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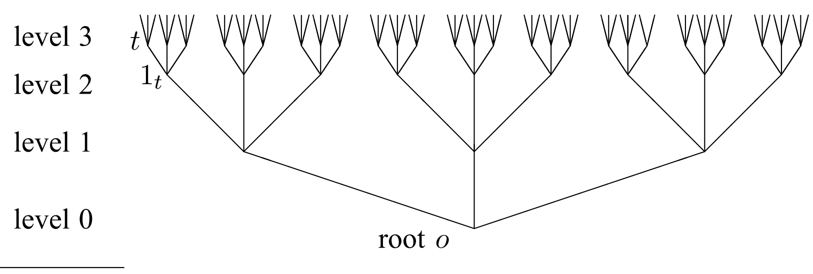
# STRONG LAWS OF LARGE NUMBERS AND ASYMPTOTIC EQUIPARTITION PROPERTY FOR THE ASYMPTOTIC $N$-BRANCH MARKOV CHAINS INDEXED BY A CAYLEY TREE*

# СИЛЬНІ ЗАКОНИ ВЕЛИКИХ ЧИСЕЛ ТА ВЛАСТИВІСТЬ АСИМПТОТИЧНО РІВНОМІРНОГО РОЗБИТТЯ ДЛЯ АСИМПТОТИЧНИХ $N$-ГІЛКОВИХ МАРКОВСЬКИХ ЛАНЦЮГІВ, ІНДЕКСОВАНИХ ДЕРЕВОМ КЕЙЛІ

We introduce a concept of asymptotic $N$-branch Markov chains indexed by a Cayley tree. Then we study the strong law of large numbers and the asymptotic equipartition property for the introduced Markov chains with a finite state space, which generalizes a class of results for inhomogeneous Markov chains indexed by a Cayley tree and nonsymmetric Markov chains indexed by a binary tree.

Введено поняття асимптотичних $N$-гілкових марковських ланцюгів, індексованих деревом Кейлі. Вивчається сильний закон великих чисел та властивість асимптотично рівномірного розбиття для введених марковських ланцюгів зі скінченним простором станів, що узагальнює клас результатів для неоднорідних марковських ланцюгів, індексованих деревом Кейлі, та несиметричних, індексованих бінарним деревом.

**1. Introduction.** A tree is a graph $T$ which is connected without circuits. Thus, $T$ is a tree if and only if, for any given vertices $\sigma, t \in T$, $\sigma \neq t$, there exists a unique path $\sigma = z_1, z_2, \ldots, z_m = t$ from $\sigma$ to $t$ with distinct nodes $z_1, z_2, \ldots, z_m$. The distance between $\sigma$ and $t$ is defined to be $m-1$, which is the number of edges connecting $\sigma$ and $t$. Select a vertex as root $o$. For any vertices $\sigma$ and $t$ of the tree $T$, we write $\sigma \leq t$ if $\sigma$ is on the unique path from $o$ to $t$. We use $\sigma \wedge t$ to represent the furthest vertex from $o$, which satisfies $\sigma \wedge t \leq t$ and $\sigma \wedge t \leq \sigma$. For any vertex $t \in T$, we denote by $|t|$ the distance between $o$ and $t$. The set of all vertices with distance $n$ from $o$ is called $n$th level of $T$. We denote the predecessor of $t$ by $1_t$, and we regard $t$ as a son of $1_t$. In this paper, we only investigate the Cayley tree $T_{C,N}$, where the root has $N$ neighbors and all other vertices have $N+1$ neighbors. That is, there are $N$ branches on the next level for each vertex of such tree. In order to distinguish them, we call them the first branch, the second branch, $\ldots$, and the $N$th branch, respectively. For any vertex $t \in T$, if the son of $t$ is on the $i$th branch, we call it the $i$th vertex. For better understanding, we draw a Cayley tree $T_{C,3}$ (see Figure).

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Let $T$ be a Cayley tree $T_{C,N}$, we denote by $T^i$, $i = 1, 2, \ldots, N$, the subgraph of $T$ containing all the $i$th vertices, $T^{(n)}$ the subtree comprised of level 0 (the root $o$) through level $n$, and $T_n^i = T^i \cap T^{(n)}$, the set of all the $i$th vertices on $T^{(n)}$. In addition, we use $L_n$ to represent the set of all vertices on level $n$.

Suppose that $S$ is the subgraph of $T$, let $X^S = \{X_t, t \in S\}$ and $|S|$ be the number of vertices of $S$, $x^S$ the realization of $X^S$.

**Definition 1** [6]. *Assume that $T$ is a infinite tree which is local finite, and $G = \{0, 1, \ldots, b-1\}$ is a finite state space. Let $X = \{X_t, t \in T\}$ be a collection of $G$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Let*

$$p = (p(x)), \quad x \in G, \tag{1}$$

*be a distribution on $G$ and*

$$P_t = (P_t(y \mid x)), \qquad x, y \in G, \quad t \in T, \tag{2}$$

*be a collection of transition matrices on $G^2$. If for any vertex $t \in T$,*

$$\begin{aligned} \boldsymbol{P}\big(X_t = y \mid X_{1_t} = x \ \ and \ \ X_\sigma = x_\sigma \ \ for \ \ t \wedge \sigma \leq 1_t\big) = \\ = \boldsymbol{P}(X_t = y \mid X_{1_t} = x) = P_t(y|x) \quad \forall x, y \in G, \end{aligned} \tag{3}$$

*and*

$$\boldsymbol{P}(X_o = x) = p(x) \quad \forall x \in G \tag{4}$$

*hold, then $X = \{X_t, t \in T\}$ is said to be $G$-valued nonhomogeneous Markov chains indexed by $T$ with the initial distribution* (1) *and transition matrices* (2), *or tree-indexed nonhomogeneous Markov chains with state space $G$. Let $\{P_n, n \geq 1\}$ be a sequence of transition matrices. If as $t \in L_n$, $P_t = P_n$, $X = \{X_t, t \in T\}$ will be called tree-indexed level-nonhomogeneous Markov chains with state space $G$* (*see* [11]).

