltonian flows (5.27).

As in Section 3 the Proposition 5.3 above makes it possible to describe the Bäcklund transformations between two special solution sets for the dispersionless heavenly equations resulting from the Lax compatibility condition (5.26). Let a diffeomorphism $\xi \in \text{Diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$ be such that a seed loop differential form $\bar{l}(\lambda, x) \in \mathcal{G}^* \simeq \Lambda^1(\mathbb{C} \times \mathbb{T}^n)$ satisfies the invariance condition

$$\bar{l}(\xi(x; \mu)) = k\bar{l}(\bar{x}) \tag{5.28}$$

for some non-zero constant $k \in \mathbb{C} \setminus \{0\}$, any $x = (\lambda, x)$ and $\bar{x} = (\mu, x) \in \mathbb{C} \times \mathbb{T}^n$ and arbitrarily an chosen parameter $\mu \in \mathbb{C}$. As the seed element $\bar{l}(\xi(x; \mu)) \in \Lambda^1_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$ satisfies simultaneously the system of compatible equations (5.26), the loop diffeomorphism $\xi \in \text{Diff}_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$, found analytically from the invariance condition (5.28), satisfies the compatible system of vector field equations

$$\frac{\partial}{\partial t}\xi = \nabla h^{(t)}(l), \quad \frac{\partial}{\partial y}\xi = \nabla h^{(y)}(l),$$

giving rise to the Bäcklund type relationships for the coefficients of the seed loop differential form $\bar{l} \in \mathcal{G}^* \simeq \Lambda^1_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)$.

The following examples demonstrate the analytical applicability of the devised above Lie-algebraic scheme for construction a wide class of nonlinear multidimensional heavenly type integrable Hamiltonian systems on functional spaces.

5.3. Example: Einstein–Weyl metric equation. Define $\bar{\mathcal{G}}^* = \text{diff}_{\text{hol}}(\mathbb{T}^1 \times \mathbb{C})$ and take the seed element

$$\bar{l} = (u_x \lambda - 2u_x v_x - u_y) dx + (\lambda^2 - v_x \lambda + v_y + v_x^2) d\lambda,$$

which generates with respect to the metric (5.16) (as before for $q = 0$) the gradient of the Casimir invariants $h^{(1)}, h^{(2)} \in I(\bar{\mathcal{G}}^*)$ in the form

$$\nabla h^{(2)}(l) \sim \lambda^2(0, 1)^\top + (-u_x, v_x)^\top \lambda + (u_y, u - v_y)^\top + O(\lambda^{-1}),$$

$$\nabla h^{(1)}(l) \sim \lambda(0, 1)^\top + (-u_x, v_x)^\top + (u_y, -v_y)^\top \lambda^{-1} + O(\lambda^{-2}),$$

as $|\lambda| \rightarrow \infty$ at $p_t = 2, p_y = 1$. For the gradients of the Casimir functions $h^{(1)}, h^{(2)} \in I(\bar{\mathcal{G}}^*)$, determined by (5.21), one can easily obtain the corresponding Hamiltonian vector field generators

$$\nabla h^{(t)}(\bar{l}) := \left\langle \nabla h^{(2)}(l)_+, \frac{\partial}{\partial x} \right\rangle = (\lambda^2 + \lambda v_x + u - v_y) \frac{\partial}{\partial x} + (-\lambda u_x + u_y) \frac{\partial}{\partial \lambda}, \tag{5.29}$$

$$\nabla h^{(y)}(\bar{l}) = \left\langle \nabla h^{(1)}(l)_+, \frac{\partial}{\partial x} \right\rangle = (\lambda + v_x) \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \lambda},$$

satisfying the compatibility condition (5.24), which is equivalent to the set of equations

$$\begin{aligned} u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\ v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} &= 0, \end{aligned} \tag{5.30}$$

describing general integrable Einstein–Weyl metric equations [20].

