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V. Mazorchuk* (Uppsala Univ., Sweden),
X. Zhang (East China Normal Univ., Shanghai, China)

SIMPLE TRANSITIVE 2-REPRESENTATIONS FOR TWO NON-FIAT 2-CATEGORIES OF PROJECTIVE FUNCTORS

ПРОСТІ ТРАНЗИТИВНІ 2-ЗОБРАЖЕННЯ ДВОХ НЕФІАТНИХ 2-КАТЕГОРІЙ ПРОЕКТИВНИХ ФУНКТОРІВ

We show that any simple transitive 2-representation of the 2-category of projective endofunctors for the quiver algebra of $\Bbbk(\bullet \longrightarrow \bullet)$ and for the quiver algebra of $\Bbbk(\bullet \longrightarrow \bullet \longrightarrow \bullet)$ is equivalent to a cell 2-representation.

Показано, що будь-яке просте транзитивне 2-зображення 2-категорії проективних ендофункторів для алгебри сагайдака $\Bbbk(\bullet \longrightarrow \bullet)$ та алгебри сагайдака $\Bbbk(\bullet \longrightarrow \bullet \longrightarrow \bullet)$ еквівалентне клітинковому 2-зображенню.

1. Introduction and description of the results. Classification problems are interesting and important problems in the classical representation theory. For example, classifications of various classes of simple or indecomposable modules over different classes of algebras played significant role in both development and applications of modern representation theory.

Higher representation theory is a recent direction of representation theory that takes its origins from the papers [2, 3, 18, 19]. Of particular interest in higher representation theory is the study of so-called finitary 2-categories as the latter are natural 2-analogues of finite dimensional algebras. Initial abstract study of finitary 2-categories and the corresponding 2-representation theory was done in [12 – 17, 20].

As an outcome of this study, one interesting and important class of 2-representations, called *simple transitive* 2*-representations*, was defined in [16]. These 2-representations are natural 2-analogues of usual simple modules over algebras. Therefore the problem of classification of simple transitive 2-representations is natural and interesting. In several cases, it turns out that simple transitive 2-representations can be classified, see, for example, various results in [5, 16, 17, 22, 23]. We also refer the reader to [6, 7, 11, 21] to related questions and applications. In particular, in [7], classification of simple transitive 2-representations for the 2-category of Soergel bimodules over the coinvariant algebra of the symmetric group was crucially used for classification of projective functors in parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$.

The most basic example of a 2-category is the 2-category $\mathscr{C}_A$ of projective functors for a finite-dimensional algebra A over an algebraically closed field $\Bbbk$, defined in [12] (Subsection 7.3). In [14, 17], it is shown that categories of the form $\mathscr{C}_A$ essentially exhausts a natural class of "simple" finitary 2-categories possessing weak involutions. For such 2-categories, it was shown in [16, 17] that simple transitive 2-representations are exactly the cell 2-representations, defined in [12]. Existence of a weak involution on a 2-category restricts the classification result to the case when A is a self-injective algebra.

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The aim of the present paper to classify simple transitive 2-representations of $\mathscr{C}_A$ for the smallest possible non-self-injective algebra, namely the path algebra A of the quiver $1 \longrightarrow 2$, over an algebraically closed field $\Bbbk$. It turns out that our approach also extends, with a significantly increased amount of technical work, to the quiver algebra of $\Bbbk(\bullet \longrightarrow \bullet \longrightarrow \bullet)$, where, as usual, the dotted arrow depicts the corresponding zero relation. Our main result is the following theorem, we refer the reader to Sections 2 for details on all definitions.

Theorem 1. *For $A = \Bbbk(\bullet \to \bullet)$ or $A = \Bbbk(\bullet \longrightarrow \bullet \longrightarrow \bullet)$, any simple transitive 2-representation of the 2-category $\mathscr{C}_A$ is equivalent to a cell 2-representation.*

Despite of the fact that the formulation of Theorem 1 is rather similar to the corresponding statement in the case when A is self-injective, considered in [16, 17], our approach to the proof is fairly different, since the general approach outlined in [16, 17] does not apply. Our approach, rather, has many similarities with the approach in [23] and is mostly based on a careful analysis of all possible cases.

In Section 2, we collect all necessary preliminaries for 2-representation theory of the 2-category $\mathscr{C}_A$. In Section 3, we prove some general results about 2-representations of $\mathscr{C}_A$ under the additional assumption that the algebra A has a non-zero projective injective module.

In Sections 4, 5 and 6, we collect the proof of Theorem 1 in the case $A = \Bbbk(\bullet \to \bullet)$. In more details, Section 4 contains preliminaries on $\mathscr{C}_A$ for $A = \Bbbk(\bullet \to \bullet)$. Section 5 contains combinatorial results on certain integer matrices which allow us to specify three essentially different cases which we have to deal with. In Section 6 we prove Theorem 1 for $A = \Bbbk(\bullet \to \bullet)$.

In Sections 7, 8, 9 and 10, we collect the proof of Theorem 1 in the second case of the algebra $A = \Bbbk(\bullet \longrightarrow \bullet \longrightarrow \bullet)$. In more details, in Section 7 one finds preliminaries on $\mathscr{C}_A$. Sections 8 and 9 are devoted to finding out which integer matrix captures the combinatorics of a faithful simple transitive 2-representation of $\mathscr{C}_A$. Finally, Section 10 completes the proof of Theorem 1 for the algebra $\Bbbk(\bullet \longrightarrow \bullet \longrightarrow \bullet)$.

2. 2-Category $\mathscr{C}_A$ and its 2-representations. ***2.1. Notation and conventions.*** Throughout the paper we work over an algebraically closed field $\Bbbk$ and abbreviate $\otimes_\Bbbk$ by $\otimes$. Unless explicitly stated otherwise, by a module, we mean a *left* module. We compose maps from right to left. For a 1-morphism F, we denote by id_{F} the identity 2-morphism for F.

2.2. 2-Category $\mathscr{C}_A$. We refer the reader to [9–11] for generalities on 2-categories. A 2-category is a category which is enriched over the monoidal category $\mathbf{Cat}$ of small categories.

Let A be a connected, basic, finite dimensional $\Bbbk$-algebra and $\mathcal{C}$ a small category equivalent to A-mod. Consider the 2-category $\mathscr{C}_A$ (which depends on $\mathcal{C}$) defined as follows:

$\mathscr{C}_A$ has one object $\mathtt{i}$ which we identify with $\mathcal{C}$;

1-morphisms in $\mathscr{C}_A$ are endofunctors of $\mathcal{C}$ isomorphic to functors given by tensoring with A-A-bimodules from the additive closure of both ${}_AA_A$ and ${}_AA \otimes A_A$;

2-morphisms in $\mathscr{C}_A$ are natural transformations of functors.

The 2-category $\mathscr{C}_A$ is *finitary* in the sense of [12] (Subsection 2.2).

2.3. 2-Representations $\mathscr{C}_A$. We consider the 2-category $\mathscr{C}_A$-afmod of all *finitary 2-representations* of $\mathscr{C}_A$. In this 2-category, an object is a strict additive functorial action of $\mathscr{C}_A$, denoted $\mathbf{M}$ or similar, on an additive, idempotent split, Krull – Schmidt, $\Bbbk$-linear category $\mathbf{M}(\mathtt{i})$ with finitely many isomorphism classes of indecomposable objects and finite dimensional morphism spaces;

1-morphisms are 2-natural transformations;

2-morphisms are modifications.

We refer the reader to [14] for details. Two 2-representations are called *equivalent* provided that there is a 2-natural transformation between them whose restriction to each object is an equivalence of categories.

We also consider the 2-category $\mathscr{C}_A\text{-mod}$ defined similarly using functorial action on categories equivalent to module categories of finite dimensional $\Bbbk$-algebras. There is the diagrammatically defined abelianization 2-functor

$$\overline{\cdot}: \ \mathscr{C}_A\text{-afmod} \to \mathscr{C}_A\text{-mod}.$$

Given a functorial action of $\mathscr{C}_A$ on some $\mathbf{M}(\mathtt{i})$ as above, the 2-functor $\overline{\cdot}$ defines component-wise a functorial action of $\mathscr{C}_A$ on the Abelian category $\overline{\mathbf{M}(\mathtt{i})}$ whose objects are diagrams of the form $X \to Y$ over $\mathbf{M}(\mathtt{i})$ and morphisms are given by the obvious commutative squares in which one mods out the projective homotopy relations. We refer the reader to [13] (Subsection 4.2) for details.

A finitary 2-representation of $\mathscr{C}_A$ is called *transitive* provided that, for any indecomposable objects X and Y in $\mathbf{M}(\mathtt{i})$, there is a 1-morphism F in $\mathscr{C}_A$ such that Y is isomorphic to a direct summand of $\mathbf{M}(\mathrm{F})\,X$.