***Remark* 1.** If $P$ is a strictly positive $2 \times 2$ transition matrix, then it has a unique stationary distribution $\pi$ such that $\pi P = \pi$ and $P$ is $\pi$-reversible. Spitzer [8] defined Markov chain fields on a tree with two state space with a transition matrix $P$. Benjamini and Peres [2] proposed a concept of tree-indexed homogeneous Markov chains with countable state space. Yang and Ye [11] presented a concept of tree-indexed level-nonhomogeneous Markov chains. Dong, Yang and Bai [6] put forward full-nonhomogeneous Markov chains indexed by a tree. Obviously, tree-indexed level-nonhomogeneous Markov chains are the special case of tree-indexed full-nonhomogeneous Markov chains.

**Definition 2.** *Suppose $T$ is a Cayley tree $T_{C,N}$, and $X = \{X_t, t \in T\}$ is a tee-indexed nonhomogeneous Markov chains taking values in $G$ with transition matrices $\{P_t, t \in T\}$ defined by Definition* 1. *Let $P_1, P_2, \ldots, P_N$ be $N$ transition matrices. If for any $k, l \in G$, we have*

$$\lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t \in T_n^i} |P_t(k \mid l) - P_i(k \mid l)| = 0, \quad i = 1, 2, \ldots, N, \tag{5}$$

*then $X = \{X_t, t \in T\}$ is said to be asymptotic $N$-branch Markov chains indexed by $T$. If $P_i = P$, $i = 1, 2, \ldots, N$, then equation* (5) *is equivalent to the equation*

$$\lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{T^{(n)}\setminus\{o\}} |P_t(k \mid l) - P(k \mid l)| = 0 \quad \forall k, l \in G, \tag{6}$$

*and $X = \{X_t, t \in T\}$ is said to be asymptotic homogeneous Markov chains indexed by $T$. If for all $t \in T^i$, $P_t = P_i$, $i = 1, 2, \ldots, N$, then $X = \{X_t, t \in T\}$ is said to be $N$-branch Markov chains indexed by $T$.*

***Remark* 2.** Let $P_1$ and $P_2$ be two $2 \times 2$ positive transition matrices on state space $\{0, 1\}$ with a common stationary distribution $\pi$ ($\pi P_1 = \pi$, $\pi P_2 = \pi$). Berger and Ye [3] introduced the concept of nonsymmetric Markov chain fields (NSMC) indexed by a binary tree with transition matrices $P_1$ and $P_2$. Dong and Yang [5] defined nonsymmetric Markov chain indexed by a binary tree with a finite-state space $G$ and any transition matrices $P_1$ and $P_2$ not necessarily having a common stationary distribution. In this paper, we present a concept of asymptotic $N$-branch Markov chains indexed by a Cayley tree, which is a generalization of nonsymmetric Markov chains indexed by a binary tree [5].

There have been lots of works on the subject of tree-indexed processes. Benjamini and Peres [2] given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for such Markov chains. Berger and Ye [3] studied the existence of entropy rate for some stationary random fields on a homogenous tree. Ye and Berger [12, 13] by using Pemantle's [7] results and a combinatorial approach, have studied the asymptotic equipartition property (AEP) in the sense of convergence in probability for $PPG$-invariant and ergodic random fields on a homogeneous tree. Yang and Liu [9] investigated the strong law of large numbers for Markov chain fields on a homogeneous tree. Yang [10] studied some strong limit theorems for countable homogeneous Markov chains indexed by a homogeneous tree, the strong law of large numbers and AEP for finite homogeneous Markov chains indexed by a homogeneous tree. Yang and Ye [11] studied the strong law of large numbers and AEP for finite level-nonhomogeneous Markov chains indexed by a homogeneous tree. Huang and Yang [4] studied the strong law of large numbers and AEP for Markov chains indexed by an infinite tree which is uniformly bounded. Bao and Ye [1] studied the strong law of large numbers and AEP for nonsymmetric Markov chain fields indexed by a binary tree with a two-state space, where $P_1$ and $P_2$ have a common stationary distribution. Dong and Yang [5] studied the strong law of large numbers and AEP for nonsymmetric Markov chains with a finite-state space indexed by a binary tree. Dong et al. [6] studied the strong law of large numbers and AEP for asymptotic homogeneous Markov chains indexed by a Cayley tree with a finite-state space.

In this paper, we aim at using the results of [6] to study the strong law of large numbers and AEP for asymptotic $N$-branch Markov chains indexed by a Cayley tree with a finite-state space. In this paper, we first introduce a concept of asymptotic $N$-branch Markov chains indexed by a Cayley tree. Then some lemmas will be presented in Section 2. Finally, we will prove the main results of this paper in Section 3. In fact, the results in this paper extend the results in existing literature, such as Dong and Yang [5], and Dong et al. [6].