As is well known [32], the invariant reduction of (5.30) at $v = 0$ gives rise to the famous dispersionless Kadomtsev–Petviashvili equation

$$(u_t + uu_x)_x + u_{yy} = 0, \quad (5.31)$$

for which the reduced vector field representation (5.25) follows from (5.29) and is given by the vector fields

$$\begin{aligned} \nabla h^{(t)}(\bar{l})_+ &= (\lambda^2 + u) \frac{\partial}{\partial x} + (-\lambda u_x + u_y) \frac{\partial}{\partial \lambda}, \\ \nabla h^{(y)}(\bar{l})_+ &= \lambda \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \lambda}, \end{aligned} \quad (5.32)$$

satisfying the compatibility condition (5.24), equivalent to the equation (5.31). In particular, one derives from (5.25) and (5.32) the vector field compatibility relationships

$$\begin{aligned} \frac{\partial \psi}{\partial t} + (\lambda^2 + u) \frac{\partial \psi}{\partial x} + (-\lambda u_x + u_y) \frac{\partial \psi}{\partial \lambda} &= 0, \\ \frac{\partial \psi}{\partial y} + \lambda \frac{\partial \psi}{\partial x} - u_x \frac{\partial \psi}{\partial \lambda} &= 0, \end{aligned}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{C}; \mathbb{C})$ and any $y, t \in \mathbb{R}, (x, \lambda) \in \mathbb{T}^1 \times \mathbb{C}$.

5.4. The modified Einstein – Weyl metric equation. This equation system is

$$\begin{aligned} u_{xt} &= u_{yy} + u_x u_y + u_x^2 w_x + u u_{xy} + u_{xy} w_x + u_{xx} a, \\ w_{xt} &= u w_{xy} + u_y w_x + w_x w_{xy} + a w_{xx} - a_y, \end{aligned}$$

where $a_x := u_x w_x - w_{xy}$, and was recently derived in [63]. In this case we take also $\bar{\mathcal{G}}^* = \text{diff}_{\text{hol}}(\mathbb{T}^1 \times \mathbb{C})$, yet for a seed element $\bar{l} \in \bar{\mathcal{G}}$ we choose the form

$$\begin{aligned} \bar{l} &= [\lambda^2 u_x + (2u_x w_x + u_y + 3u u_x) \lambda + 2u_x \partial_x^{-1} u_x w_x + 2u_x \partial_x^{-1} u_y + \\ &\quad + 3u_x w_x^2 + 2u_y w_x + 6u u_x w_x + 2u u_y + 3u^2 u_x - 2a u_x] dx + \\ &\quad + [\lambda^2 + (w_x + 3u) \lambda + 2\partial_x^{-1} u_x w_x + 2\partial_x^{-1} u_y + w_x^2 + 3u w_x + 3u^2 - a] d\lambda, \end{aligned}$$

which with respect to the metric (5.16) (as before for $q = 0 = p$) generates two Casimir invariants $h^{(1)}, h^{(2)} \in I(\bar{\mathcal{G}}^*)$, whose gradients, as follows from (5.21), equal to

$$\begin{aligned} \nabla h^{(2)}(l) &\sim \lambda^2 [(u_x, -1)^\top + (u u_x + u_y, -u + w_x)^\top \lambda^{-1} + (0, u w_x - a)^\top \lambda^{-2}] + O(\lambda^{-1}), \\ \nabla h^{(1)}(l) &\sim \lambda [(u_x, -1)^\top + (0, w_x)^\top \lambda^{-1}] + O(\lambda^{-1}), \end{aligned} \quad (5.33)$$

as $|\lambda| \rightarrow \infty$ at $p_y = 1, p_t = 2$. The suitable positive projections

$$\begin{aligned} \nabla h^{(y)}(l) &:= \nabla h^{(1)}(l)_+ = (u_x \lambda, -\lambda + w_x)^\top, \\ \nabla h^{(t)}(l) &:= \nabla h^{(2)}(l)_+ = (u_x \lambda^2 + (u u_x + u_y) \lambda, -\lambda^2 + (w_x - u) \lambda + u w_x - a)^\top. \end{aligned}$$