A transitive 2-representation $\mathbf{M}$ is called *simple* provided that $\mathbf{M}(\mathtt{i})$ does not have non-zero proper $\mathscr{C}_A$-invariant ideals.

For simplicity, we will often use the module notation $\mathrm{F}\,X$ instead of the representation notation $\mathbf{M}(\mathrm{F})\,X$.

We note that each strict monoidal category can be viewed as a 2-category with one object. With this identification, the above notion of 2-representation corresponds to the notion of strict monoidal functor. One can view 2-representations (with a fixed target) as homomorphism 2-categories in an appropriate version of the 3-category 2-$\mathbf{Cat}$ (here our choice of the level of strictness for transformations corresponds to *strong* transformations in the language of [9] (see [14], Subsection 2.3) for details).

2.4. Cells in $\mathscr{C}_{\mathbf{A}}$. Let $1 = e_1 + e_2 + \ldots + e_n$ be a primitive decomposition of $1 \in A$. Up to isomorphism, indecomposable 1-morphisms in $\mathscr{C}_A$ are given by tensoring with ${}_AA_A$ or with ${}_AAe_i \otimes e_jA_A$, where $i, j = 1, 2, \ldots, n$. We fix a representative F_0 in the isomorphism class of 1-morphisms which correspond to tensoring with ${}_AA_A$. For $i, j = 1, 2, \ldots, n$, we fix a representative F_{ij} in the isomorphism class of 1-morphisms which correspond to tensoring with ${}_AAe_i \otimes e_jA_A$. The set of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}_A$ has the natural structure of a multisemigroup (see [13], Section 3) and [8]. Combinatorics of this structure is encoded into so-called *left, right* and *two-sided cells* (see [13], Section 3). Two 1-morphisms F and G belong to the same *left cell* provided that there exist 1-morphisms K_1 and K_2 such that F is isomorphic to a direct summand of $\mathrm{K}_1 \circ \mathrm{G}$ and G is isomorphic to a direct summand of $\mathrm{K}_2 \circ \mathrm{F}$. Right and two-sided cells are defined similarly using composition on the right or from both sides, respectively.

For $\mathscr{C}_A$, the two sided-cells are

$$\mathcal{J}_0 := \{\mathrm{F}_0\} \qquad \text{and} \qquad \mathcal{J} := \{\mathrm{F}_{ij} : i, j = 1, 2, \ldots, n\}.$$

The two-sided cell $\{\mathrm{F}_0\}$ is a left and a right cell as well. Other left cells are

$$\{\mathrm{F}_{ij} : i = 1, 2, \ldots, n\}, \quad j = 1, 2, \ldots, n.$$

Other right cells are

$$\{\mathrm{F}_{ij} : j = 1, 2, \ldots, n\}, \quad i = 1, 2, \ldots, n.$$

As usual, we have

$$\mathrm{F}_{ij} \circ \mathrm{F}_{st} = \mathrm{F}_{it}^{\oplus \dim(e_j A e_s)}. \tag{1}$$

We set

$$\mathrm{F} := \bigoplus_{i,j=1}^{n} \mathrm{F}_{ij}$$

and note that

$$\mathrm{F} \circ \mathrm{F} \cong \mathrm{F}^{\oplus \dim(A)}. \tag{2}$$

All 1-morphisms in the additive closure of F are called *projective endofunctors* of $\mathcal{C}$. Similarly for A-mod.

As usual, we will say that a pair $(\mathrm{F}_{ij}, \mathrm{F}_{st})$ of 1-morphisms is a pair of *adjoint* 1-morphisms provided that there exist 2-morphisms

$$\alpha : \mathrm{F}_{ij} \circ \mathrm{F}_{st} \to \mathrm{F}_0 \qquad \text{and} \qquad \beta : \mathrm{F}_0 \to \mathrm{F}_{st} \circ \mathrm{F}_{ij}$$

such that

$$(\alpha \circ_0 \mathrm{id}_{\mathrm{F}_{ij}}) \circ_1 (\mathrm{id}_{\mathrm{F}_{ij}} \circ_0 \beta) = \mathrm{id}_{\mathrm{F}_{ij}} \qquad \text{and} \qquad (\mathrm{id}_{\mathrm{F}_{st}} \circ_0 \alpha) \circ_1 (\beta \circ_0 \mathrm{id}_{\mathrm{F}_{st}}) = \mathrm{id}_{\mathrm{F}_{st}}.$$

The 2-category $\mathscr{C}_A$ is $\mathcal{J}$-simple in the sense that any non-zero two-sided 2-ideal of $\mathscr{C}_A$ contains the identity 2-morphisms for all 1-morphisms given by projective endofunctors (see [1, 13]).

2.5. Cell 2-representations. The first example of a finitary 2-representation of $\mathscr{C}_A$ is the *principal* 2-representation $\mathbf{P} := \mathscr{C}_A(\mathtt{i}, {}_-)$. This has a unique maximal $\mathscr{C}_A$-invariant ideal and the corresponding quotient is the cell 2-representation $\mathbf{C}_{\mathcal{L}}$, where $\mathcal{L} = \{\mathrm{F}_0\}$.

For any other left cell $\mathcal{L}$, the additive closure of elements in $\mathcal{L}$ gives a 2-subrepresentation of $\mathbf{P}$. This 2-subrepresentation again has a unique maximal $\mathscr{C}_A$-invariant ideal and the corresponding quotient is the cell 2-representation $\mathbf{C}_{\mathcal{L}}$. This latter cell 2-representation is equivalent to the defining action of $\mathscr{C}_A$ on the category A-proj of projective objects in A-mod (see [12] for details).

2.6. Matrices in the Grothendieck group. Let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}_A$ and $X_1, X_2, \ldots, X_k$ be a fixed complete and irredundant list of representatives of isomorphism classes of indecomposable objects in $\mathbf{M}(\mathtt{i})$. For a 1-morphism G in $\mathscr{C}_A$, we denote by $[\mathrm{G}]$ the $k \times k$ matrix with non-negative integer coefficients where, for $i, j = 1, 2, \ldots, k$, the coefficient in the intersection of the ith row and the jth column gives the number of indecomposable direct summands of $\mathbf{M}(\mathrm{G})\, X_j$ which are isomorphic to X_i. Note that $[\mathrm{G} \oplus \mathrm{H}] = [\mathrm{G}] + [\mathrm{H}]$ and $[\mathrm{G} \circ \mathrm{H}] = [\mathrm{G}][\mathrm{H}]$.

2.7. Action on simple transitive 2-representations. The following statement is proved in [16] (Lemma 12).

Lemma 1. *Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_A$. Then, for any non-zero object $X \in \overline{\mathbf{M}}(\mathtt{i})$, the object $\mathrm{F}\, X$ is projective in $\overline{\mathbf{M}}(\mathtt{i})$.*

The following statement is proved in [16] (Lemma 13).

Lemma 2. *Let B be a finite dimensional $\Bbbk$-algebra and G an exact endofunctor of B-mod. Assume that G sends each simple object of B-mod to a projective object. Then G is a projective functor.*

3. Existence of a projective-injective module guarantees exactness of the action. ***3.1. Exactness of the action of some projective functors.*** Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_A$. Consider its abelianization $\overline{\mathbf{M}}$. For $\overline{\mathbf{M}}(\mathtt{i})$, let $L_1, L_2, \ldots, L_k$ be a complete and irredundant list of representatives of isomorphism classes of simple objects. For $i \in \{1, 2, \ldots, k\}$, denote by P_i the indecomposable projective cover of L_i and by I_i the indecomposable injective envelope of L_i.

Lemma 3. *Let Q be a finite dimensional $\Bbbk$-algebra and K a right exact endofunctor of Q-mod. Then the following conditions are equivalent*:

(a) *The functor K sends projective objects to projective objects.*

(b) *The right adjoint K' of K is exact.*

Proof. By adjunction, for a projective generator $P \in Q\text{-mod}$, we have a natural isomorphism

$$\mathrm{Hom}_Q(\mathrm{K}P, {}_-) \cong \mathrm{Hom}_Q(P, \mathrm{K}'{}_-). \tag{3}$$

If $\mathrm{K}P$ is projective, the left-hand side of (3) is exact. Hence, the right-hand side is also exact. As P is a projective generator, the functor $\mathrm{Hom}_Q(P, {}_-)$ detects any non-zero homology. This forces K' to be exact. Therefore, (a) implies (b).

Conversely, assume that K' is exact. Then the right-hand side of (3) is exact. Hence, the left-hand side is exact. This means that $\mathrm{K}P$ is projective. Therefore, (b) implies (a). The claim follows.