**2. Some lemmas.** Before the main results, we first present some lemmas which the main results based.

**Lemma 1** ([6], Theorem 1). *Suppose that $T$ is a Cayley tree, and $X = \{X_t, t \in T\}$ is a nonhomogeneous Markov chain indexed by $T$ with state space $G$ defined by Definition 1. Let $\{g_t(x, y), t \in T\}$ be a collection of functions defined on $G^2$, $\{a_n, n \geq 1\}$ be a sequence of nonnegative random variables. Let $\alpha > 0$, $B$ and $D(\alpha)$ be two events defined as follows*:

$$B = \left\{\omega \;\Big|\; \lim_{n\to\infty} a_n = \infty\right\}, \tag{7}$$

$$D(\alpha) = \left\{\omega \;\Big|\; \limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T^{(n)}\setminus\{o\}} E[g_t^2(X_{1_t}, X_t)e^{\alpha|g_t(X_{1_t}, X_t)|}|X_{1_t}] = M(\omega) < \infty\right\} \bigcap B. \tag{8}$$

*Let*

$$H_n(\omega) = \sum_{t\in T^{(n)}\setminus\{o\}} g_t(X_{1_t}, X_t), \tag{9}$$

$$G_n(\omega) = \sum_{t\in T^{(n)}\setminus\{o\}} E[g_t(X_{1_t}, X_t)|X_{1_t}]. \tag{10}$$

*Then we have*

$$\lim_{n\to\infty} \frac{H_n(\omega) - G_n(\omega)}{a_n} = 0 \quad \textit{a.e.}, \quad \omega \in D(\alpha). \tag{11}$$

***Remark* 3.** It is easy to see that if $\{g_t(x, y), t \in T\}$ are uniformly bounded and $a_n = |T^{(n)}|$, then $D(\alpha) = \Omega\ \forall\alpha > 0$.

***Corollary* 1** ([6], Corollary 1). *Let $T$ be a Cayley tree, and $X = \{X_t, t \in T\}$ be a nonhomogeneous Markov chain indexed by $T$ with state space $G$ defined by Definition 1. If $\{g_t(x, y), t \in T\}$ are uniformly bounded functions defined on $G^2$, then we have*

$$\lim_{n\to\infty} \frac{H_n(\omega) - G_n(\omega)}{|T^{(n)}|} = 0 \quad \textit{a.e.} \tag{12}$$

Let $A$ be a subset of $T$. We define

$$S_k(A) = |\{t \in A, X_t = k\}|,$$

that is, $S_k(A)$ is the number of $k$ in $X^A$. Let $I_k(j)$ be Kronecker delta-functions $\left(I_k(j) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}\right)$ and $\delta^i(t)$ be indicator functions of the set $T_i$ $(\delta^i(t) = I_{T_i}(t))$, $i = 1, 2, \ldots, N$. Obviously, we have

$$S_k(T^{(n)}) = \sum_{t\in T^{(n)}} I_k(X_t), \tag{13}$$

$$S_k(T_n^i) = \sum_{t\in T_n^i} I_k(X_t) = \sum_{t\in T^{(n)}\setminus\{o\}} I_k(X_t)\delta^i(t), \tag{14}$$

and

$$S_k(T^{(n)}) = \sum_{i=1}^{N} S_k(T_n^i) - I_k(X_o), \tag{15}$$

where $i$ ranges from 1 to $N$.

**Lemma 2.** *Assume that $T$ is a Cayley tree $T_{C,N}$, and $X = \{X_t, t \in T\}$ is an asymptotic $N$-branch Markov chain indexed by $T$ with state space $G$ defined by Definition* 2. *Let $S_l(T^{(n)})$ and $S_k(T_n^i)$, $i = 1, 2, \ldots, N$, be defined as* (13) *and* (14), *respectively. Then we have*

$$\lim_{n\to\infty} \left\{ \frac{S_k(T_n^i)}{|T^{(n)}|} - \sum_{l=0}^{b-1} \frac{S_l(T^{(n-1)})}{N|T^{(n-1)}|} P_i(k \mid l) \right\} = 0 \quad a.e., \quad i = 1, 2, \ldots, N. \tag{16}$$

***Proof.*** Letting $g_t(x, y) = I_k(y)\delta^i(t)$, $i = 1, 2, \ldots, N$, in Corollary 1, it is easy to show that $\{g_t(x, y), t \in T\}$ are uniformly bounded functions. Since

$$H_n(\omega) = \sum_{t\in T^{(n)}\setminus\{o\}} I_k(X_t)\delta^i(t) = \sum_{t\in T_n^i} I_k(X_t) = S_k(T_n^i) \tag{17}$$

and

$$G_n(\omega) = \sum_{t\in T^{(n)}\setminus\{o\}} E[g_t(X_{1_t}, X_t) \mid X_{1_t}] =$$

$$= \sum_{t\in T^{(n)}\setminus\{o\}} \sum_{x_t\in G} I_k(x_t)\delta^i(t) P_t(x_t \mid X_{1_t}) =$$

$$= \sum_{t\in T^{(n)}\setminus\{o\}} \sum_{l=0}^{b-1} \delta^i(t) I_l(X_{1_t}) P_t(k \mid l), \tag{18}$$

it follows from (17), (18) and Corollary 1 that

$$\lim_{n\to\infty} \frac{1}{|T^{(n)}|} \left\{ S_k(T_n^i) - \sum_{t\in T^{(n)}\setminus\{o\}} \sum_{l=0}^{b-1} \delta^i(t) I_l(X_{1_t}) P_t(k \mid l) \right\} = 0 \quad \text{a.e.} \tag{19}$$

