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FORCED FREQUENCY LOCKING FOR DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL FORCINGS

ВИМУШЕНЕ ЗАМИКАННЯ ЧАСТОТИ ДЛЯ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ДИСТРИБУТИВНИМИ ФОРСУВАННЯМИ

This paper deals with forced frequency locking, i.e., the behavior of periodic solutions to autonomous differential equations under the influence of small periodic forcings. We show that, although the forcings are allowed to be discontinuous (e.g., step-function-like) or even distributional (e.g., Dirac-function-like), the forced frequency locking happens as in the case of smooth forcings, and we derive formulas for the locking cones and for the asymptotic phases as in the case of smooth forcings.

Розглянуто вимушене замикання частоти, тобто поведінку періодичних розв'язків автономних диференціальних рівнянь під впливом малих періодичних форсувань. Показано, що, незважаючи на той факт, що ці форсування можуть бути розривними (типу східчастих функцій) або навіть дистрибутивними (типу дельта-функцій), вимушені замикання частоти відбуваються, як і у випадку гладких форсувань, і можна отримати формули для конусів замикання та асимптотичних фаз, як і у випадку гладких форсувань.

1. Introduction and main results. This paper is dedicated to A. M. Samoilenko on the occasion of his 80th birthday. The author gratefully acknowledges many years of friendship and scientific cooperation with Anatolii Mykhailovych, in particular of scientific cooperation concerning forced frequency locking [9, 11 – 13].

The following phenomenon is usually called forced frequency locking (or injection locking or master-slave synchronization or master-slave entrainment): If x_0 is a T_0-periodic solution to an autonomous evolution equation and if this equation is forced by a periodic forcing with intensity $\varepsilon \approx 0$ and period $T \approx T_0$, then generically the following is true: If the pair (ε, T) belongs to a certain open conus-like subset of the plane, then there exist T-periodic solutions $x(t) \approx x_0(tT_0/T + \varphi)$ to the forced equation. Moreover, if x_0 is an exponentially orbitally stable periodic solution to the unforced equation, then at least one of the locked periodic solutions to the forced equation is exponentially stable.

Forced frequency locking appears in many areas of natural sciences, and it is used in diverse applications in technology. Moreover, since a long time it is mathematically rigorously described for ODEs and parabolic PDEs with smooth forcings (see, e.g., [5 – 7, 14, 15, 17]). It turns out that this phenomenon appears also in dissipative hyperbolic PDEs, functional-differential equations (cf. [8]) as well as in evolution equations with discontinuous or even Dirac-function-like forcings, but for those cases there is no rigorous mathematical description available up to now.

As an example, in the present paper we describe forced frequency locking for smooth differential equations with possibly distributional forcings of the type

$$\dot{x} = f(x) + \varepsilon g(T). \tag{1.1}$$

Here $x : \mathbb{R} \to \mathbb{R}^n$ is a T-periodic function to be determined, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2-smooth vector field, $\varepsilon > 0$ and $T > 0$ are the intensity and the period of the forcing, respectively, and $g(T)$ is a linear functional on the vector space of continuous T-periodic functions $\mathbb{R} \to \mathbb{R}^n$, which works as

$$\langle g(T), y\rangle := \int\limits_0^T h(t/T)\cdot y(t)\,dt + d\cdot y(0) \quad \text{for all continuous } T\text{-periodic} \quad y\colon \mathbb{R}\to\mathbb{R}^n$$

with $h \in L^1((0,1);\mathbb{R}^n)$ and $d \in \mathbb{R}^n$, where "$\cdot$" is the Euclidean scalar product in $\mathbb{R}^n$. Hence, we suppose that the Dirac-function-like part of the forcing is supported in the times $t = 0,\ \pm1,\ \pm2, \ldots,$ and the function h describes the shape of the regular part of the forcing (over one period).

