UDC 512.5

X. He, S. Li (Dep. Math., Guangxi Univ., Nanning, Guangxi, China),
Y. Wang (Lingnan College and Dep. Math., Sun Yat-sen Univ., Guangzhou, China)

GROUPS ALL CYCLIC SUBGROUPS OF WHICH ARE BNA-SUBGROUPS*
ГРУПИ, ВСІ ЦИКЛІЧНІ ПІДГРУПИ ЯКИХ Є BNA-ПІДГРУПАМИ

Suppose that G is a finite group and H is a subgroup of G. We say that H is a BNA-subgroup of G if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$. The BNA-subgroups of G are between normal and abnormal subgroups of G. We obtain some new characterizations for finite groups based on the assumption that all cyclic subgroups are BNA-subgroups.

Нехай G — скінченна група, а H — підгрупа групи G. Говоримо, що H — BNA-підгрупа групи G, якщо $H^x = H$ або $x \in \langle H, H^x \rangle$ для всіх $x \in G$. BNA-підгрупи групи G знаходяться між нормальними та анормальними підгрупами G. Отримано деякі нові характеристики скінченних груп на основі припущення, що всі циклічні підгрупи є BNA-підгрупами.

1. Introduction. Throughout this article, all groups are finite. We use conventional notions and notation. The reader is referred to [1]. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$, G' is the derived subgroup of G, $H \lessdot G$ means that H is a maximal subgroup of G.

A class $\mathcal{F}$ of groups is called a formation provided that (i) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for all normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ (see [1], Chapter VI). Throughout this article, we use $\mathcal{U}$ to denote the formation of all supersoluble groups. Clearly, $\mathcal{U}$ is saturated [1].

An interesting investigation in finite groups theory is to determine the structure of finite groups using the generalized normal subgroups. Normal subgroup is a fundamental concept. As it is well known, groups whose all subgroups are normal are called Dedekind groups. Restricting the system of groups on which the condition of normality is imposed, one can obtain some generalizations of Dedekind groups. For instance, groups whose all subgroups are quasinormal are called Iwasawa groups, and groups all subgroups of which are subnormal are called nilpotent groups.

Recall that a subgroup H of G is said to be an *abnormal subgroup* if $x \in \langle H, H^x \rangle$ for all $x \in G$. There have been several papers on normal subgroups and abnormal subgroups. A. Fattahi classified the finite groups with only normal and abnormal subgroups [2]. G. Ebert and S. Bauman studied the finite groups whose subgroups are either subnormal or abnormal [3]. Cuccia showed that if G is a finite group and, for every minimal subgroup X of G, $C_G(X)$ is either subnormal or abnormal, then G is soluble [4]. Recently, Liu and Li in [5] classified CLT-groups with normal or abnormal subgroups. The concept of abnormal subgroup is, in a sense, opposite of that of normal subgroup. Precisely speaking, G has only a subgroup, itself, that is both normal and abnormal in G. Each maximal subgroup of G is either normal or abnormal. In [6], we analyze and introduce the following generalization of normal subgroup and abnormal subgroup.

* Supported by the NNSF of China (No. 11401116, No. 11226045, No. 11361006, No. 11461004), NSF of Guangxi (No. 2014GXNSFBA118003).

Definition 1.1 ([6], Definition 1.1). *A subgroup H of G is called a BNA-subgroup (Between Normal subgroup and Abnormal subgroup) of G if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$, H is also said to be BNA-normal in G.*

In this article, we investigate the groups whose all cyclic subgroups of prime power order are BNA-subgroups and prove the following main result.

Theorem 1.1. *All cyclic subgroups of G of prime power order are BNA-subgroups of G if and only if one of the following statements holds:*

(1) *G is a Dedekind group.*

(2) *$G = G' \rtimes \langle x \rangle$, where G' is a Hall subgroup of G and is Abelian, x induces a power automorphism of order p on G', p is the smallest prime of $\pi(G)$.*

(3) *G is a finite group whose every subgroup is a BNA-subgroup of G.*

(4) *G is a BNA-subgroup transitive group.*

2. Preliminaries. In this section, we collect some known results which are needed in the proof of our results.

Lemma 2.1 ([6], Lemma 2.1). *Let $H \leq K \leq G$ and $N \trianglelefteq G$. Suppose that H is a BNA-subgroup of G. Then the following statements hold:*

(1) *H is a BNA-subgroup of K.*

(2) *HN is a BNA-subgroup of G.*

(3) *HN/N is a BNA-subgroup of G/N.*

(4) *Any maximal subgroup of G is a BNA-subgroup of G.*

Lemma 2.2 ([6], Lemma 2.2). *Let H be a BNA-subgroup of G. Then the following statements hold:*

(1) *The normal closure $H^G = H$ or $H^G = G$.*

(2) *If, in addition, H is subnormal in G, then H is normal in G.*

Lemma 2.3. *Let H be a BNA-subgroup of G. If there exists a proper normal subgroup M of G with $H \leq M$, then $H \trianglelefteq G$.*

Proof. It is easy to follow from (1) of Lemma 2.2.