of the gradients (5.33) generate the Hamiltonian flows (5.26), giving rise to the compatible Lax – Sato vector field system

$$\frac{\partial\psi}{\partial y} + (-\lambda + w_x)\frac{\partial\psi}{\partial x} + u_x\lambda\frac{\partial\psi}{\partial\lambda} = 0,$$

$$\frac{\partial\psi}{\partial t} + \left[-\lambda^2 + (w_x - u)\lambda + uw_x - a\right]\frac{\partial\psi}{\partial x} + (u_x\lambda^2 + (uu_x + u_y)\lambda)\frac{\partial\psi}{\partial\lambda} = 0,$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{C}; \mathbb{C})$ and any $y, t \in \mathbb{R}, (x, \lambda) \in \mathbb{T}^1 \times \mathbb{C}$.

5.5. Example: The Dunajski heavenly equations. This equation, suggested in [18], generalizes the corresponding antiself-dual vacuum Einstein equation, which is related to the Plebański metric and the celebrated Plebański [52] second heavenly equation (4.16). To study the integrability of the Dunajski equations

$$\begin{aligned} u_{x_1t} + u_{yx_2} + u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2 - v &= 0, \\ v_{x_1t} + v_{x_2y} + u_{x_1x_1}v_{x_2x_2} - 2u_{x_1x_2}v_{x_1x_2} &= 0, \end{aligned} \tag{5.34}$$

where $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2), (y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$, we define $\bar{\mathcal{G}}^* := \text{diff}_{\text{hol}}^*(\mathbb{C} \times \mathbb{T}^n)$ and take the following as a seed element $\bar{l} \in \bar{\mathcal{G}}^*$

$$\bar{l} = (v_{x_1} - u_{x_1x_1} + u_{x_1x_2} + \lambda) dx_1 + (v_{x_2} - u_{x_1x_2} + u_{x_2x_2} + \lambda) dx_2 + \lambda d\lambda.$$

With respect to the metric (5.16) (as before for $q = 0$), the gradients of two functionally independent Casimir invariants $h^{(p_y)}, h^{(p_t)} \in I(\bar{\mathcal{G}}^*)$ can be obtained as $|\lambda| \rightarrow \infty$ in the asymptotic form as

$$\begin{aligned} \nabla h^{(p_y)}(l) &\sim (0, -1, 0)^\top + \lambda^{-1}(u_{x_2x_2}, -u_{x_1x_2}, v_{x_2})^\top + O(\lambda^{-2}), \\ \nabla h^{(p_t)}(l) &\sim (1, 0, 0)^\top + \lambda^{-1}(-u_{x_1x_2}, u_{x_1x_1}, -v_{x_1})^\top + O(\lambda^{-2}). \end{aligned} \tag{5.35}$$

Upon calculating the Hamiltonian vector field generators at $p_t = 1 = p_y$:

$$\begin{aligned} \nabla h^{(y)}(l) &:= (\lambda^{p_y} \nabla h^{(p_y)}(l))_+ = (u_{x_2x_2}, -\lambda - u_{x_1x_2}, v_{x_2})^\top, \\ \nabla h^{(t)}(l) &:= (\lambda^{p_t} \nabla h^{(p_t)}(l))_+ = (\lambda - u_{x_1x_2}, u_{x_1x_1}, -v_{x_1})^\top, \end{aligned}$$

following from the Casimir functions gradients (5.35), one easily obtains the following vector fields