Of course, (1.1) has to be understood in the weak sense: A T-periodic function $x \in L^\infty(\mathbb{R};\mathbb{R}^n)$ is called a solution to (1.1) if

$$\int\limits_0^T \big(x(t)\cdot\dot{y}(t) + \big(f(x(t)) + \varepsilon h(t/T)\big)\cdot y(t)\big)\,dt + \varepsilon\, d\cdot y(0) = 0 \tag{1.2}$$

$$\text{for all smooth } T\text{-periodic} \quad y\colon \mathbb{R}\to\mathbb{R}^n.$$

In order to describe our results we introduce a new scaled time and new scaled unknown functions as follows:

$$t_{\text{new}} := \frac{1}{T}t_{\text{old}}, \qquad x_{\text{new}}(t_{\text{new}}) := x_{\text{old}}(t_{\text{old}}).$$

Then (1.2) is transformed into

$$\int\limits_0^1 \big(x(t)\cdot\dot{y}(t) + T\big(f(x(t)) + \varepsilon h(t)\big)\cdot y(t)\big)\,dt + \varepsilon\, d\cdot y(0) = 0 \tag{1.3}$$

$$\text{for all smooth 1-periodic} \quad y\colon \mathbb{R}\to\mathbb{R}^n.$$

Problem (1.3) can be formally written in the form

$$\dot{x}(t) = T\big(f(x(t)) + \varepsilon h(t)\big) + \varepsilon d\sum_{k\in\mathbb{Z}}\delta(t+k),$$

where δ is the Dirac function supported in zero.

Let us formulate our assumptions. We suppose that there exists a non-constant 1-periodic solution x_0 to the unforced problem, i.e., to (1.1) with $\varepsilon = 0$, or, what is the same, to (1.3) with $\varepsilon = 0$ and $T = 1$:

$$\dot{x}_0(t) = f(x_0(t)), \qquad x_0(t+1) = x_0(t), \qquad \dot{x}_0 \neq 0. \tag{1.4}$$

We are going to show that generically the following is true: For all $(\varepsilon, T) \approx (0,1)$, which belong to a certain open conus-like subset of the plane, there exist 1-periodic solutions

$$x(t) \approx x_0(t+\varphi) \tag{1.5}$$

to (1.3). Moreover, we describe how this conus-like subset of the plane looks like and how the asymptotic phase φ depends on the period T. Finally, we prove certain local uniqueness result for (1.3). We solve (1.3) for $\varepsilon \approx 0$ and $T \approx 1$ and (1.5) by means of a Liapunov – Schmidt reduction, by scaling techniques and by means of the implicit function theorem. Remark that we do not use any results about an initial value problem corresponding to (1.1).

From assumption (1.4) it follows that $x = \dot{x}_0$ is a nontrivial solution to the linear homogeneous boundary-value problem

$$\dot{x}(t) = f'(x_0(t))x(t), \qquad x(t+1) = x(t). \tag{1.6}$$

We suppose that

$$\text{any solution to (1.6) is a scalar multiple of } \dot{x}_0. \tag{1.7}$$

In other words, we suppose that the vector space of all solutions to (1.6) is one-dimensional. Therefore the vector space of all solutions to the adjoint linear homogeneous boundary-value problem

$$-\dot{x}(t) = f'(x_0(t))^T x(t), \qquad x(t+1) = x(t) \tag{1.8}$$

is one-dimensional also. Here $f'(x_0(t))^T$ is the transposed to $f'\big(x_0(t)\big)$ matrix. We suppose that

$$\text{there exists a solution } x = x_* \text{ to (1.8) with } \int\limits_0^1 x_*(t) \cdot \dot{x}_0(t)\, dt = 1. \tag{1.9}$$

In other words, we suppose that the eigenvalue $\lambda = 0$ to the eigenvalue problem $\dot{x}(t) = \big(\lambda + f'(x_0(t))\big)x(t),\ x(t+1) = x(t)$, is not only geometrically simple, but also algebraically simple. Finally, we introduce a 1-periodic function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\varphi) := -\int\limits_0^1 x_*(t+\varphi) \cdot h(t)\, dt - x_*(\varphi) \cdot d \tag{1.10}$$

and, for given $\varepsilon_0 > 0$ and $\tau_0 \in \mathbb{R}$, open conus-like sets

$$K(\varepsilon_0, \tau_0) := \big\{(\varepsilon, T) \in \mathbb{R}^2 :\ \varepsilon \in (0, \varepsilon_0),\ T = 1 + \varepsilon\tau,\ \tau \in (\tau_0 - \varepsilon_0, \tau_0 + \varepsilon_0)\big\}$$

and the Banach space $L^\infty_{\mathrm{per}} := \{x \in L^\infty(\mathbb{R}; \mathbb{R}^n) : x(t+1) = x(t) \text{ for almost all } t \in \mathbb{R}\}$ with its norm $\|x\|_\infty := \operatorname{esssup}\big\{\|x(t)\| : t \in \mathbb{R}\big\}$. Here $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