Lemma 4. *Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left A-modules Ae_s and $\mathrm{Hom}_\Bbbk(e_tA, \Bbbk)$ are isomorphic. Then, for any $i \in \{1, 2, \ldots, n\}$, the pair $(\mathrm{F}_{it}, \mathrm{F}_{si})$ is a pair of adjoint 1-morphisms.*

Proof. The functor F_{it} is given by tensoring with the A-A-bimodule $Ae_i \otimes e_tA$. The right adjoint of this functor is thus the functor $\mathrm{Hom}_A(Ae_i \otimes e_tA, {}_-)$. By the computation in [12] (Subsection 7.3), the exact functor $\mathrm{Hom}_A(Ae_i \otimes e_tA, {}_-)$ is isomorphic to the functor of tensoring with the A-A-bimodule

$$\mathrm{Hom}_\Bbbk(e_tA, \Bbbk) \otimes e_iA.$$

The injective A-module $I_t \cong \mathrm{Hom}_\Bbbk(e_tA, \Bbbk)$ is isomorphic to the projective A-module Ae_s, by assumption. Therefore, $\mathrm{Hom}_\Bbbk(e_tA, \Bbbk) \otimes e_iA$ is isomorphic to $Ae_s \otimes e_iA$. This means that F_{si} is isomorphic to the right adjoint of F_{it}. The claim follows.

***Corollary* 1.** *Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left A-modules Ae_s and $\mathrm{Hom}_\Bbbk(e_tA, \Bbbk)$ are isomorphic. Then, for any $i \in \{1, 2, \ldots, n\}$ and any 2-representation $\mathbf{N}$ of $\mathscr{C}_A$, the pair $(\mathbf{N}(\mathrm{F}_{it}), \mathbf{N}(\mathrm{F}_{si}))$ is a pair of adjoint functors.*

Proof. This follows directly from Lemma 4 and definitions.

***Corollary* 2.** *Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left A-modules Ae_s and $\mathrm{Hom}_\Bbbk(e_tA, \Bbbk)$ are isomorphic. Then, for any $i \in \{1, 2, \ldots, n\}$ and any finitary 2-representation $\mathbf{N}$ of $\mathscr{C}_A$, the functor $\overline{\mathbf{N}}(\mathrm{F}_{si})$ is exact.*

Proof. This follows from the definitions by combining Lemma 3 and Corollary 1.

3.2. Auxiliary lemma.

Lemma 5. *Let Q be a finite dimensional $\Bbbk$-algebra and K, H and G be three endofunctors of Q-mod. Assume that*:

(a) H *is a projective functor*;

(b) K *is right exact*;

(c) K *sends projective objects to projective objects*;

(d) $\mathrm{K} \circ \mathrm{H} \cong \mathrm{G}$.

Then G is a projective functor.

Proof. By assumption (a), the functor H is given by tensoring with the Q-Q-bimodule $X \otimes Y$, for some projective left Q-module X and some projective right Q-module Y. By assumption (b), K is given by tensoring with some Q-Q-bimodule V. Using assumption (d), the Q-Q-bimodule that determines the functor G is given by

$$V \otimes_Q (X \otimes Y) \cong (V \otimes_Q X) \otimes Y. \tag{4}$$

By assumption (c), $V \otimes_Q X$ is a projective left Q-module. This implies that (4) is a projective Q-Q-bimodule and hence G is a projective functor.

3.3. Exactness of the action.

Proposition 1. *Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left A-modules Ae_s and $\mathrm{Hom}_{\Bbbk}(e_t A, \Bbbk)$ are isomorphic. Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_A$. Then the functor $\overline{\mathbf{M}}(\mathrm{F})$ is exact.*

Proof. Let B be a finite dimensional algebra such that $\overline{\mathbf{M}}(\mathtt{i})$ is equivalent to $B\text{-mod}$.

For $i \in \{1, 2, \ldots, n\}$, consider the 1-morphism F_{si}. By Corollary 2, the functor $\overline{\mathbf{M}}(\mathrm{F}_{si})$ is exact. By Lemma 1, $\overline{\mathbf{M}}(\mathrm{F}_{si})$ sends any object in $\overline{\mathbf{M}}(\mathtt{i})$ to a projective object. Therefore, by Lemma 2, $\overline{\mathbf{M}}(\mathrm{F}_{si})$ is a projective endofunctor of $B\text{-mod}$.

Now, for any $j \in \{1, 2, \ldots, n\}$, we have

$$\mathrm{F}_{js} \circ \mathrm{F}_{si} \cong \mathrm{F}_{ji}^{\oplus k},$$

where $k = \dim(e_s A e_s) > 0$. Therefore, $\overline{\mathbf{M}}(\mathrm{F}_{ji}^{\oplus k})$ is a projective functor for $B\text{-mod}$ by Lemma 5. By additivity, $\overline{\mathbf{M}}(\mathrm{F}_{ji})$ is a projective functor for $B\text{-mod}$ as well. In particular, $\overline{\mathbf{M}}(\mathrm{F}_{ji})$ is exact. The claim follows.

4. The algebra $\Bbbk(\bullet \to \bullet)$. Let $\Bbbk$ be an algebraically closed field. Denote by A the path algebra, over $\Bbbk$, of the quiver $1 \xrightarrow{\alpha} 2$. The algebra A has basis e_1, e_2 and α and the multiplication table $(x, y) \mapsto x \cdot y$ is given by

$x \backslash y$	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_2	α
α	α	0	0

Note that $e_1 A e_2 = 0$ as A contains no paths from 2 to 1. Note also that the left A-modules Ae_1 and $\mathrm{Hom}_{\Bbbk}(e_2 A, \Bbbk)$ are isomorphic.

Let $\mathcal{C}$ be a small category equivalent to $A\text{-mod}$. Consider the corresponding finitary 2-category $\mathscr{C}_A$. Up to isomorphism, indecomposable 1-morphisms in $\mathscr{C}_A$ are F_0 and F_{ij}, where $i, j = 1, 2$. Note that formula (2) for A reads $\mathrm{F} \circ \mathrm{F} = \mathrm{F}^{\oplus 3}$. Using (1), the table of compositions for the functors F_{ij} (up to isomorphism) is as follows:

$\circ$	F_{11}	F_{12}	F_{21}	F_{22}
F_{11}	F_{11}	F_{12}	0	0
F_{12}	F_{11}	F_{12}	F_{11}	F_{12}
F_{21}	F_{21}	F_{22}	0	0
F_{22}	F_{21}	F_{22}	F_{21}	F_{22}

(5)

Set $\mathcal{J}_0 := \{\mathrm{F}_0\}$ and $\mathcal{J} := \{\mathrm{F}_{ij} : i, j = 1, 2\}$. Note that the 2-category $\mathscr{C}_A$ is not weakly fiat in the sense of [13, 17] as the algebra A is not self-injective.

As $\mathscr{C}_A$ is $\mathcal{J}$-simple and A has trivial center, the only proper non-zero quotient of $\mathscr{C}_A$ contains just the identity 1-morphism (up to isomorphism) and its scalar endomorphisms (cf. [14]). Therefore this quotient is fiat with strongly regular $\mathcal{J}$-classes and hence it has a unique, up to equivalence, simple transitive 2-representation, namely $\mathbf{C}_{\mathcal{L}_0}$, where $\mathcal{L}_0 = \mathcal{J}_0$ (see [16], Theorem 18). This means that, in order to prove Theorem 1 for A, it is enough to consider *faithful* 2-representations of $\mathscr{C}_A$.

From the formula

$$\mathrm{Hom}_{A\text{-}A}(Ae_i \otimes e_j A, Ae_s \otimes e_t A) \cong e_i A e_s \otimes e_t A e_j, \tag{6}$$

for all $i, j, s, t \in \{1, 2\}$, we get the following table of $\mathrm{Hom}_{\mathscr{C}_A(\mathtt{i})}(X, Y)$ (up to isomorphism), where X and Y are indecomposable 1-morphisms:

$X \setminus Y$	F_{11}	F_{12}	F_{21}	F_{22}
F_{11}	$\Bbbk$	$\Bbbk$	0	0
F_{12}	0	$\Bbbk$	0	0
F_{21}	$\Bbbk$	$\Bbbk$	$\Bbbk$	$\Bbbk$
F_{22}	0	$\Bbbk$	0	$\Bbbk$

(7)

5. Integer matrices for $\Bbbk(\bullet \to \bullet)$. ***5.1. Integer matrices satisfying $M^2 = 3M$.*** In this section we classify all square matrices M with positive integer coefficients which satisfy $M^2 = 3M$.