$$\begin{aligned} \nabla h^{(t)}(\bar{l}) &:= \left\langle \nabla h^{(t)}(l), \frac{\partial}{\partial \mathbf{x}} \right\rangle = u_{x_2x_2} \frac{\partial}{\partial x_1} - (\lambda + u_{x_1x_2}) \frac{\partial}{\partial x_2} + v_{x_2} \frac{\partial}{\partial \lambda}, \\ \nabla h^{(y)}(\bar{l}) &:= \left\langle \nabla h^{(y)}(l), \frac{\partial}{\partial \mathbf{x}} \right\rangle = (\lambda - u_{x_1x_2}) \frac{\partial}{\partial x_1} + u_{x_1x_1} \frac{\partial}{\partial x_2} - v_{x_1} \frac{\partial}{\partial \lambda}, \end{aligned}$$

satisfying the Lax–Sato compatibility condition (5.24)

$$\begin{aligned} \frac{\partial\psi}{\partial t} + u_{x_2x_2} \frac{\partial\psi}{\partial x_1} - (\lambda + u_{x_1x_2}) \frac{\partial\psi}{\partial x_2} + v_{x_2} \frac{\partial\psi}{\partial\lambda} &= 0, \\ \frac{\partial\psi}{\partial y} + (\lambda - u_{x_1x_2}) \frac{\partial\psi}{\partial x_1} + u_{x_1x_1} \frac{\partial\psi}{\partial x_2} - v_{x_1} \frac{\partial\psi}{\partial\lambda} &= 0, \end{aligned}$$

equivalent to the the Dunajski [18] equation (5.34) and satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^2; \mathbb{C})$, any $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$ and all $\lambda \in \mathbb{C}$. As was mentioned in [9], the Dunajski equations (5.34) generalize both the dispersionless Kadomtsev–Petviashvili and Plebański second heavenly equations, and is also a Lax–Sato integrable quasi-Hamiltonian system.

6. Integrability, bi-Hamiltonian structures and the classical Lagrange–d’Alembert principle. It is evident that all evolution flows like (3.6) or (5.26) are Hamiltonian with respect to the second Lie–Poisson bracket (5.17) on the adjoint loop space $\tilde{\mathcal{G}}^* = \widetilde{\text{diff}}^*(\mathbb{T}^n)$ or on the subspace $\tilde{\mathcal{G}}^* = \text{diff}_{\text{hol}}^*(\mathbb{C} \times \mathbb{T}^n)$, respectively. Moreover, they are poly-Hamiltonian on the corresponding functional manifolds, as the related bilinear forms (3.1) and (5.16) are marked by integers $p \in \mathbb{Z}$. This leads to [69] an infinite hierarchy of compatible Poisson structures on the phase spaces, isomorphic, respectively, to the orbits of a chosen seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ or of a seed element $\bar{l} \in \tilde{\mathcal{G}}^*$. Taking also into account that all these Hamiltonian flows possess an infinite hierarchy of commuting nontrivial conservation laws, one can prove their formal complete integrability under some naturally formulated constraints. The corresponding analytical expressions for the infinite hierarchy of conservation laws can be retrieved from the asymptotic expansion (3.12) for Casimir functional gradients by employing the well-known [5, 6, 40, 69] formal homotopy technique.

As an arbitrary heavenly equation is a Hamiltonian system with respect to both evolution parameters $t, y \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}$, one can construct [3, 5, 6, 40, 54] its suitable Lagrangian (or quasi-Lagrangian) representation under some natural constraints. Thus, it is possible to retrieve the corresponding Poisson structures related to both these evolution parameters $t, y \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}$, which, as follows from the Lie-algebraic analysis in Section 3, are compatible to each other. In this way, one can show that any heavenly type equation is a bi-Hamiltonian integrable system on the corresponding functional manifold. It should be mentioned here that this property was introduced by Sergyeyev in (*arXiv: 1501.01955*), published in [36], and rediscovered and applied in detail in [66] for investigating the integrability properties of the general heavenly equation (4.17), first suggested by Schief in [60] and later studied by Doubrov and Ferapontov in [17].