Lemma 2.4 ([6], Lemma 2.3). *Let H be a BNA-subgroup of G. Then*:

(1) *$N_G(H) \leq \langle H, H^x \rangle$ whenever $H^x \neq H$.*

(2) *If $N_G(H) \leq K \leq G$, then K is an abnormal subgroup of G.*

Lemma 2.5. *Let $H \leq G$. Then H is a BNA-subgroup of G if and only if $N_G(H)$ is abnormal in G and $N_G(H) \leq \langle H, H^x \rangle$ whenever $x \notin N_G(H)$.*

Proof. The *necessity* is clear by Lemma 2.4. We now prove the *sufficiency.* Let x be an element of G such that $H^x \neq H$. It's sufficient to show that $x \in \langle H, H^x \rangle$.

In fact, by hypothesis, $N_G(H) \leq \langle H, H^x \rangle$. On the other hand, $(H^x)^{x^{-1}} = H \neq H^x$. Applying again the hypothesis, we have that $N_G(H^x) \leq \langle H^x, (H^x)^{x^{-1}} \rangle = \langle H^x, H \rangle$. So $\langle N_G(H), N_G(H^x) \rangle \leq \leq \langle H, H^x \rangle$. Thus it suffices to show that $x \in \langle N_G(H), N_G(H^x) \rangle$. Indeed, $N_G(H)$ is abnormal in G by hypothesis, which requires $x \in \langle N_G(H), N_G(H^x) \rangle$.

Lemma 2.6 ([7], Proposition 3.11). *Let $\mathcal{F}_1 = LF(F_1)$ and $\mathcal{F}_2 = LF(F_2)$, where F_i is both an integrated and full formation function of $\mathcal{F}_i$, $i = 1, 2$. Then the following statements are equivalent*:

(1) *$\mathcal{F}_1 \subseteq \mathcal{F}_2$.*

(2) *$F_1(p) \subseteq F_2(p)$ for all $p \in P$.*

3. Main results.

Theorem 3.1. *Suppose that all minimal subgroups and cyclic subgroups of order* 4 *of* G *are* BNA*-subgroups of* G*. Then* G *is supersoluble.*

Proof. Let p be the smallest prime of $\pi(G)$ and P be a Sylow p-subgroup of G. If X is a minimal subgroup or a cyclic subgroup of order 4 in P (if it exists), then X is subnormal in $N_G(P)$. It follows from Lemma 2.2 that X is normal in $N_G(P)$. Theorem 1 of [8] implies that G is p-nilpotent. Let H be the complement of P in G. Then $G/H \cong P$ is supersoluble. It follows from Lemma 2.3 that all minimal subgroups of H are normal in G. Thus G is supersoluble.

In fact, we have the following stronger result.

Theorem 3.2. *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$, H *a normal subgroup of* G *such that* $G/H \in \mathcal{F}$*. Suppose that all minimal subgroups and cyclic subgroups of order* 4 *of* H *are* BNA*-subgroups of* G*. Then* $G \in \mathcal{F}$*.*

Proof. It is easy to see that H is supersoluble by Theorem 3.1. Let q be the largest prime dividing $|H|$ and Q a Sylow q-subgroup of H. Clearly, $Q \trianglelefteq H$ and so $Q \lhd G$. Consider the factor groups $(G/Q, H/Q)$. Since $(G/Q)/(H/Q) \cong G/H \in \mathcal{F}$, it follows by Lemma 2.1 that $(G/Q, H/Q)$ satisfy the hypotheses of the theorem. Therefore $G/Q \in \mathcal{F}$ by induction on G/Q. Hypotheses of the theorem implies that arbitrary minimal subgroup L of Q is a BNA-subgroup of G. It follows by Lemma 2.2 that $L \trianglelefteq G$. Let $F_i,\ i = 1, 2,$ be the integrated and full formation function such that $\mathcal{U} = LF(F_1)$ and $\mathcal{F} = LF(F_2)$, respectively. Since L is a normal subgroup of G of prime order, $G/C_G(L) \in F_1(q)$. It follows by Lemma 2.6 that $G/C_G(L) \in F_2(q)$ and so $L \leq Z_\infty^{\mathcal{F}}(G)$. Therefore $G \in \mathcal{F}$ by Theorem 4 of [9].