Proposition 2. *Let M be a $k \times k$ matrix, for some k, with positive integer coefficients, satisfying $M^2 = 3M$. Then M is one of the following matrices*:

$$M_1 := (3), \qquad M_2 := \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \qquad M_3 := \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \qquad M_4 := \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix},$$

$$M_5 := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad M_6 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Proof. Clearly, we have $M_i^2 = 3M_i$, for each $i = 1, 2, 3, 4, 5, 6$. So, we need to show that no other square matrix with positive integer coefficients satisfies $M^2 = 3M$.

Let M be a $k \times k$ matrix, for some k, with positive integer coefficients satisfying $M^2 = 3M$. Then M is diagonalizable (as $x^2 - 3x$ has no multiple roots) and the only possible eigenvalues for M are 0 and 3. From the Perron–Frobenius theorem it follows that the Perron–Frobenius eigenvalue 3 must have multiplicity one. Therefore, M has rank one and trace three. As all entries in M are positive integers, we get $k \leq 3$.

If $k = 1$, then, clearly, $M = M_1$.

If $k = 3$, then all diagonal entries in M are 1. As all 2×2 minors in M should have determinant zero and positive integer entries, it follows that all entries in M are 1 and thus $M = M_6$.

If $k=2$, then the two diagonal entries in M are 1 and 2. As the determinant of M is zero, the two remaining entries are also 1 and 2. Therefore, $M=M_i$ for some $i\in\{2,3,4,5\}$.

Proposition 2 is proved.

5.2. The matrix* $[\mathrm{F}]$ *for a faithful simple transitive* 2*-representation. Let $\mathbf{M}$ be a finitary, simple, transitive and faithful 2-representation of $\mathscr{C}_A$. Let $M:=[\mathrm{F}]$ be the matrix of $\mathbf{M}(\mathrm{F})$ and, for $i,j=1,2$, let $M_{ij}:=[\mathrm{F}_{ij}]$ be the matrix of $\mathbf{M}(\mathrm{F}_{ij})$. Note that $M=M_{11}+M_{12}+M_{21}+M_{22}$.

The symmetric group S_k acts on $\mathrm{Mat}_{k\times k}(\mathbb{Z})$ by conjugation with permutation matrices. This action corresponds to permutation of basis elements, whenever the matrix on which we act represents an endomorphism of some free $\mathbb{Z}$-module. We will call this action the *permutation action*.

Proposition 3. *In order to respect the multiplication rule* (5), *up to the permutation action, we have the following three possibilities*:

(a) $M=M_2$ *and*

$$M_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad M_{12}=\begin{pmatrix}1&1\\0&0\end{pmatrix},\qquad M_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad M_{22}=\begin{pmatrix}0&0\\1&1\end{pmatrix};$$

(b) $M=M_3$ *and*

$$M_{11}=\begin{pmatrix}0&1\\0&1\end{pmatrix},\qquad M_{12}=\begin{pmatrix}1&0\\1&0\end{pmatrix},\qquad M_{21}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad M_{22}=\begin{pmatrix}1&0\\0&0\end{pmatrix};$$

(c) $M=M_6$ *and*

$$M_{11}=\begin{pmatrix}1&0&0\\1&0&0\\0&0&0\end{pmatrix},\qquad M_{12}=\begin{pmatrix}0&1&1\\0&1&1\\0&0&0\end{pmatrix},$$

$$M_{21}=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\qquad M_{22}=\begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}.$$

Proof. As $\mathbf{M}$ is simple, transitive and faithful, we get that M has positive integer entries. As $\mathrm{F}\circ\mathrm{F}=\mathrm{F}^{\oplus 3}$, we have $M=M_i$ for some $i\in\{1,2,3,4,5,6\}$, by Proposition 2. As M is the sum of four non-zero matrices (corresponding to all F_{ij}) each of which has non-negative integer entries, we have $M\neq M_1$. The case $M=M_4$ reduces to the case $M=M_3$ by swapping the basis elements. The case $M=M_5$ reduces to the case $M=M_2$ by swapping the basis elements. It is easy to check that the cases (a), (b) and (c) listed in the formulation satisfy (5).

Assume $M=M_2$. Note, from (5), that F_{11}, F_{12} and F_{22} are idempotent, while F_{21} is nilpotent. Therefore M_{11}, M_{12}, M_{22} must have non-zero diagonals, while the diagonal for M_{21} should be zero. From $M_{11}M_{22}=0$ it follows that M_{11} and M_{22} cannot have common diagonal entries. In any case, this means that M_{12} has the non-zero diagonal entry in the left upper corner. Let us first assume the following:

$$M_{11}=\begin{pmatrix}0&*\\ *&1\end{pmatrix},\qquad M_{12}=\begin{pmatrix}1&*\\ *&0\end{pmatrix},\qquad M_{21}=\begin{pmatrix}0&*\\ *&0\end{pmatrix},\qquad M_{22}=\begin{pmatrix}1&*\\ *&0\end{pmatrix}.$$

From $M_{11}M_{21} = 0$, we get

$$M_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}.$$

This, however, contradicts $M_{11}M_{12} = M_{12}$. Now assume

$$M_{11} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}.$$

From $M_{11}M_{21} = M_{11}M_{22} = 0$, we have

$$M_{11} = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & 1 \\ * & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 \\ * & 1 \end{pmatrix}.$$

From $M_{12}M_{22} = M_{12}$, we obtain

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Assume $M = M_3$. Note, from (5), that $\mathrm{F}_{11}, \mathrm{F}_{12}$ and F_{22} are idempotent, while F_{21} is nilpotent. Therefore M_{11}, M_{12}, M_{22} must have non-zero diagonals, while the diagonal for M_{21} should be zero. From $M_{11}M_{22} = 0$ it follows that M_{11} and M_{22} cannot have common diagonal entries. In any case this means that M_{12} has the non-zero diagonal entry in the left upper corner. Let us first assume the following:

$$M_{11} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}.$$

From $M_{11}M_{21} = 0$, we get

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix}.$$

This, however, contradicts $M_{21}M_{12} = M_{22}$. Now assume the following:

$$M_{11} = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}.$$

From $M_{11}M_{21} = M_{11}M_{22} = 0$, we have

$$M_{11} = \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & * \\ 1 & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}.$$

From $M_{11}M_{12} = M_{12}$, we obtain

$$M_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume $M = M_6$. Note, from (5), that F_{11}, F_{12} and F_{22} are idempotent, while F_{21} is nilpotent. Therefore M_{11}, M_{12}, M_{22} must have non-zero diagonals, while the diagonal for M_{21} should be zero. Therefore, up to permutation of basis vectors, we may assume that

$$M_{11} = \begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & * & * \\ * & 1 & * \\ * & * & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 1 \end{pmatrix}.$$

From $M_{11}M_{21} = M_{11}M_{22} = 0$ we thus get that the last column of M_{11} must be zero and the first row of both M_{21} and M_{22} must be zero. Since the M_{ij}'s add up to M, the rightmost element in the first row of M_{12} must be 1:

$$M_{11} = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & * & 1 \\ * & 1 & * \\ * & * & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & * \\ * & * & 1 \end{pmatrix}.$$

From $M_{11}M_{12} = M_{12}$ it follows that the second row of M_{11} cannot be zero, which yields

$$M_{11} = \begin{pmatrix} 1 & * & 0 \\ 1 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & * & 1 \\ 0 & 1 & * \\ * & * & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ * & * & 1 \end{pmatrix}.$$

Now $M_{11}M_{12} = M_{12}$ implies that the first and the second rows of M_{12} should coincide, moreover, the first element in the third row in M_{12} should be zero and also the third row in M_{11} and thus also in M_{12} must be zero:

$$M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 1 \end{pmatrix}.$$

Now, from $M_{21}M_{11} = M_{21}$, we have

$$M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 1 & 1 \end{pmatrix}.$$

Finally, from $M_{12}M_{21} = M_{11}$, we obtain

$$M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Proposition 3 is proved.

6. Proof of Theorem 1 for $\Bbbk(\bullet \to \bullet)$. Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_A$. Let B be a basic finite dimensional algebra such that $\mathbf{M}(\mathtt{i})$ is equivalent to B-proj.

As the left A-modules Ae_1 and $\mathrm{Hom}_{\Bbbk}(e_2A, \Bbbk)$ are isomorphic, from Proposition 1 it follows that the functor $\overline{\mathbf{M}}(\mathrm{F})$ is exact. From Lemmas 1 and 2 we thus obtain that $\overline{\mathbf{M}}(\mathrm{F})$ is a projective endofunctor of B-mod.

***Case* 1.** Assume that $M = M_3$ and the M_{ij}'s are thus given by Proposition 3 (b). Let $\overline{\mathbf{M}}$ be the abelianization of $\mathbf{M}$. As usual, we write P_1 and P_2 for indecomposable projectives in $\mathbf{M}(\mathtt{i})$ and L_1 and L_2 for their respective simple tops. Let ϵ_1 and ϵ_2 be the corresponding primitive idempotents in B. For $i, j = 1, 2$, denote by G_{ij} an endofunctor of $\overline{\mathbf{M}}(\mathtt{i})$ which corresponds to tensoring with $B\epsilon_i \otimes \epsilon_j B$.