In his book “*Mecanique analytique*”, v. 1-2, published in 1788 in Paris, J. L. Lagrange formulated *one of the basic, most general, differential variational principles of classical mechanics, expressing necessary and sufficient conditions for the correspondence of the real motion of a system of material points, subjected by ideal constraints, to the applied active forces*. Within the d’Alembert–Lagrange principle the positions of the system in its real motion are compared with infinitely close positions permitted by the constraints at the given moment of time.

According to the d’Alembert–Lagrange principle, during a real motion of a system of $N \in \mathbb{Z}_+$ particles with masses $m_j \in \mathbb{R}_+$, $j = \overline{1, N}$, the sum of the elementary works performed by the given active forces $F^{(j)}$, $j = \overline{1, N}$, and by the forces of inertia for all *the possible particle displacements* $\delta x^{(j)} \in \mathbb{E}^3$, $j = \overline{1, N}$, **is equal to or less than zero:**

$$\sum_{j=\overline{1, N}} \left\langle F^{(j)} - m_j \frac{d^2 x^{(j)}}{dt^2}, \delta x^{(j)} \right\rangle \leq 0 \quad (6.1)$$

at any moment of time $t \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in the three-dimensional Euclidean space \mathbb{E}^3 . *The equality in (6.1) is valid for the possible reversible displacements*, the symbol \leq is valid for *the possible irreversible displacements* $\delta x^{(j)} \in \mathbb{E}^3$, $j = \overline{1, N}$. Equation (6.1) is *the general equation of the dynamics of systems with ideal constraints*; it comprises all the equations and laws of motion, so that one can say that all dynamics is reduced to this single general formula.

This principle, established by J. L. Lagrange by generalization of the principle of virtual displacements with the aid of the classical d’Alembert principle. For systems subject to bilateral constraints J. L. Lagrange based himself on formula (6.1) to deduce the general properties and laws of motion of bodies, as well as the equations of motion, which he applied to solve a number of problems in

dynamics including the problems of motions of non-compressible, compressible and elastic liquids, thus combining “dynamics and hydrodynamics as branches of the same principle and as conclusions drawn from a single general formula”.

As it was first demonstrated in the work [25], in the last case of *generalized reversible motions* of a compressible elastic liquid, located in a one-connected open domain $\Omega_t \subset \mathbb{R}^n$ with the smooth boundary $\partial\Omega_t$, $t \in \mathbb{R}$, in space \mathbb{R}^n , $n \in \mathbb{Z}_+$, the expression (6.1) can be rewritten as

$$\delta W(t) := \int_{\Omega_t} \langle l(x(t); \lambda), \delta x(t) \rangle d^n x(t) = 0 \tag{6.2}$$

for all $t \in \mathbb{R}$, where $l(x(t); \lambda) \in \tilde{T}^*(\mathbb{R}^n)$ is the corresponding virtual vector “*reaction force*”, exerted by the ambient medium on the liquid and called a *seed element*, which is here assumed to depend meromorphically on a constant complex parameter $\lambda \in \mathbb{C}$. If now to suppose that the evolution of liquid points $x(t) \in \Omega_t$ is determined for any parameters $\lambda \neq \mu \in \mathbb{C}$ by the generating *gradient type vector field*

$$\frac{dx(t)}{dt} = \frac{\mu}{\mu - \lambda} \nabla h(l(\mu))(t; x(t)) \tag{6.3}$$

and the Cauchy data

$$x(t)|_{t=0} = x^{(0)} \in \Omega_0$$

for an arbitrarily chosen open one-connected domain $\Omega_0 \subset \mathbb{T}^n$ with the smooth boundary $\partial\Omega_0 \subset \mathbb{R}^n$ and a smooth functional $h : \tilde{T}^*(\mathbb{R}^n) \rightarrow \mathbb{R}$, the Lagrange–d’Alembert principle says: the infinitesimal virtual work (6.2) equals zero for all moments of time, that is $\delta W(t) = 0 = \delta W(0)$ for all $t \in \mathbb{R}$. To check that it is really zero, let us calculate the temporal derivative of the expression (6.2):