Theorem 3.3. *Let* G *be a nilpotent group. If all cyclic subgroups of* G *of prime power order are* BNA*-subgroups of* G*, then* G *is either Abelian or Hamiltonian.*

Proof. Let H be an arbitrary cyclic subgroup of G. It is easy to see that $H \trianglelefteq G$ by Lemma 2.2. Therefore all subgroups of G are normal and the result follows.

Recall that G is called a $\mathcal{T}$-group if $H \lhd K \lhd G$ always implies that $H \lhd G$ [10, p. 388].

We give the following definition.

Definition 3.1 *Let* H *and* K *be subgroups of* G*. We call that* G *is a* BNA*-subgroup transitive group if* H *is a* BNA*-subgroup of* K *and* K *is a* BNA*-subgroup of* G *always imply that* H *is a* BNA*-subgroup of* G*.*

Theorem 3.4. *All cyclic subgroups of* G *are* BNA*-subgroups of* G *if and only if one of the following statements holds:*

(1) *G is a Dedekind group.*

(2) $G = G' \rtimes \langle x \rangle$, *where* G' *is a Hall subgroup of* G *and is Abelian,* x *induces a power automorphism of order* p *on* G', p *is the smallest prime of* $\pi(G)$.

(3) *G is a finite group whose every subgroup is a BNA-subgroup of G.*

(4) *G is a BNA-subgroup transitive group.*

In fact, we can show the following stronger result.

Theorem 3.5. *All cyclic subgroups of* G *of prime power order are* BNA*-subgroups of* G *if and only if one of the following statements holds*:

(1) *G is a Dedekind group.*

(2) $G = G' \rtimes \langle x \rangle$, *where* G' *is a Hall subgroup of* G *and is Abelian,* x *induces a power automorphism of order* p *on* G', p *is the smallest prime of* $\pi(G)$.

(3) *G is a finite group whose every subgroup is a BNA-subgroup of G.*

(4) *G is a BNA-subgroup transitive group.*

Proof. *Necessity.* It follows by the hypotheses of the theorem and Theorem 3.1 that G is supersoluble. If G is nilpotent, then G is a Dedekind group by Theorem 3.3. Therefore we need only investigate the case that G is nonnilpotent.

Let K be an arbitrary proper subgroup of G which is normal in G. Then every cyclic subgroup C of K of prime power order satisfies that $C^G \leq K < G$. It is clear that $C^G = C$ by Lemma 2.2, that is, C is normal in G. In particular, all subgroups of K are normal in G, therefore K is a Dedekind group.

Let p denote the smallest prime dividing the order of G. Since G is supersoluble, there is a normal subgroup M in G of index p. Then

$$G = MP, \qquad (*)$$

where $P = \langle x\rangle$ with $x^p \in M$. By the above argument, every subgroup of M is normal in G.

Consider the following two cases:

***Case* 1.** M is non-Abelian.

In this case, M is an Hamiltonian group. It follows by [10, p. 139] that

$$M = Q_8 \times A \times B,$$

where Q_8 is the quaternion group of order 8, A is an elementary Abelian 2-group and B is an Abelian group of odd order. In particular, M is of even order. Since p is the smallest prime dividing the order of G, then $p = 2$ and $|G : M| = 2$. Let T be a Sylow 2-subgroup of G which contains P. Since T is a nilpotent group and satisfies the hypotheses of the theorem, it is clear that all subgroups of T are normal in T. Of course, $T \leq N_G(P)$ and T is an Hamiltonian group, $T = Q_8 \times P$, where $P = \langle x\rangle$ with $x^2 = 1$.

Denote $K := [B]P$. If P is nonnormal in G, since $T \leq N_G(P)$, then there exists $b \in B$ such that $P^b \neq P$. Lemma 2.4 implies that $N_G(P) \leq \langle P, P^b\rangle$. Moreover, every subgroup of M is normal in G by the above argument. Hence $P\langle b\rangle$ is a subgroup. Consequently, $T \leq \langle P, P^b\rangle \leq P\langle b\rangle$, which gives that $T = P$, contrary to the fact that T is non-Abelian. Therefore, $G = M \times P$ and we conclude that G is an Hamiltonian group.

***Case* 2.** M is Abelian.

According to $(*)$, we can get that $G = MP$, where $P = \langle x\rangle$ with $x^p \in M$, p is the smallest prime dividing the order of G.

Since G is nonnilpotent, it is easy to see that P is nonnormal in G. Thus, there is an element c in M of prime power order such that $[c, P] \neq 1$. Then c is not a p-element. It follows by Lemma 2.4 that $N_G(P) \leq \langle P, P^c\rangle$. Furthermore, every subgroup of M is normal in G by the above argument. Hence $P\langle c\rangle$ is a subgroup of G. Consequently, $N_G(P) \leq P\langle c\rangle$ and so P is a Sylow p-subgroup of G. Let X be an arbitrary subgroup of M. It is clear that $X^x = X$. Therefore x induces an automorphism with fixed point free, this implies that $|P| = p$. Obviously, $M = G'$.