Since

$$\left| \sum_{t\in T^{(n)}\setminus\{o\}} \sum_{l=0}^{b-1} \delta^i(t) I_l(X_{1_t}) P_t(k \mid l) - \sum_{l=0}^{b-1} S_l(T^{(n-1)}) P_i(k \mid l) \right| =$$

$$= \left| \sum_{l=0}^{b-1} \sum_{t\in T^{(n)}\setminus\{o\}} \delta^i(t) I_l(X_{1_t}) P_t(k \mid l) - \sum_{l=0}^{b-1} \sum_{t\in T^{(n-1)}} I_l(X_t) P_i(k \mid l) \right| =$$

$$= \left| \sum_{l=0}^{b-1} \sum_{t\in T^{(n)}\setminus\{o\}} \delta^i(t) I_l(X_{1_t}) P_t(k \mid l) - \sum_{l=0}^{b-1} \sum_{t\in T^{(n)}\setminus\{o\}} \delta^i(t) I_l(X_{1_t}) P_i(k \mid l) \right| \le$$

$$\le \sum_{l=0}^{b-1} \sum_{t\in T_n^i} |P_t(k \mid l) - P_i(k \mid l)| \tag{20}$$

and $X = \{X_t, t \in T\}$ is a tree-indexed asymptotic $N$-branch Markov chain, we derive equation (16) from equations (5), (19), (20) and $\lim_{n\to\infty} \dfrac{|T^{(n)}|}{|T^{(n-1)}|} = N$.

Lemma 2 is proved.

**3. Strong law of large numbers and asymptotic equipartition property.** In this section, we will prove the main results of this paper — the strong law of large number and $AEP$ for the tree-indexed asymptotic $N$-branch Markov chains.

**Theorem 1.** *Suppose $T$ is a Cayley tree $T_{C,N}$. Let $X = \{X_t, t \in T\}$ be an asymptotic $N$-branch Markov chain indexed by $T$ with state space $G$ defined by Definition* 2, *and let $S_k(T^{(n)})$ be defined as* (13). *If transition matrix $P = \dfrac{1}{N}(P_1 + P_2 + \ldots + P_N)$ is ergodic, then we have*

$$\lim_{n\to\infty} \frac{S_k(T^{(n)})}{|T^{(n)}|} = \pi(k) \quad a.e. \quad \forall k \in G, \tag{21}$$

*where $\pi = (\pi(0), \ldots, \pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Accumulating the equation (16) together with respect to $i$ from 1 to $N$ and according to equation (15), we have

$$\lim_{n\to\infty} \left\{ \frac{S_k(T^{(n)})}{|T^{(n)}|} - \sum_{l=0}^{b-1} \frac{S_l(T^{(n-1)})}{|T^{(n-1)}|} P(k \mid l) \right\} = 0 \quad \text{a.e.} \tag{22}$$

Similar to the last part of the proof of Theorem 2 in Yang and Ye [12], by (22), we obtain

$$\lim_{n\to\infty} \left\{ \frac{S_m(T^{(n+M)})}{|T^{(n+M)}|} - \sum_{l=0}^{b-1} \frac{S_l(T^{(n-1)})}{|T^{(n-1)}|} P^{(M+1)}(m|l) \right\} = 0 \quad \text{a.e.} \tag{23}$$

Noticing that

$$\frac{1}{|T^{(n-1)}|} \sum_{l=0}^{b-1} S_l(T^{(n-1)}) = 1 \tag{24}$$

and

$$\lim_{M\to\infty} P^{(M+1)}(k \mid l) = \pi(k), \quad k \in G, \tag{25}$$

we derive equation (21) from equations (23) – (25).

Theorem 1 is proved.

***Corollary 2.*** *Under the conditions of Theorem* 1, *let $S_k(T_n^i)$, $i = 1, 2, \ldots, N$, be defined as* (14). *Then we have*

$$\lim_{n\to\infty} \frac{S_k(T_n^i)}{|T^{(n)}|} = \frac{1}{N} \sum_{l=0}^{b-1} \pi(l) P_i(k \mid l) \quad a.e. \quad i = 1, 2, \ldots, N. \tag{26}$$

***Proof.*** This corollary follows from Lemma 2 and Theorem 1 directly.

***Corollary* 3.** *Let $T$ be a Cayley tree $T_{C,N}$, $X = \{X_t, t \in T\}$ be a $N$-branch Markov chain indexed by $T$ with state space $G$, and $S_k(T_n^i)$, $i = 1, 2, \ldots, N$, be defined as* (14). *If transition matrix $P = \dfrac{1}{N}(P_1 + P_2 + \ldots + P_N)$ is ergodic, then equation* (26) *holds, where $\pi = (\pi(0), \ldots, \pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $P_t = P_i$ for any $t \in T^i$, $i = 1, 2, \ldots, N$, in Corollary 2. Then this corollary is obtained directly.

***Corollary* 4** ([5], Corollary 2)**.** *Suppose that $T$ is a binary tree, and $X = \{X_t, t \in T\}$ is a nonsymmetric Markov chain indexed by a binary tree with state space $G$. Let $S_k(T_n^i)$, $i = 1, 2$, be defined as* (16). *If transition matrix $P = \dfrac{1}{2}(P_1 + P_2)$ is ergodic, then we have*

$$\lim_{n\to\infty} \frac{S_k(T_n^i)}{|T^{(n)}|} = \frac{1}{2} \sum_{l=0}^{b-1} \pi(l) P_i(k \mid l) \quad \text{a.e.} \quad i = 1, 2. \tag{27}$$