Now we formulate our results:

Theorem 1.1. *Suppose* (1.4), (1.7) *and* (1.9), *and let* $(\varepsilon_k, T_k, x_k) \in (0,\infty)^2 \times L^\infty_{\mathrm{per}},\ k \in \mathbb{N}$, *be a sequence of solutions to* (1.3) *with*

$$\lim_{k\to\infty}\left(\varepsilon_k + |T_k - 1| + \inf_{\varphi\in\mathbb{R}} \|x_k - x_0(\cdot - \varphi)\|_\infty\right) = 0.$$

Then there exist $\varphi_0 \in [0,1]$ *and a subsequence* $(\varepsilon_{k_l}, T_{k_l}, x_{k_l}),\ l \in \mathbb{N}$, *such that*

$$\lim_{l\to\infty} \frac{T_{k_l} - 1}{\varepsilon_{k_l}} = \Phi(\varphi_0) \tag{1.11}$$

and

$$\lim_{l\to\infty} \big\|x_{k_l} - x_0(\cdot + \varphi_0)\big\|_\infty = 0. \tag{1.12}$$

Theorem 1.2. *Suppose* (1.4), (1.7) *and* (1.9), *and let* $\varphi_0 \in [0,1]$ *and* $\tau_0 \in \mathbb{R}$ *be given such that*

$$\Phi(\varphi_0) = \tau_0, \qquad \Phi'(\varphi_0) \neq 0. \tag{1.13}$$

Then the following is true:

(i) **existence and local uniqueness:** *there exist* $\varepsilon_0 > 0$ *and* $\delta > 0$ *such that for all* $(\varepsilon, T) \in \in K(\varepsilon_0, \tau_0)$ *there exists exactly one solution* $x = \hat{x}(\varepsilon, T)$ *to* (1.3) *with*

$$\big\|x - x_0(\cdot + \varphi_0)\big\|_\infty < \delta;$$

(ii) **smooth dependence:** *the map* $(\varepsilon, T) \in K(\varepsilon_0, \tau_0) \mapsto \hat{x}(\varepsilon, T) \in L^\infty_{\text{per}}$ *is* C^1*-smooth*;

(iii) **asymptotic behavior:** *it holds*

$$\sup_{(\varepsilon,T)\in K(\varepsilon_0,\tau_0)} \frac{1}{\varepsilon} \inf_{\varphi\in\mathbb{R}} \big\|\hat{x}(\varepsilon, T) - x_0(\cdot + \varphi)\big\|_\infty < \infty; \tag{1.14}$$

(iv) **asymptotic phases:** *there exists* $\hat{\varphi} \in C^2\big([\tau_0 - \varepsilon_0, \tau_0 + \varepsilon_0]; \mathbb{R}\big)$ *with* $\hat{\varphi}(\tau_0) = \varphi_0$ *such that for all* $\tau \in (\tau_0 - \varepsilon_0, \tau_0 + \varepsilon)$ *it holds*

$$\lim_{\varepsilon\to 0} \big\|\hat{x}(\varepsilon, 1 + \varepsilon\tau) - x_0(\cdot + \hat{\varphi}(\tau))\big\|_\infty = 0 \qquad \textit{and} \qquad \Phi\big(\hat{\varphi}(\tau)\big) = \tau. \tag{1.15}$$

***Remark* 1.1.** If $(\varepsilon, T) \in K(\varepsilon_0, \tau_0)$, then $T = 1 + \varepsilon\tau$ with $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in (\tau_0 - \varepsilon_0, \tau_0 + \varepsilon)$, i.e.,

$$\tau = \frac{T - 1}{\varepsilon}$$

is a scaled period parameter. Hence, the so-called phase equation $\Phi(\varphi) = \tau$ describes the relationship between the scaled period τ and the corresponding asymptotic phase $\varphi = \hat{\varphi}(\tau)$ (cf. (1.15)) of the solution family $\hat{x}(\varepsilon, T)$.