From the form of M_{21}, we see that F_{21} acts via G_{12}. Similarly, F_{22} acts via G_{11}. From the matrices M_{21} and M_{22} it follows that

$$[P_1 : L_1] = 1, \qquad [P_1 : L_2] = 0, \qquad [P_2 : L_1] = 0, \qquad [P_2 : L_2] = 1.$$

This means that $B \cong \Bbbk \oplus \Bbbk$. Therefore, all G_{ij} are isomorphisms between the corresponding $\Bbbk$-mod components. From the matrices M_{12} and M_{21} it thus follows directly that there are no nonzero homomorphisms from F_{21} to F_{12}. This contradicts (7) and hence Case 1 cannot occur.

***Case* 2.** Assume that $M = M_2$ and the M_{ij}'s are thus given by Proposition 3 (a). Let $\overline{\mathbf{M}}$ be the abelianization of $\mathbf{M}$. As usual, we write P_1 and P_2 for indecomposable projectives in $\mathbf{M}(\mathtt{i})$ and L_1 and L_2 for their respective simple tops. Let ϵ_1 and ϵ_2 be the corresponding primitive idempotents in B. For $i, j = 1, 2$, denote by G_{ij} an endofunctor of $\overline{\mathbf{M}}(\mathtt{i})$ which corresponds to tensoring with $B\epsilon_i \otimes \epsilon_j B$.

From the form of M_{11}, we see that F_{11} acts via G_{11}. Similarly, F_{21} acts via G_{21}.

From the form of M_{12}, we see that F_{12} acts either via G_{12} or via G_{11} or via $\mathrm{G}_{12} \oplus \mathrm{G}_{11}$. However, we already know that the matrix of G_{11} is M_{11}. This leaves us with possibilities G_{12} or $\mathrm{G}_{12} \oplus \mathrm{G}_{11}$ for F_{12}.

Assume that F_{12} acts via $\mathrm{G}_{12} \oplus \mathrm{G}_{11}$. We already know the matrix of G_{11}, so the matrix of G_{12} is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This and the matrix M_{11} imply that

$$\mathrm{G}_{11}\, P_1 \cong P_1, \qquad \mathrm{G}_{11}\, P_2 = 0, \qquad \mathrm{G}_{12}\, P_1 = 0, \qquad \mathrm{G}_{12}\, P_2 \cong P_2.$$

Therefore,

$$[P_1 : L_1] = 1, \qquad [P_1 : L_2] = 0, \qquad [P_2 : L_1] = 0, \qquad [P_2 : L_2] = 1$$

and we have $B \cong \Bbbk \oplus \Bbbk$. This leads to the same contradiction as in Case 1 above. Therefore, F_{12} acts via G_{12}. Similarly one shows that F_{22} acts via G_{22}.

From the matrices for all G_{ij}'s it follows that the Cartan matrices of A and B coincide which implies that A and B are isomorphic (that is special for our case, but the algebra A is very small, so this claim is clear). Furthermore, all F_{ij}'s act via the corresponding G_{ij}. It now follows by the usual arguments (see [16], Proposition 9), that $\overline{\mathbf{M}}$ is equivalent to a cell 2-representation of $\mathscr{C}_A$.

***Case* 3.** Assume that $M = M_6$ and the M_{ij}'s are thus given by Proposition 3 (c). Let $\overline{\mathbf{M}}$ be the abelianization of $\mathbf{M}$. As usual, we write P_1, P_2 and P_3 for indecomposable projectives in $\mathbf{M}(\mathtt{i})$ and L_1, L_2 and L_3 for their respective simple tops. Let ϵ_1, ϵ_2 and ϵ_3 be the corresponding primitive idempotents in B. For $i, j = 1, 2, 3$, denote by G_{ij} an endofunctor of $\overline{\mathbf{M}}(\mathtt{i})$ which corresponds to tensoring with $B\epsilon_i \otimes \epsilon_j B$.

From the form of M_{21}, we see that F_{21} acts via G_{31}. From the form of M_{11}, we see that F_{11} acts via $\mathrm{G}_{11} \oplus \mathrm{G}_{21}$. This implies

$$[P_1 : L_1] = 1, \qquad [P_2 : L_1] = [P_3 : L_1] = 0.$$

From the form of M_{22}, we see that F_{22} acts either via G_{32} or via G_{33} or via $\mathrm{G}_{32} \oplus \mathrm{G}_{33}$. In the latter case, we have that the matrices of G_{32} and G_{33} are, respectively:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that

$$[P_2 : L_2] = 1, \qquad [P_1 : L_2] = [P_3 : L_2] = 0$$

and

$$[P_3 : L_3] = 1, \qquad [P_1 : L_3] = [P_2 : L_3] = 0.$$

This implies that $B \cong \Bbbk \oplus \Bbbk \oplus \Bbbk$ and leads to a similar contradiction as in Case 1.

***Subcase* 3.1.** Assume that F_{22} acts via G_{32}. This implies

$$[P_2 : L_2] = [P_3 : L_2] = 1, \qquad [P_1 : L_2] = 0.$$

From $[P_1 : L_2] = 0$ we have $\epsilon_2 B \epsilon_1 = 0$. This means that

$$\mathrm{Hom}_{B\text{-}B}(B\epsilon_3 \otimes \epsilon_1 B, B\epsilon_3 \otimes \epsilon_2 B) = 0,$$

that is, $\mathrm{Hom}(\mathrm{G}_{31}, \mathrm{G}_{32}) = 0$. This contradicts $\mathrm{Hom}_{\mathscr{C}}(\mathrm{F}_{21}, \mathrm{F}_{22}) \neq 0$, see (7).

***Subcase* 3.2.** Assume that F_{22} acts via G_{33}. This implies

$$[P_3 : L_3] = [P_2 : L_3] = 1, \qquad [P_1 : L_3] = 0.$$

From $[P_1 : L_3] = 0$ we obtain $\epsilon_3 B \epsilon_1 = 0$. This means that

$$\mathrm{Hom}_{B\text{-}B}(B\epsilon_3 \otimes \epsilon_1 B, B\epsilon_3 \otimes \epsilon_3 B) = 0,$$

that is, $\mathrm{Hom}(\mathrm{G}_{31}, \mathrm{G}_{33}) = 0$. This contradicts $\mathrm{Hom}_{\mathscr{C}}(\mathrm{F}_{21}, \mathrm{F}_{22}) \neq 0$, see (7). The proof is now complete.

7. The algebra $\Bbbk(\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet)/(\beta\alpha)$. Let $\Bbbk$ be an algebraically closed field. Denote by A the path algebra, over $\Bbbk$, of the quiver

$$\Bbbk\Big(1 \xrightarrow{\alpha} 2 \xrightarrow{\alpha} 3\Big) \qquad \text{modulo the relations} \qquad \beta\alpha = 0.$$

The algebra A has basis e_1, e_2, e_3, α and β and the multiplication table $(x, y) \mapsto x \cdot y$ is given by

$x\backslash y$	e_1	e_2	e_3	α	β
e_1	e_1	0	0	0	0
e_2	0	e_2	0	α	0
e_3	0	0	e_3	0	β
α	α	0	0	0	0
β	0	β	0	0	0

Note that $e_1 A e_2 = 0$, $e_1 A e_3 = 0$, $e_2 A e_3 = 0$ and $e_3 A e_1 = 0$. Note also that the left A-modules Ae_1 and $\mathrm{Hom}_{\Bbbk}(e_2 A, \Bbbk)$ are isomorphic and the left A-modules Ae_2 and $\mathrm{Hom}_{\Bbbk}(e_3 A, \Bbbk)$ are isomorphic.