***Proof.*** Let $N = 2$ in Corollary 3. Then this corollary is derived directly.

***Corollary* 5** [1]**.** *Let $T$ be a binary tree and $G = \{0, 1\}$. Suppose that $P_1$ and $P_2$ be two strictly positive $2 \times 2$ transition matrices on $G^2$ which have a common stationary distribution $\pi = (\pi(0), \pi(1))$. Let $X = \{X_t, t \in T\}$ be a nonsymmetric Markov chain field on a binary tree taking values in $G$ with transition matrices $P_1$ and $P_2$, and let $S_k(T_n^i)$, $i = 1, 2$, be defined as* (14). *Then we have*

$$\lim_{n\to\infty} \frac{S_k(T_n^i)}{|T^{(n)}|} = \frac{1}{2} \sum_{l=0}^{1} \pi(l) P_i(k \mid l) \quad \text{a.e.}, \quad i = 1, 2. \tag{28}$$

***Proof.*** Letting $P = \dfrac{1}{2}(P_1 + P_2)$, it is easy to show that $P$ is ergodic and has the stationary distribution $\pi$. Then this corollary follows from Corollary 3 immediately.

***Corollary* 6** ([6], Theorem 2)**.** *Let $T$ be a Cayley tree, and $X = \{X_t, t \in T\}$ be a tree-indexed nonhomogeneous Markov chain with state space $G$ defined by Definition* 1. *Assume that $P = (P(k \mid l))$, $k, l \in G$, be a transition matrix, and $P$ is ergodic. Let $S_k(T^{(n)})$ be defined as* (13). *If* (6) *holds, then we have*

$$\lim_{n\to\infty} \frac{S_k(T^{(n)})}{|T^{(n)}|} = \pi(k) \quad \text{a.e.} \quad \forall k \in G, \tag{29}$$

*where $\pi = (\pi(0), \ldots, \pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $P_i = P$ for any $i$, $1 \le i \le N$ in Theorem 1. Then we know that equation (6) is equivalent to equation (5). Thus this corollary is obtained directly.

Let $A$ be a subset of $T$ that does not contain $o$, and let

$$S_{k,m}(A) = \big|\{t \in A, (X_{1_t}, X_t) = (k, m)\}\big|.$$

Obviously, we have the following formulas:

$$S_{k,m}(T^{(n)} \backslash \{o\}) = \sum_{t \in T^{(n)} \backslash \{o\}} I_m(X_t) I_k(X_{1_t}), \tag{30}$$

$$S_{k,m}(T_n^i) = \sum_{t \in T^{(n)} \setminus \{o\}} I_m(X_t) I_k(X_{1_t}) \delta^i(t), \tag{31}$$

$$S_{k,m}(T^{(n)} \setminus \{o\}) = \sum_{i=1}^{N} S_{k,m}(T_n^i). \tag{32}$$

**Theorem 2.** *Suppose that $T$ is a Cayley tree, and $X = \{X_t, t \in T\}$ is an asymptotic $N$-branch Markov chain indexed by $T$ with state space $G$ defined by Definition* 2. *Let $S_{k,m}(T_n^i)$, $i = 1, 2, \ldots, N$, be defined as* (31), *and transition matrix $P = \dfrac{1}{N}(P_1 + P_2 + \ldots + P_N)$. If $P$ is ergodic*, *then we have*

$$\lim_{n\to\infty} \frac{S_{k,m}(T_n^i)}{|T^{(n)}|} = \frac{1}{N} \pi(k) P_i(m \mid k) \quad a.e., \quad i = 1, 2, \ldots, N, \tag{33}$$

*where $\pi = (\pi(0), \ldots, \pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $g_t(x, y) = I_k(x) I_m(y) \delta^i(t)$, $i = 1, 2, \ldots, N$, in Corollary 1. Then we get

$$H_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} I_k(X_{1_t}) I_m(X_t) \delta^i(t) = S_{k,m}(T_n^i) \tag{34}$$

and

$$G_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) \mid X_{1_t}] =$$

$$= \sum_{t \in T^{(n)} \setminus \{o\}} \sum_{x_t \in G} I_k(X_{1_t}) I_m(x_t) \delta^i(t) P_t(x_t \mid X_{1_t}) =$$

$$= \sum_{t \in T_n^i} I_k(X_{1_t}) P_t(m \mid k). \tag{35}$$

Obviously, $\{g_t(x, y), t \in T\}$ are uniformly bounded functions. Then it follows from Corollary 1 that

$$\lim_{n\to\infty} \frac{S_{k,m}(T_n^i) - \sum_{t \in T_n^i} I_k(X_{1_t}) P_t(m \mid k)}{|T^{(n)}|} = 0 \quad \text{a.e.} \tag{36}$$

Noticing that

$$\left| \sum_{t \in T_n^i} I_k(X_{1_t}) P_t(m \mid k) - S_k(T^{(n-1)}) P_i(m \mid k) \right| =$$

$$= \left| \sum_{t \in T_n^i} I_k(X_{1_t}) P_t(m \mid k) - \sum_{t \in T_n^i} I_k(X_{1_t}) P_i(m \mid k) \right| \leq$$

$$\leq \sum_{t \in T_n^i} |P_t(m \mid k) - P_i(m \mid k)|, \tag{37}$$

we have by (5), (36) and (37)

$$\lim_{n\to\infty}\frac{S_{k,m}(T_n^i)-S_k(T^{(n-1)})P_i(m\mid k)}{|T^{(n)}|}=0\quad \text{a.e.} \tag{38}$$

According to Theorem 1, (38) and $\lim_{n\to\infty}\dfrac{|T^{(n)}|}{|T^{(n-1)}|}=N$, we derive equation (33) immediately.

Theorem 2 is proved.