***Remark* 1.2.** If $d = 0$ and $h \in C^1\big([0,1]; \mathbb{R}^n\big)$ with $h(0) = h(1)$ and $h'(0) = h'(1)$ (i.e., if the forcing is C^1-smooth), and if x_0 is an exponentially orbitally stable periodic solution to $\dot{x}(t) = = f(x(t))$, then the following is true: if $\Phi'(\varphi_0) > 0$ (or $\Phi'(\varphi_0) < 0$), then $\hat{x}(\varepsilon, T)$ is an exponentially stable (or unstable) periodic solution to $\dot{x}(t) = T\big(f(x(t)) + \varepsilon h(t)\big)$ (for all $(\varepsilon, T) \in K(\varepsilon_0, \tau_0)$ with sufficiently small ε_0) (cf., e.g., [7], Theorem 3, or [10], Theorem 5.1). In other words: the phase equation $\Phi(\varphi) = \tau$ describes not only the relationship between the scaled period and the corresponding asymptotic phase, but also the stability of the locked periodic solutions $\hat{x}(\varepsilon, T)$. It is an **open problem** if a similar result is true in the case of general distributional forcings.

***Remark* 1.3.** Assertion (ii) of Theorem 1.2 claims that the data-to-solution map $\hat{x}$ of the problem (1.3) is C^1-smooth from $K(\varepsilon_0, \tau_0)$ into L^∞_{per}. But the corresponding data-to-solution map

$$(\varepsilon, T) \in K(\varepsilon_0, \tau_0) \mapsto \tilde{x}(\varepsilon, T) \in L^\infty_{\text{per}} \qquad \text{with} \quad \tilde{x}(\varepsilon, T)(t) := \hat{x}(\varepsilon, T)(t/T)$$

to (1.2) is not smooth, even not continuous, in general!

***Remark* 1.4.** Assertion (iii) of Theorem 1.2 claims that $\hat{x}(\varepsilon, 1 + \varepsilon\tau)$ tends, for $\varepsilon \to 0$, to a phase shift of x_0, and this phase shift depends on τ. In particular, $\hat{x}(\varepsilon, T)$ does not converge, for $(\varepsilon, T) \to (0, 1)$ with $(\varepsilon, T) \in K(\varepsilon_0, \tau_0)$, in L^∞_{per}.

2. Transformation of (1.3) into an equation in L^∞_{per}. In this section we show that the variational problem (1.3) is equivalent to a smooth equation in the Banach space L^∞_{per}. For that reason we introduce a function $G\colon \mathbb{R}^2 \to \mathbb{R}$, which is 1-periodic with respect to both its variables, by defining it in the square $[0,1]\times[0,1]$ as below and then extending it 1-periodically on $\mathbb{R}^2$:

$$G(t,s) := \begin{cases} \dfrac{e}{1-e}e^{t-s} & \text{for} \quad 0 \le t \le s \le 1, \\ \dfrac{1}{1-e}e^{t-s} & \text{for} \quad 0 \le s < t \le 1. \end{cases}$$

It is easy to verify that G is a Green's function, i.e., for any 1-periodic functions $y, z\colon \mathbb{R}\to\mathbb{R}^n$ it holds

$$\dot y - y = z \qquad \text{and only if} \qquad y(t) = \int\limits_0^1 G(t,s)z(s)\,ds \tag{2.1}$$

and

$$\int\limits_0^1 \left((\dot y(t) - y(t)) \int\limits_0^1 G(s,t) z(s)\,ds \right) dt = \int\limits_0^1 y(t)z(t)\,dt. \tag{2.2}$$

Lemma 2.1. *A function $x \in L^\infty_{\mathrm{per}}$ is a solution to* (1.3) *if and only if for almost all $t\in\mathbb{R}$ it holds*

$$x(t) + \int\limits_0^1 G(s,t)\left(x(s) + T(f(x(s)) + \varepsilon h(s))\right)ds + \varepsilon G(0,t)\,d = 0. \tag{2.3}$$