Let $\mathcal{C}$ be a small category equivalent to A-mod. Consider the corresponding finitary 2-category $\mathscr{C}_A$. Up to isomorphism, indecomposable 1-morphisms in $\mathscr{C}_A$ are F_0 and F_{ij}, where $i, j = 1, 2, 3$. Note that formula (2) for A reads $\mathrm{F} \circ \mathrm{F} \cong \mathrm{F}^{\oplus 5}$. Using (1), the table of compositions for the functors F_{ij} (up to isomorphism) is as follows:

$\circ$	F_{11}	F_{12}	F_{13}	F_{21}	F_{22}	F_{23}	F_{31}	F_{32}	F_{33}
F_{11}	F_{11}	F_{12}	F_{13}	0	0	0	0	0	0
F_{12}	F_{11}	F_{12}	F_{13}	F_{11}	F_{12}	F_{13}	0	0	0
F_{13}	0	0	0	F_{11}	F_{12}	F_{13}	F_{11}	F_{12}	F_{13}
F_{21}	F_{21}	F_{22}	F_{23}	0	0	0	0	0	0
F_{22}	F_{21}	F_{22}	F_{23}	F_{21}	F_{22}	F_{23}	0	0	0
F_{23}	0	0	0	F_{21}	F_{22}	F_{23}	F_{21}	F_{22}	F_{23}
F_{31}	F_{31}	F_{32}	F_{33}	0	0	0	0	0	0
F_{32}	F_{31}	F_{32}	F_{33}	F_{31}	F_{32}	F_{33}	0	0	0
F_{33}	0	0	0	F_{31}	F_{32}	F_{33}	F_{31}	F_{32}	F_{33}

(8)

Set $\mathcal{J}_0 := \{\mathrm{F}_0\}$ and $\mathcal{J} := \{\mathrm{F}_{ij} : i, j = 1, 2, 3\}$. Note that the 2-category $\mathscr{C}_A$ is not weakly fiat in the sense of [13, 17] as the algebra A is not self-injective.

As $\mathscr{C}_A$ is $\mathcal{J}$-simple and A has trivial center, the only proper non-zero quotient of $\mathscr{C}_A$ contains just the identity 1-morphism (up to isomorphism) and its scalar endomorphisms (cf. [14]). Therefore this quotient is fiat with strongly regular $\mathcal{J}$-classes and hence it has a unique, up to equivalence, simple transitive 2-representation, namely $\mathbf{C}_{\mathcal{L}_0}$, where $\mathcal{L}_0 = \mathcal{J}_0$ (see [16], Theorem 18). This means that, in order to prove Theorem 1 for A, it is enough to consider *faithful* 2-representations of $\mathscr{C}_A$.

From (6), we get the following table of $\mathrm{Hom}_{\mathscr{C}_A(\mathtt{i})}(X, Y)$ (up to isomorphism), where X and Y are indecomposable 1-morphisms:

$X \setminus Y$	F_{11}	F_{12}	F_{13}	F_{21}	F_{22}	F_{23}	F_{31}	F_{32}	F_{33}
F_{11}	$\Bbbk$	$\Bbbk$	0	0	0	0	0	0	0
F_{12}	0	$\Bbbk$	$\Bbbk$	0	0	0	0	0	0
F_{13}	0	0	$\Bbbk$	0	0	0	0	0	0
F_{21}	$\Bbbk$	$\Bbbk$	0	$\Bbbk$	$\Bbbk$	0	0	0	0
F_{22}	0	$\Bbbk$	$\Bbbk$	0	$\Bbbk$	$\Bbbk$	0	0	0
F_{23}	0	0	$\Bbbk$	0	0	$\Bbbk$	0	0	0
F_{31}	0	0	0	$\Bbbk$	$\Bbbk$	0	$\Bbbk$	$\Bbbk$	0
F_{32}	0	0	0	0	$\Bbbk$	$\Bbbk$	0	$\Bbbk$	$\Bbbk$
F_{33}	0	0	0	0	0	$\Bbbk$	0	0	$\Bbbk$

(9)

8. Integer matrices for $\Bbbk(\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet)/(\beta\alpha)$. ***8.1. Integer matrices satisfying $M^2 = 5M$.*** In this section we classify all square matrices M with positive integer coefficients which satisfy $M^2 = 5M$.

Proposition 4. *Let M be a $k \times k$ matrix, for some k, with positive integer coefficients, satisfying $M^2 = 5M$. Then, up to permutation action, M is one of the following matrices*:

$$N_1 := (5), \qquad N_2 := \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}, \qquad N_3 := \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}, \qquad N_4 := \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix},$$

$$N_5 := \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}, \qquad N_6 := \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}, \qquad N_7 := \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}, \qquad N_8 := \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix},$$

$$N_9 := \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}, \qquad N_{10} := \begin{pmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad N_{11} := \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

$$N_{12} := \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \qquad N_{13} := \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \qquad N_{14} := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$N_{15} := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}, \qquad N_{16} := \begin{pmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We note an important difference with Proposition 2: to make our list shorter, Proposition 4 gives classification only up to permutation action.

Proof. Clearly, we have $N_i^2 = 5N_i$, for each $i = 1, 2, \ldots, 16$. So, we need to show that any other square matrix with positive integer coefficients satisfying $M^2 = 5M$ can be reduced to one of the above using permutation action.

Let M be a $k \times k$ matrix, for some k, with positive integer coefficients satisfying $M^2 = 5M$. Then M is diagonalizable (as $x^2 - 5x$ has no multiple roots) and the only possible eigenvalues for M are 0 and 5. From the Perron – Frobenius theorem it follows that the Perron – Frobenius eigenvalue 5 must have multiplicity one. Therefore, M has rank one and trace five. As all entries in M are positive integers, we get $k \leq 5$. Using the permutation action, we may assume that the entries on the main diagonal of M weakly decrease from the top left corner to the bottom right corner.

If $k = 1$, then, clearly, $M = N_1$.

If $k = 2$, then the diagonal of M is either $(4, 1)$ or $(3, 2)$. In the first case, as the determinant of M is zero, the two remaining entries are either 2 and 2 or 4 and 1. This gives $M = N_2$, $M = N_3$ or $M = N_4$. In the second case, as the determinant of M is zero, the two remaining entries are either 2 and 3 or 1 and 6. This gives $M = N_5$, $M = N_6$, $M = N_7$ or $M = N_8$.

If $k = 3$, then the diagonal of M is either $(3, 1, 1)$ or $(2, 2, 1)$. In the first case, as M has rank one, any 2×2 minor in M has determinant zero. This means that all entries which are neither in the first row nor in the first column are equal to 1. If the first row contains more than one entry different from 1, then all entires in this row are 3 and we get $M = N_{10}$. If the first column contains more than one entry different from 1, then all entires in this column are 3 and we get $M = N_9$.

In the second case write

$$M = \begin{pmatrix} 2 & m_{12} & m_{13} \\ m_{21} & 2 & m_{23} \\ m_{31} & m_{32} & 1 \end{pmatrix}.$$

Then $m_{32}m_{23} = 2$, $m_{31}m_{13} = 2$ and $m_{21}m_{12} = 4$. Hence, both (m_{32}, m_{23}) and (m_{31}, m_{13}) are in $\{(1,2),(2,1)\}$. We can choose them independently and the fact that M has rank one then uniquely determines the pair (m_{21}, m_{12}). This gives us $M = N_{11}$, $M = N_{12}$ and $M = N_{13}$ and also the possibility

$$M = N'_{12} := \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

which reduces to $M = N_{12}$ by permutation action.

If $k = 4$, then the diagonal of M is $(2,1,1,1)$. As M has rank one, any 2×2 minor in M has determinant zero. This means that all entries which are neither in the first row nor in the first column are equal to 1. If the first row contains more than one entry different from 1, then all entires in this row are 2 and we get $M = N_{16}$. If the first column contains more than one entry different from 1, then all entires in this column are 2 and we obtain $M = N_{15}$.

If $k = 5$, then all diagonal entries in M are 1. As all 2×2 minors in M should have determinant zero and positive integer entries, it follows that all entries in M are 1 and thus $M = N_{14}$.

Proposition 4 is proved.

8.2. Filtering "easy cases" out. Let $\mathbf{M}$ be a finitary, simple, transitive and faithful 2-representation of $\mathscr{C}_A$. Let $M := [\mathrm{F}]$ be the matrix of $\mathbf{M}(\mathrm{F})$ and, for $i, j = 1, 2, 3$, let $M_{ij} := [\mathrm{F}_{ij}]$ be the matrix of $\mathbf{M}(\mathrm{F}_{ij})$. We have $M_{ij} \neq 0$, for all $i, j = 1, 2, 3$. By Proposition 4, up to permutation action, we have $M = N_i$, for some $i \in \{1, 2, \ldots, 16\}$ as in Proposition 4. Note that trace of M is five.

As usual, we call "position (i, j)" the intersection of the ith row and the jth column of a matrix.

From now on, we assume that $\overline{\mathbf{M}}(\mathtt{i})$ is equivalent to $B\text{-mod}$, for some basic algebra B. Let L_1, $L_2, \ldots,$ L_k be a complete and irredundant list of representatives of isomorphism classes of simple objects in $\overline{\mathbf{M}}(\mathtt{i})$. For $i \in \{1, 2, \ldots, k\}$, denote by P_i the indecomposable projective cover of L_i and by I_i the indecomposable injective envelope of L_i. The matrices M_{ij} are given with respect to this fixed ordering of isomorphism classes of simple objects.