Now we have finished the proof of (1) and (2).

By the above proof, it is easy to see that every subgroup of G is a BNA-subgroup of G. Therefore, (3) holds.

(3) obviously implies (4).

Now we have finished the proof of the necessity.

Sufficiency. Suppose that G is a Dedekind group, then any subgroup of G is normal in G and so every cyclic subgroup of G of prime power order is a BNA-subgroup of G.

Suppose that $G = G' \rtimes \langle x \rangle$, where G' is a Hall subgroup of G and is Abelian, x induces a power automorphism of order p on G', p is the smallest prime of $\pi(G)$. Let L be an arbitrary cyclic subgroup of G of prime power order. Then L has the following two cases:

$$L \leq G' \text{ or } L \leq \langle x^g \rangle \text{ for some } g \in G'.$$

If $L \leq G'$, then L is normal in G and so is a BNA-subgroup of G.

If $L < \langle x^g \rangle$, then $L \times G' \leq \langle (x^g)^p \rangle G' \lhd G$ by Lemma 2.8 of [11]. Since $\langle (x^g)^p \rangle G' \lhd G$, then $\langle (x^g)^p \rangle \lhd G$. Thus $L \lhd G$ and then L is a BNA-subgroup of G. If $L = \langle x^g \rangle$, then L is pronormal in G and so $N_G(L)$ is abnormal in G by Theorem 7 of [12]. Obviously, $N_G(L) = L$. If not, there is a subgroup of G' which centralizes L and is also centralized by L. This contradicts with x induces a power automorphism of G' of order p. It follows from Lemma 2.5 that L is a BNA-subgroup of G.

Suppose that G is a finite group whose every subgroup is a BNA-subgroup of G. Then the result is obvious.

Suppose that G is a BNA-subgroup transitive group. Let H be a cyclic subgroup of G of prime power order. It is easy to know that any maximal subgroup of G is a BNA-subgroup of G by Lemma 2.1. We know that G has an ascending series of maximal subgroups

$$H = H_0 \lessdot H_1 \lessdot H_2 \lessdot \ldots \lessdot H_{r-1} \lessdot H_r = G,$$

where H_{i-1} is a maximal subgroup of H_i.

It follows by Lemma 2.1 that maximal subgroup H_{i-1} of H_i is a BNA-subgroup of H_i for $i = 1, 2, \ldots, r$. Since G is a BNA-subgroup transitive group, then H is a BNA-subgroup of G.

This completes the proof of the theorem.

References

1. *Huppert B.* Endliche Gruppen I. – Berlin etc.: Springer-Verlag, 1967. – 793 p.
2. *Fattahi A.* Groups with only normal and abnormal subgroups // J. Algebra. – 1974. – **28**, № 1. – P. 15 – 19.
3. *Ebert G., Bauman S.* A note on subnormal and abnormal chains // J. Algebra. – 1975. – **36**, № 2. – P. 287 – 293.
4. *Cuccia P., Liotta M.* A condition on the minimal subgroups of a finite group // Boll. Unione mat. ital. – 1982. – **1**, № 6. – P. 303 – 308.
5. *Liu J., Li S., He J.* CLT-groups with normal or abnormal subgroups // J. Algebra. – 2012. – **362**. – P. 99 – 106.
6. *He X., Li S., Wang Y.* On BNA-normality and solvability of finite groups // Rend. Semin. mat. Univ. Padova. – 2014, available online at http://rendiconti.math.unipd.it/forthcoming/downloads/HeLiWang_logo.pdf.
7. *Doerk K., Hawkes T. O.* Finite soluble groups. – Berlin; New York: Walter De Gruyter, 1992.
8. *Li S.* On minimal subgroups of finite groups // Communs Algebra. – 1994. – **22**, № 6. – P. 1913 – 1918.
9. *Ballester-Bolinches A., Pedraza-Aguilera M. C.* On minimal subgroup of finite groups // Acta Math. Hung. – 1996. – **73**, № 4. – P. 335 – 342.
10. *Robinson D. J. S.* A course in the theory of groups. – Berlin; New York: Springer-Verlag, 1993.
11. *Wei H., Wang Y.* On C^*-normality and its properties // J. Group Theory. – 2007. – **10**. – P. 211 – 223.
12. *Baand M. S., Borevich Z. I.* On arrangement of intermediate subgroups // Rings and Linear Groups. – Krasnodar: Kubansk. Univ., 1988. – P. 14 – 41.

Received 17.12.13,
after revision — 06.08.16