$$\begin{aligned} \frac{d}{dt} \delta W(t) &= \frac{d}{dt} \int_{\Omega_t} \langle l(x(t); \lambda), \delta x(t) \rangle d^n x(t) = \\ &= \frac{d}{dt} \int_{\Omega_0} \langle l(x(t); \lambda), \delta x(t) \rangle \left| \frac{\partial(x(t))}{\partial x_0} \right| d^n x^{(0)} = \\ &= \int_{\Omega_0} \frac{d}{dt} \left(\langle l(x(t); \lambda), \delta x(t) \rangle \left| \frac{\partial(x(t))}{\partial x_0} \right| \right) d^n x^{(0)} = \\ &= \int_{\Omega_0} \left[\frac{d}{dt} \langle l(x(t); \lambda), \delta x(t) \rangle + \langle l(x(t); \lambda), \delta x(t) \rangle \operatorname{div} \tilde{K}(\mu) \right] \left| \frac{\partial(x(t))}{\partial x_0} \right| d^n x^{(0)} = \\ &= \int_{\Omega_t} \left[\frac{d}{dt} \langle l(x(t); \lambda), \delta x(t) \rangle + \langle l(x(t); \lambda), \delta x(t) \rangle \operatorname{div} \tilde{K}(\mu) \right] d^n x(t) = 0, \end{aligned}$$

if the condition

$$\frac{d}{dt} \langle l(x(t); \lambda), \delta x(t) \rangle + \langle l(x(t); \lambda), \delta x(t) \rangle \operatorname{div} \tilde{K}(\mu; \lambda) = 0 \tag{6.4}$$

holds for all $t \in \mathbb{R}$, where

$$\tilde{K}(\mu; \lambda) := \frac{\mu}{\mu - \lambda} \nabla h(\tilde{l}(\mu)) = \frac{\mu}{\mu - \lambda} \left\langle \nabla h(l(\mu)), \frac{d}{dx} \right\rangle \tag{6.5}$$

is a vector field on \mathbb{R}^n , corresponding to the evolution equations (6.3). Taking into account that the full temporal derivative $d/dt := \partial/\partial t + L_{\tilde{K}(\mu; \lambda)}$, where $L_{\tilde{K}(\mu; \lambda)} = i_{\tilde{K}(\mu; \lambda)}d + di_{\tilde{K}(\mu; \lambda)}$ denotes the well known [1, 5, 24] Cartan expression for the Lie derivation along the vector field (6.5), can be represented as $\mu, \lambda \rightarrow \infty, |\lambda/\mu| < 1$ in the asymptotic form

$$\frac{d}{dt} \sim \sum_{j \in \mathbb{Z}_+} \mu^{-j} \frac{\partial}{\partial t_j} + \sum_{j \in \mathbb{Z}_+} \mu^{-j} L_{\tilde{K}_j(\lambda)},$$

the equality (6.4) can be *equivalently rewritten* as an infinite hierarchy of the following evolution equations:

$$\partial \tilde{l}(\lambda) / \partial t_j := -ad_{\tilde{K}_j(\lambda)_+}^* \tilde{l}(\lambda) \tag{6.6}$$

for every $j \in \mathbb{Z}_+$ on the space of differential 1-forms $\tilde{\Lambda}^1(\mathbb{R}^n) \simeq \tilde{\mathcal{G}}^*$, where $\tilde{l}(\lambda) := \langle l(x; \lambda), dx \rangle \in \tilde{\Lambda}^1(\mathbb{R}^n) \simeq \tilde{\mathcal{G}}^*$, $\tilde{\mathcal{G}} := \widehat{\text{diff}}(\mathbb{R}^n)$ is the Lie algebra of the corresponding loop diffeomorphism group $\widehat{\text{Diff}}(\mathbb{R}^n)$. As from (6.5) one easily finds that

$$\tilde{K}_j(\lambda) = \nabla h^{(j)}(\tilde{l}) \tag{6.7}$$

for $\lambda \in \mathbb{C}$ and any $j \in \mathbb{Z}_+$, the evolution equations (6.6) transform equivalently into

$$\partial \tilde{l}(\lambda) / \partial t_j := -ad_{\nabla h^{(j)}(\tilde{l})_+}^* \tilde{l}(\lambda), \tag{6.8}$$

allowing to formulate the following important Adler – Kostant – Symes type [5 – 7, 56, 57, 69] proposition.