Lemma 6. (i) *All diagonal elements in M_{13}, M_{21}, M_{31} and M_{32} are zero.*

(ii) *Each of the matrices M_{11}, M_{12}, M_{22}, M_{23} and M_{33}, has exactly one entry equal to 1 on the diagonal and all other diagonal entries are zero.*

Proof. From (8), we see that F_{ij} is idempotent if and only if

$$(i, j) \in \{(1,1),(1,2),(2,2),(2,3),(3,3)\}$$

and $\mathrm{F}_{ij}^2 = 0$ otherwise. As the trace of a non-zero idempotent with non-negative coefficients is non-zero, each idempotent M_{ij} has trace at least one. As trace of M is five, it follows that all idempotents M_{ij} have trace one. This proves claim (ii). Claim (i) follows from claim (ii) as M is the sum of the M_{ij}'s.

***Corollary* 3.** *The matrix M cannot be equal to N_i, where $i \in \{1, 2, \ldots, 10\}$.*

Proof. Each of the matrices N_i, where $i \in \{1, 2, \ldots, 10\}$, contains a diagonal element which is greater than or equal to 3. If $M = N_i$ would be possible, at least three idempotents M_{ij} would have this diagonal element non-zero. But then any product of any two such matrices would be non-zero. However, from (8) we have that, for any three different idempotents F_{ij}, one of the products of two of these elements is zero. The obtained contradiction completes the proof.

8.3. Auxiliary adjunction. We will need the following easy observations:

Lemma 7. *Let D be a finite dimensional algebra and (G, H) an adjoint pair of right exact endofunctors of D*-mod*. Let L and L' be simple D-modules and P and P' their corresponding indecomposable projective covers. Assume that L' appears in the top of* $\mathrm{G}\,P$. *Then* $\mathrm{H}\,P' \neq 0$.

Proof. By adjunction, we have

$$0 \neq \mathrm{Hom}_B(\mathrm{G}\,P, L') \cong \mathrm{Hom}_B(P, \mathrm{H}\,L'),$$

which implies $\mathrm{H}\,L' \neq 0$. As H is right exact, this forces $\mathrm{H}P' \neq 0$.

Lemma 8. *We have the following pairs of adjoint 1-morphisms in $\mathscr{C}_A$:*

$$(\mathrm{F}_{33}, \mathrm{F}_{23}), \qquad (\mathrm{F}_{23}, \mathrm{F}_{22}), \qquad (\mathrm{F}_{22}, \mathrm{F}_{12}), \qquad (\mathrm{F}_{12}, \mathrm{F}_{11}).$$

Proof. As both, the left A-modules Ae_1 and $\mathrm{Hom}_{\Bbbk}(e_2A, \Bbbk)$ are isomorphic and the left A-modules Ae_2 and $\mathrm{Hom}_{\Bbbk}(e_3A, \Bbbk)$ are isomorphic, the claim follows from Lemma 4.

8.4. Idempotent integral matrices of rank one. Recall from [4] (Theorem 2) that, up to permutation action, idempotent matrices of rank one with non-negative integer entries have the form

$$\begin{pmatrix} \mathbf{0} & v & vw^t \\ \mathbf{0} & 1 & w^t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ denotes the zero matrix (of an appropriate size), and v and w are arbitrary vectors with non-negative integer entries. In particular, if the diagonal entry 1 is in the ith row, then the whole matrix can be written as the product of its ith column with its ith row.

8.5. Filtering matrices N_{14}, N_{15} and N_{16} out.

Proposition 5. *The matrix M cannot be equal to N_{14}, N_{15} or N_{16}.*

Proof. Assume that the diagonals of the matrices M_{11} and M_{12} are different. This means that M_{11} has 1 in row i, that M_{12} has 1 in row j, and that $i \neq j$.

Then $\mathrm{F}_{12}\,P_j$ has P_j as a direct summand. Therefore, by combining Lemmas 7 and 8, we have that M_{11} must have a non-zero element in column j. From Subsection 8.4 it follows that M_{11} has a non-zero entry in position (i, j).

As M_{11} has a non-zero entry in position (i, j) and the matrix M_{12} has 1 in position (j, j), it follows that $M_{11}M_{12}$ has a non-zero entry in position (i, j). From (8), we have $M_{11}M_{12} = M_{12}$, which means that M_{12} has a non-zero entry in position (i, j). This already means that the case $M = N_{14}$ is not possible.

Assume $M = N_{15}$. Then we must have $j = 1$. Exactly the same argument as above applied to M_{22} and M_{23} shows that M_{23} has 1 in position $(1, 1)$. This contradicts $M_{23}M_{12} = 0$ as follows from (8). Therefore, $M = N_{15}$ is not possible.

Assume $M = N_{16}$. Then we must have $i = 1$. Exactly the same argument as above applied to M_{22} and M_{23} shows that M_{22} has 1 in position $(1, 1)$. This contradicts $M_{11}M_{22} = 0$ as follows from (8). Therefore, $M = N_{16}$ is not possible. This completes the proof.

The remaining cases for M will be studied on a case-by-case basis.

9. Filtering matrices N_{11} and N_{12} out. ***9.1. Statement.*** The main aim of this section is to prove the following proposition.

Proposition 6. *The matrix M cannot be equal to N_{11} or N_{12}.*

We start with the following observation.

Lemma 9. *The only unordered pairs of idempotent 1-morphisms of the form F_{ij} such that the product of any two elements in the pair is non-zero are*

$$\{\mathrm{F}_{11}, \mathrm{F}_{12}\}, \qquad \{\mathrm{F}_{12}, \mathrm{F}_{22}\}, \qquad \{\mathrm{F}_{22}, \mathrm{F}_{23}\}, \qquad \{\mathrm{F}_{23}, \mathrm{F}_{33}\}.$$

Proof. This follows directly from (8).

9.2. Proof for $M = N_{11}$. We will arrange matrices M_{ij}, where $i, j = 1, 2, 3$, as follows:

$$\begin{matrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33}. \end{matrix} \tag{10}$$

Assume $M = N_{11}$. The diagonal elements in N_{11} are $(2, 2, 1)$. Therefore, two pairs of idempotent matrices of the form M_{ij} would have common diagonal elements. Any product of matrices in any such pair would be non-zero. Therefore, using Lemma 9, we have three cases to consider.

***Case* 1.** Suppose first that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{11}, \mathrm{F}_{12}\}$ and $\{\mathrm{F}_{22}, \mathrm{F}_{23}\}$. Up to permutation action, we may assume that

$$\begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & * & * \\ * & 1 & * \\ * & * & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & * & * \\ * & 1 & * \\ * & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 1 \end{pmatrix}.$$

Using all possible zero products which appear in (8), we obtain that the M_{ij}'s look as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\ *&0&0\\ *&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ *&1&0\\ *&*&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ 0&1&*\\ 0&*&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\ 0&0&0\\ *&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ 0&0&0\\ *&*&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ 0&0&0\\ 0&*&1\end{pmatrix}.$$

As all matrices must be non-zero and add up to M, we obtain

$$\begin{pmatrix}1&0&0\\ 0&0&0\\ 0&0&0\end{pmatrix},\quad \begin{pmatrix}1&*&0\\ 0&0&0\\ 0&0&0\end{pmatrix},\quad \begin{pmatrix}0&*&2\\ 0&0&0\\ 0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\ *&0&0\\ 0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ *&1&0\\ 0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ 0&1&2\\ 0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\ 0&0&0\\ 1&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ 0&0&0\\ 0&1&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\ 0&0&0\\ 0&0&1\end{pmatrix}.$$

This contradicts $M_{13}M_{31} = M_{11}$ which is a consequence of (8). Therefore, this case is not possible.

***Case* 2.** Suppose now that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{12}, \mathrm{F}_{22}\}$ and $\{\mathrm{F}_{23}, \mathrm{F}_{33}\}$. Up to permutation action, we may assume that the M_{ij}'s look as follows:

$$\begin{pmatrix}0&*&*\\ *&0&*\\ *&*&1\end{pmatrix},\quad \begin{pmatrix}1&*&*\\ *&0&*\\ *&*&0\end{pmatrix},\quad \begin{pmatrix}0&*&*\\ *&0&*\\ *&*&0\end{pmatrix},$$

$$\begin{pmatrix}0&*&*\\ *&0&*\\ *&*&0\end{pmatrix},\quad \begin{pmatrix}1&*&*\\ *&0&*\\ *&*&0\end{pmatrix},\quad \begin{pmatrix}0&*&*\\ *&1&*\\ *&*&0\end{pmatrix},$$

$$\begin{pmatrix}0&*&*\\ *&0&*\\ *&*&0\end{pmatrix},\quad \begin{pmatrix}0&*&*\\ *&0&*\\ *&*&0\end{pmatrix},\quad \begin{pmatrix}0&*&*\\ *&1&*\\ *&*&0\end{pmatrix}.$$

Using all possible zero products which appear in (8), we obtain that the M_{ij}'s look as follows:

$$\begin{pmatrix}0&0&*\\ 0&0&0\\ 0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&*\\ 0&0&0\\ *&0&0\end{pmatrix},\quad \begin{pmatrix}0&*&0\\ 0&0&0\\ 0&*&0\end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & * \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here we have that P_2 is a direct summand of $\mathrm{F}_{23}\, P_2$. From Lemmas 7 and 8, it follows that $\mathrm{F}_{22}\, P_2 \neq 0$, which is a contradiction. Therefore, this case cannot occur either.