***Corollary* 7.** *Under conditions of Theorem* 2, *let* $S_{k,m}(T^{(n)}\backslash\{o\})$ *be as* (32). *Then we have*

$$\lim_{n\to\infty}\frac{S_{k,m}(T^{(n)}\backslash\{o\})}{|T^{(n)}|}=\frac{1}{N}\pi(k)\sum_{i=1}^{N}P_i(m\mid k)\quad \textit{a.e.} \tag{39}$$

***Proof.*** Combining (32) and (33), we obtain (39) directly.

***Corollary* 8.** *Suppose that $T$ is a Cayley tree, and $X=\{X_t,t\in T\}$ is a $N$-branch Markov chain indexed by $T$ with state space $G$. Let $S_{k,m}(T^{(n)}\backslash\{o\})$ be as* (32), *and transition matrix $P=\dfrac{1}{N}(P_1+P_2+\ldots+P_N)$. If $P$ is ergodic, then* (39) *holds, where $\pi=(\pi(0),\ldots,\pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $P_t=P_i$ for any $t\in T^i$, $i=1,2,\ldots,N$, in Corollary 7. Then we obtain this result directly.

***Corollary* 9** ([5], Corollary 4)**.** *Let $T$ be a binary tree, and $X=\{X_t,t\in T\}$ be a nonsymmetric Markov chain indexed by $T$ with state space $G$. Let $S_{k,m}(T^{(n)}\backslash\{o\})$ be as* (32), *and transition matrix $P=\dfrac{1}{2}(P_1+P_2)$. If $P$ is ergodic, then we have*

$$\lim_{n\to\infty}\frac{S_{k,m}(T^{(n)}\backslash\{o\})}{|T^{(n)}|}=\frac{1}{2}\pi(k)\sum_{i=1}^{2}P_i(m\mid k)\quad \textit{a.e.}, \tag{40}$$

*where $\pi=(\pi(0),\ldots,\pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $N=2$ in Corollary 8. Then this result is derived directly.

***Corollary* 10** ([6], Theorem 2)**.** *Suppose that $T$ is a Cayley tree, and $X=\{X_t,t\in T\}$ is a nonhomogeneous Markov chain indexed by $T$ with state space $G$ defined by Definition* 1. *Let $S_{k,m}(T^{(n)}\backslash\{o\})$ be defined as* (32). *If transition matrix $P$ is ergodic and (6) holds, then we have*

$$\lim_{n\to\infty}\frac{S_{k,m}(T^{(n)}\backslash\{o\})}{|T^{(n)}|}=\pi(k)P(m\mid k)\quad \textit{a.e.}, \tag{41}$$

*where $\pi$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $P_i=P$ for all $i$, $1\le i\le N$, in Corollary 7. Then we obtain that (6) is equivalent to (5). Thus this corollary is derived immediately.

Let $T$ be a Cayley tree, and let $X=\{X_t,t\in T\}$ be a stochastic process indexed by tree $T$ with state space $G$. Denote $P\big(x^{T^{(n)}}\big)=\mathbf{P}\big(X^{T^{(n)}}=x^{T^{(n)}}\big)$. If we set

$$f_n(\omega)=-\frac{1}{|T^{(n)}|}\ln P\big(X^{T^{(n)}}\big), \tag{42}$$

then $f_n(\omega)$ is said to be the entropy density of $X^{T^{(n)}}$. If $X = \{X_t, t \in T\}$ is a $G$-valued nonhomogeneous Markov chain indexed by $T$ with initial distribution (1) and transition matrices (2) defined by Definition 1, then we have (see [6], (38))

$$P\big(x^{T^{(n)}}\big) = \mathbf{P}\big(X^{T^{(n)}} = x^{T^{(n)}}\big) = P(X_o)\prod_{i=1}^{N}\prod_{t\in T_n^i} P_t(x_t \mid x_{1_t}). \tag{43}$$

In this case, we have

$$f_n(\omega) = -\frac{1}{|T^{(n)}|}\left\{\ln P(X_o) + \sum_{i=1}^{N}\sum_{t\in T_n^i}\ln P_t(X_t \mid X_{1_t})\right\}. \tag{44}$$

The convergence of $f_n(\omega)$ to a constant (in a sense of $L_1$ convergence, convergence in probability, a.e. convergence) is called the Shannon – McMillan theorem or the entropy theorem or the AEP in information theory.

**Lemma 3** ([6], Lemma 3). *Suppose that $T$ is a Cayley tree. Let $\varphi(x)$ be a bounded function on interval $\Delta$ and continuous at $x = a (a \in \Delta)$. Let $\{a_t, t \in T\}$ be a collection of real numbers. If*

$$\lim_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}}|a_t - a| = 0, \tag{45}$$

*then we obtain*

$$\lim_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}}|\varphi(a_t) - \varphi(a)| = 0. \tag{46}$$

**Theorem 3.** *Suppose that $T$ is a Cayley tree, and $X = \{X_t, t \in T\}$ is an asymptotic $N$-branch Markov chain indexed by $T$ with state space $G$ defined by Definition 2. Let $f_n(\omega)$ be defined as (44). If transition matrix $P = \dfrac{1}{N}(P_1 + P_2 + \ldots + P_N)$ is ergodic, then*

$$\lim_{n\to\infty} f_n(\omega) = -\frac{1}{N}\sum_{i=1}^{N}\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\pi(l)P_i(k \mid l)\ln P_i(k \mid l) \quad a.e. \tag{47}$$