Proposition 6.1. *The evolution equations (6.8) are completely integrable commuting to each other Hamiltonian flows on the adjoint loop space $\tilde{\mathcal{G}}^*$ for a seed element $\tilde{l}(\lambda) \in \tilde{\mathcal{G}}^*$, generated by Casimir functionals $h^{(j)} \in I(\tilde{\mathcal{G}}^*)$, naturally determined by conditions $ad_{\nabla h^{(j)}(\tilde{l})}^* \tilde{l}(\lambda) = 0, j \in \mathbb{Z}_+$, with respect to the modified Lie – Poisson bracket on the adjoint space $\tilde{\mathcal{G}}^*$*

$$\{(\tilde{l}, \tilde{X}), (\tilde{l}, \tilde{Y})\} := (\tilde{l}, [\tilde{X}, \tilde{Y}]_{\mathcal{R}}),$$

defined for any $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{G}}$ by means of the canonical \mathcal{R} -structure on the loop Lie algebra $\tilde{\mathcal{G}}$:

$$[\tilde{X}, \tilde{Y}]_{\mathcal{R}} := [\tilde{X}_+, \tilde{Y}_+] - [\tilde{X}_-, \tilde{Y}_-],$$

where “ \tilde{Z}_{\pm} ” means the positive (+)/(-)-negative part of a loop Lie algebra element $\tilde{Z} \in \tilde{\mathcal{G}}$ subject to the loop parameter $\lambda \in \mathbb{C}$.

If, for instance, to consider the first two flows from (6.8) in the form

$$\begin{aligned} \partial \tilde{l}(\lambda) / \partial t_1 &:= \partial \tilde{l}(\lambda) / \partial y = -ad_{\nabla h^{(y)}(\tilde{l})}^* \tilde{l}(\lambda), \\ \partial \tilde{l}(\lambda) / \partial t_2 &:= \partial \tilde{l}(\lambda) / \partial t = -ad_{\nabla h^{(t)}(\tilde{l})}^* \tilde{l}(\lambda), \end{aligned}$$

where, by definition,

$$\nabla h^{(y)}(\tilde{l}) := \nabla h^{(1)}(\tilde{l})_+, \quad \nabla h^{(t)}(\tilde{l}) := \nabla h^{(t)}(\tilde{l})_+,$$

which are, by construction, commuting to each other, from their compatibility condition one ensues some system of nonlinear equations in partial derivatives on the coefficients of the seed element $\tilde{l}(\lambda) \in \tilde{\mathcal{G}}^*$. As the latter is, evidently, equivalent to the Lax–Sato compatibility condition for the corresponding vector fields $\nabla h^{(y)}(\tilde{l})$ and $\nabla h^{(t)}(\tilde{l}) \in \tilde{\mathcal{G}}$:

$$[\partial/\partial y + \nabla h^{(y)}(\tilde{l}), \partial/\partial t + \nabla h^{(t)}(\tilde{l})] = 0, \quad (6.9)$$

a resulting from (6.9) system of nonlinear equations in partial derivatives is often called of heavenly type and was before actively analyzed in a series of articles [9, 32, 36, 41–43, 60, 61, 67, 68] and recently in [9–11, 26]. These works are closely related to the problem of constructing a hierarchy of commuting to each other vector fields, analytically depending on a complex parameter $\lambda \in \mathbb{C}$, which was in general form studied and completely solved by Pfeiffer in his classical work [47, 48, 51].

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