***Case* 3.** Suppose first that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{11}, \mathrm{F}_{12}\}$ and $\{\mathrm{F}_{23}, \mathrm{F}_{33}\}$. Up to permutation action, we may assume that the M_{ij}'s look as follows:

$$\begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ * & 1 & * \\ * & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ * & 1 & * \\ * & * & 0 \end{pmatrix}.$$

Using all possible zero products which appear in (8), we obtain that the M_{ij}'s look as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & * \\ * & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Here we have that P_2 is a direct summand of $\mathrm{F}_{23}\, P_2$. From Lemmas 7 and 8, it follows that $\mathrm{F}_{22}\, P_2 \neq 0$, which is a contradiction. Therefore, this case cannot occur either.

This completes the proof of Lemma 9 for $M = N_{11}$.

9.3. Proof for $M = N_{12}$. Let

$$N'_{12} := \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

The matrix N'_{12} reduces to $M = N_{12}$ by permutation action, however, it is convenient to use the freedom of permutation action in another way, see below. Because of Lemma 9, we have three cases to consider.

***Case* 1.** Suppose first that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{11}, \mathrm{F}_{12}\}$ and $\{\mathrm{F}_{22}, \mathrm{F}_{23}\}$. Then, using permutation action and all possible zero products which appear in (8), we obtain that the M_{ij}'s look as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 1 \end{pmatrix}.$$

If $M = N_{12}$, then the $(1, 3)$-entry of M_{13} equals 2 while the $(3, 1)$-entry of M_{31} equals 1. As the $(1, 1)$-entry of M_{11} is 1, we get a contradiction to $M_{13}M_{31} = M_{11}$, which follows from (8). This implies that $M = N'_{12}$ and, using also $M_{12}M_{21} = M_{11}$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 1 \end{pmatrix}.$$

As the M_{ij}'s must add up to M, we get

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\1&0&0\\*&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\3&1&0\\*&*&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&1&2\\0&*&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\*&*&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\0&*&1\end{pmatrix}.$$

This contradicts $M_{23}M_{31} = M_{21}$, which follows from (8). Therefore, this case cannot occur.

***Case* 2.** Suppose now that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{12}, \mathrm{F}_{22}\}$ and $\{\mathrm{F}_{23}, \mathrm{F}_{33}\}$. This gives the same contradiction as in Case 2 in Subsection 9.2. Therefore, this case cannot occur either.

***Case* 3.** Suppose first that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{11}, \mathrm{F}_{12}\}$ and $\{\mathrm{F}_{23}, \mathrm{F}_{33}\}$. This gives the same contradiction as in Case 3 in Subsection 9.2. Therefore, this case cannot occur either. This completes the proof of Lemma 9.

10. Proof of Theorem 1 for $\Bbbk(\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet)/(\beta\alpha)$. ***10.1. Finding the matrices.*** Combining Proposition 4 with Corollary 3, Propositions 5 and 6, we have $M = N_{13}$. We will arrange our matrices similarly to (10).

We will need the following easy and general observation:

Lemma 10. *Let M be any of the N_m's and $i, j \in \{1, 2, 3\}$. If, for some s, the column s in the matrix M_{ij} is non-zero, then the column s is non-zero in M_{tj}, for any $t \in \{1, 2, 3\}$.*

Proof. The fact that the column s in M_{ij} is non-zero is equivalent to saying that $\mathrm{F}_{ij}\, P_s \neq 0$ (and similarly for F_{tj}). We have $\mathrm{F}_{it} \circ \mathrm{F}_{tj} \cong \mathrm{F}_{ij}$ from (8). Therefore, $\mathrm{F}_{tj}\, P_s = 0$ implies $\mathrm{F}_{ij}\, P_s = 0$ and the claim follows.

Proposition 7. *The only possibility for the M_{ij}'s is*

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\1&1&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&1&1\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}.$$

Proof. Due to Lemma 9, we have to consider three cases.

***Case* 1.** Suppose that the pairs of idempotent matrices which sharediagonal elements are $\{\mathrm{F}_{12}, \mathrm{F}_{22}\}$ and $\{\mathrm{F}_{23}, \mathrm{F}_{33}\}$. This gives the same contradiction as in Case 2 in Subsection 9.2. Therefore, this case cannot occur.

***Case* 2.** Suppose that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{11}, \mathrm{F}_{12}\}$ and $\{\mathrm{F}_{23}, \mathrm{F}_{33}\}$. This gives the same contradiction as in Case 3 in Subsection 9.2. Therefore, this case cannot occur either.

***Case* 3.** Suppose that the pairs of idempotent matrices which share diagonal elements are $\{\mathrm{F}_{11}, \mathrm{F}_{12}\}$ and $\{\mathrm{F}_{22}, \mathrm{F}_{23}\}$. Then, using permutation action and all possible zero products which appear in (8), we obtain that the M_{ij}'s look as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 1 \end{pmatrix}.$$

Using that all matrices must be non-zero and add up to M and also $M_{13}M_{31} = M_{11}$, $M_{21}M_{12} = M_{22}$ and $M_{21}M_{13} = M_{23}$, given by (8), we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 1 \end{pmatrix}.$$

Comparing the first and the second columns in M_{32} with those of M_{22} and also the third column in M_{33} with that of M_{23} and using Lemma 10 we get exactly the arrangement in the formulation of our proposition.

Proposition 7 is proved.

10.2. Connecting to the cell 2-representation. Now we know that the M_{ij}'s have the form as specifies in Proposition 7. For $i, j = 1, 2, 3$, we denote by G_{ij} the corresponding indecomposable projective endofunctor of $\overline{\mathbf{M}}(\mathtt{i})$.

From the form of M_{i1}, where $i = 1, 2, 3$, we see that F_{i1} acts via G_{i1} (up to isomorphism). Moreover, we also have $[P_i : L_1] = \delta_{i,1}$.

From the form of M_{12}, we see that F_{12} acts via either G_{12} or G_{11} or $\mathrm{G}_{12} \oplus \mathrm{G}_{11}$. However, we already know that G_{11} has matrix M_{11}. This leaves us with the only possibilities of G_{12} or $\mathrm{G}_{12} \oplus \mathrm{G}_{11}$.

Assume that F_{12} acts via $\mathrm{G}_{12} \oplus \mathrm{G}_{11}$. Then the matrix of G_{12} is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This implies that $[P_i : L_2] = \delta_{i,2}$, for $i = 1, 2, 3$.

According to (9), there is a non-zero 2-morphism $\alpha : \mathrm{F}_{21} \to \mathrm{F}_{12}$. As $\mathbf{M}$ is faithful, $\overline{\mathbf{M}}(\alpha)$ is non-zero. Evaluation of the latter at

P_3 is zero as P_3 is annihilated by both F_{21} and F_{12};

P_2 is zero as P_2 is annihilated by F_{21};

P_1 is zero as $\mathrm{F}_{12}\, P_1 \cong P_1$, $\mathrm{F}_{21}\, P_1 \cong P_2$ and we also have

$$\mathrm{Hom}_{\overline{\mathbf{M}}(\mathtt{i})}(P_2, P_1) = [P_1 : L_2] = 0,$$

by the previous paragraph.

Therefore, $\overline{\mathbf{M}}(\alpha)$ must be zero, a contradiction. Consequently, F_{12} acts via G_{12}, which also implies $[P_1 : L_2] = 1$. From this, it follows that F_{i2} acts via G_{i2}, for $i = 1, 2, 3$.

A similar argument shows that F_{i3} acts via G_{i3}, for $i = 1, 2, 3$, and that

$$[P_2 : L_3] = [P_3 : L_3] = 1, \qquad [P_3 : L_1] = [P_3 : L_2] = 0.$$

This means that $B \cong A$ and that each F_{ij} acts via the corresponding G_{ij}. It now follows by the usual arguments (see [16], Proposition 9) that $\overline{\mathbf{M}}$ is equivalent to a cell 2-representation of $\mathscr{C}_A$. The claim of Theorem 1 for the algebra $\Bbbk(\bullet \longrightarrow \bullet \longrightarrow \bullet)$ follows.

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