*holds, where $\pi = (\pi(0), \ldots, \pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** If $X = \{X_t, t \in T\}$ be a tree-indexed asymptotic $N$-branch Markov chain, then (5) holds. Let $\varphi(x) = x\ln x$ $(\varphi(0) = 0)$. It is easy to verify that $\varphi(x)$ is a continuous function on $[0, 1]$. It follows from Lemma 3 and equation (5) that for any $k, l \in G$

$$\lim_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T_n^i}|P_t(k \mid l)\ln P_t(k \mid l) - P_i(k \mid l)\ln P_i(k \mid l)| = 0 \quad \text{a.e.} \quad \forall k, l \in G, \tag{48}$$

$$i = 1, 2, \ldots, N.$$

Letting $a_n = |T^{(n)}|$, $g_t(x, y) = \delta^i(t)\ln P_t(y|x)$ $1 \le i \le N$ for all $t \in T$ in Lemma 1, we obtain

$$H_n(\omega) = \sum_{t\in T^{(n)}\setminus\{o\}}[\delta^i(t)\ln P_t(X_t \mid X_{1_t})] = \sum_{t\in T_n^i}\ln P_t(X_t \mid X_{1_t}), \tag{49}$$

$$G_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) \mid X_{1_t}] =$$

$$= \sum_{t \in T^{(n)} \setminus \{o\}} \sum_{x_t \in G} \delta^i(t) \ln P_t(x_t \mid X_{1_t}) P_t(x_t \mid X_{1_t}) =$$

$$= \sum_{t \in T_n^i} \sum_{x_t \in G} [P_t(x_t \mid X_{1_t}) \ln P_t(x_t \mid X_{1_t})] \tag{50}$$

by equations (9) and (10). Let $\alpha = \dfrac{1}{2}$. Then for any $t \in T$ and $i,\ 1 \le i \le N$, we have

$$E\big[g_t^2(X_{1_t}, X_t) e^{\alpha |g_t(X_{1_t}, X_t)|} \mid X_{1_t}\big] =$$

$$= \sum_{x_t \in G} \delta^i(t) P_t(x_t \mid X_{1_t}) e^{-\frac{1}{2}\delta^i(t) \ln P_t(x_t | X_{1_t})} \ln^2 P_t(x_t \mid X_{1_t}) \le$$

$$\le \sum_{x_t \in G} [P_t(x_t \mid X_{1_t})]^{\frac{1}{2}} \ln^2 P_t(x_t \mid X_{1_t}) \le 16be^{-2}$$

and

$$\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{o\}} E\big[g_t^2(X_{1_t}, X_t) e^{\frac{1}{2}|g_t(X_{1_t}, X_t)|} | X_{1_t}\big] \le 16be^{-2}. \tag{51}$$

By (49)–(51) and Lemma 1, we obtain $D\left(\dfrac{1}{2}\right) = \Omega$ and

$$\lim_{n \to \infty} \frac{1}{|T^{(n)}|} \left\{ \sum_{t \in T_n^i} \ln P_t(X_t \mid X_{1_t}) - \sum_{t \in T_n^i} \sum_{x_t \in G} [P_t(x_t \mid X_{1_t}) \ln P_t(x_t \mid X_{1_t})] \right\} = 0 \quad \text{a.e.} \tag{52}$$

for any $i,\ 1 \le i \le N$. Notice that

$$\Bigg| \frac{1}{|T^{(n)}|} \sum_{t \in T_n^i} \sum_{x_t \in G} P_t(x_t \mid X_{1_t}) \ln P_t(x_t \mid X_{1_t}) -$$

$$- \frac{1}{N} \sum_{l=0}^{b-1} \sum_{k=0}^{b-1} \pi(l) P_i(k \mid l) \ln P_i(k \mid l) \Bigg| \le$$

$$\le \Bigg| \frac{1}{|T^{(n)}|} \sum_{t \in T_n^i} \sum_{l=0}^{b-1} \sum_{k=0}^{b-1} I_l(X_{1_t}) P_t(k \mid l) \ln P_t(k \mid l) -$$

$$- \frac{1}{|T^{(n)}|} \sum_{t \in T_n^i} \sum_{l=0}^{b-1} \sum_{k=0}^{b-1} I_l(X_{1_t}) P_i(k \mid l) \ln P_i(k \mid l) \Bigg| +$$

$$+ \Bigg| \frac{1}{|T^{(n)}|} \sum_{t \in T_n^i} \sum_{l=0}^{b-1} \sum_{k=0}^{b-1} I_l(X_{1_t}) P_i(k \mid l) \ln P_i(k \mid l) -$$

$$-\frac{1}{N}\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\pi(l)P_i(k\mid l)\ln P_i(k\mid l)\Bigg| \leq$$

$$\leq \sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\frac{1}{|T^{(n)}|}\sum_{t\in T_n^i}|P_t(k\mid l)\ln P_t(k\mid l)-P_i(k\mid l)\ln P_i(k\mid l)|+$$

$$+\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}P_i(k\mid l)|\ln P_i(k\mid l)|\left|\frac{1}{|T^{(n)}|}\sum_{t\in T_n^i}I_l(X_{1_t})-\frac{1}{N}\pi(l)\right|=$$

$$=\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\frac{1}{|T^{(n)}|}\sum_{t\in T_n^i}|P_t(k\mid l)\ln P_t(k\mid l)-P_i(k\mid l)\ln P_i(k\mid l)|+$$

$$+\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}P_i(k\mid l)|\ln P_i(k\mid l)|\left|\frac{S_l(T^{(n-1)})}{|T^{(n)}|}-\frac{1}{N}\pi(l)\right| \tag{53}$$

for all $i,\ 1\leq i\leq N$. It follows from Theorem 1, and equations (52), (53) that

$$\lim_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T_n^i}\ln P_t(X_t\mid X_{1_t})=\frac{1}{N}\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\pi(l)P_i(k\mid l)\ln P_i(k\mid l)\quad \text{a.e.} \tag{54}$$

Accumulating the equation (54) together with respect to $i$ from 1 to $N$, we have

$$\lim_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{t\in T^{(n)}\setminus\{o\}}\ln P_t(X_t\mid X_{1_t})=\frac{1}{N}\sum_{i=1}^{N}\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\pi(l)P_i(k\mid l)\ln P_i(k\mid l)\quad \text{a.e.} \tag{55}$$

Then equation (47) follows from equations (44) and (55).

Theorem 3 is proved.

***Corollary* 11.** *Suppose that $T$ is a Cayley tree, and $X=\{X_t,t\in T\}$ is a $N$-branch Markov chain indexed by $T$ with state space $G$. Let $f_n(\omega)$ be as* (44). *If transition matrix $P=\frac{1}{N}(P_1+$ $+P_2+\ldots+P_N)$ and $P$ is ergodic, then* (47) *holds, where $\pi=(\pi(0),\ldots,\pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $P_t=P_i$ for all $t\in T^i$, $i=1,2,\ldots,N$, in Theorem 3. Then we derive this corollary immediately.

***Corollary* 12** ([5], Theorem 5)**.** *Suppose that $T$ is a binary tree, and $X=\{X_t,t\in T\}$ is a nonsymmetric Markov chain indexed by $T$ with state space $G$. Let $f_n(\omega)$ be defined as* (44). *If transition matrix $P=\frac{1}{2}(P_1+P_2)$ is ergodic, then we have*

$$\lim_{n\to\infty}f_n(\omega)=-\frac{1}{2}\sum_{i=1}^{2}\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\pi(l)P_i(k\mid l)\ln P_i(k\mid l)\quad a.e., \tag{56}$$

*where $\pi=(\pi(0),\ldots,\pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $N=2$ in Corollary 11. Then this corollary derived obviously.

***Corollary* 13** ([6], Theorem 3)**.** *Suppose that $T$ is a Cayley tree, and $X = \{X_t, t \in T\}$ is a nonhomogeneous Markov chain indexed by $T$ with state space $G$ defined by Definition* 1. *Let transition matrix $P$ is ergodic, and $f_n(\omega)$ be defined by* (44). *If* (6) *holds, then we have*

$$\lim_{n\to\infty} f_n(\omega) = -\sum_{k=0}^{b-1}\sum_{m=0}^{b-1} \pi(l)P(k \mid l)\ln P(k \mid l) \quad a.e., \tag{57}$$

*where $\pi = (\pi(0),\ldots,\pi(b-1))$ is the unique stationary distribution determined by $P$.*

***Proof.*** Let $P_i = P$, $i = 1,2,\ldots,N$, in Theorem 3. Then equation (6) is equivalent to equation (5). Thus we obtain this corollary immediately.

## References


1. *Bao Z. H., Ye Z. X.* Strong law of large numbers and asymptotic equipartition property for nonsymmetric Markov chain fields on Cayley trees // Acta Math. Sci. Ser. B. – 2007. – **27**, № 4. – P. 829 – 837.
2. *Benjamini I., Peres Y.* Markov chains indexed by trees // Ann. Probab. – 1994. – **22**, № 1. – P. 219 – 243.
3. *Berger T., Ye Z.* Entropic aspects of random fields on trees // IEEE Trans. Inform. Theory. – 1990. – **36**. – P. 1006 – 1018.
4. *Huang H. L., Yang W. G.* Strong law of large numbers for Markov chains indexed by an infinite tree with uniformly bounded degree // Sci. China. Ser. A. – 2008. – **51**, № 2. – P. 195 – 202.
5. *Dong Y., Yang W. G.* Strong law of large numbers and asymptotic equipartition probability for nonsymmetric Markov chain indexed by Cayley tree // J. Math. Res. and Exposit. – 2010. – **30**, № 6. – P. 976 – 984.
6. *Dong Y., Yang W. G., Bai J. F.* The strong law of large numbers and the Shannon – McMillan theorem for nonhomogeneous Markov chains indexed by a Cayley tree // Statist. Probab. Lett. – 2011. – **81**. – P. 1883 – 1890.
7. *Pemantle R.* Automorphism invariant measure on trees // Ann. Probab. – 1992. – **20**, № 3. – P. 1549 – 1566.
8. *Spitzer F.* Markov random fields on an infinite tree // Ann. Probab. – 1975. – **3**. – P. 387 – 398.
9. *Yang W. G., Liu W.* Strong law of large numbers for Markov chains fields on a Bethe tree // Statist. Probab. Lett. – 2000. – **49**, № 3. – P. 245 – 250.
10. *Yang W. G.* Some limit properties for Markov chains indexed by a homogeneous tree // Statist. Probab. Lett. – 2003. – **65**, № 3. – P. 241 – 250.
11. *Yang W. G., Ye Z.* The asymptotic equipartition property for Markov chains indexed by a Homogeneous tree // IEEE Trans. Inform. Theory. – 2007. – **53**, № 9. – P. 3275 – 3280.
12. *Ye Z., Berger T.* Ergodicity, regularity and asymptotic equipartition property of random fields on trees // J. Comb. Inform. Syst. Sci. – 1996. – **21**, № 2. – P. 157 – 184.
13. *Ye Z., Berger T.* Information measures for discrete random fields. – Beijing: Sci. Press, 1998.


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