Proof. Take $x\in L^\infty_{\mathrm{per}}$ and smooth 1-periodic functions $y,z\colon\mathbb{R}\to\mathbb{R}^n$ with $\dot y - y = z$. Then (2.1) yields

$$\int\limits_0^1 \Big(x(t)\cdot \dot y(t) + T\big(f(x(t)) + \varepsilon h(t)\big)\cdot y(t)\Big)\,dt + \varepsilon\, d\cdot y(0) =$$

$$= \int\limits_0^1 \Big(x(t)\cdot\big(\dot y(t) + y(t)\big) + \big(x(t) + T(f(x(t)) + \varepsilon h(t))\big)\cdot y(t)\Big)\,dt + \varepsilon\, d\cdot y(0) =$$

$$= \int\limits_0^1 \left(x(t)\cdot z(t) + \big(x(t) + T\big(f(x(t)) + \varepsilon h(t)\big)\big)\cdot \int\limits_0^1 G(t,s)z(s)\,ds \right) dt + \varepsilon\, d\cdot y(0) =$$

$$= \int\limits_0^1 z(s)\cdot \left(x(s) + \int\limits_0^1 G(t,s)\big(x(t) + T\big(f(x(t)) + \varepsilon h(t)\big)\big)\,ds + \varepsilon G(0,s)d \right) ds.$$

Hence, x is a solution to (1.3) iff the left-hand side vanishes for all smooth 1-periodic $y\colon\mathbb{R}\to\mathbb{R}^n$, and this is the case iff the right-hand side vanishes for all smooth 1-periodic $z\colon\mathbb{R}\to\mathbb{R}^n$, i.e., iff (2.3) holds for almost all $t\in\mathbb{R}$.

Lemma 2.1 is proved.

It follows from Lemma 2.1 that any solution $x \in L^\infty_{\rm per}$ to (1.3), i.e., to (2.3), is continuous up to the points $t = k,\ k \in \mathbb{Z}$, where it jumps from $x(k-0)$ to

$$x(k+0) = x(k-0) - \varepsilon d.$$

In order to analyze equation (2.3), let us introduce maps $\mathcal{F} \in C^2(\mathbb{R} \times L^\infty_{\rm per}; L^\infty_{\rm per})$ and $\mathcal{G} \in$ $\in C^\infty(\mathbb{R}; L^\infty_{\rm per})$ by defining for almost all $t \in \mathbb{R}$

$$\mathcal{F}(T,x)(t) := \int\limits_0^1 G(s,t)\big(x(s) + Tf(x(s))\big)\, ds,$$

$$\mathcal{G}(T)(t) := T\int\limits_0^1 G(s,t)h(s)\, ds + G(0,t)d.$$

Then the variational problem (1.3), which is equivalent to the integral equation (2.3), is equivalent to the abstract equation

$$x + \mathcal{F}(T,x) + \varepsilon\mathcal{G}(T) = 0. \tag{2.4}$$

For $\varphi \in \mathbb{R}$ we define $S_\varphi \in \mathcal{L}(L^\infty_{\rm per})$ by

$$S_\varphi x(t) := x(t+\varphi) \qquad \text{for almost all} \quad t \in \mathbb{R}.$$

The map $\varphi \mapsto S_\varphi$ is a representation of the rotation group $SO(2)$ on the vector space $L^\infty_{\rm per}$, but this representation is not strongly continuous because the map $\varphi \in \mathbb{R} \mapsto S_\varphi x \in L^\infty_{\rm per}$ is continuous if and only if the function x is continuous, i.e., not for all $x \in L^\infty_{\rm per}$. But the maps $\varphi \in \mathbb{R} \mapsto S_\varphi x_0 \in L^\infty_{\rm per}$ and $\varphi \in \mathbb{R} \mapsto S_\varphi x_* \in L^\infty_{\rm per}$ are C^3-smooth and C^2-smooth, respectively, because the functions x_0 and x_* are C^3-smooth and C^2-smooth, respectively (because the vector field f is supposed to be C^2-smooth). This will be used repeatedly in what follows.

It is easy to verify that

$$S_\varphi\mathcal{F}(T,x) = \mathcal{F}(T, S_\varphi x) \qquad \text{for all} \quad T,\varphi \in \mathbb{R} \qquad \text{and} \qquad x \in L^\infty_{\rm per}. \tag{2.5}$$

Because of (2.5) equation (2.4) is a symmetry breaking problem. If x is a solution to (2.4) with $\varepsilon = 0$, then $S_\varphi x$ is a solution also for all $\varphi \in \mathbb{R}$. But for $\varepsilon \neq 0$ this is not the case, in general.

We are going to use well-known techniques for treating symmetry breaking problems, which are developed, e.g., in [1–3, 10, 16]. The main ingredient for that is the Fredholm property of the operator $I + \partial_x\mathcal{F}(1,x_0)$.

Lemma 2.2. *The operator $\partial_x\mathcal{F}(1,x_0)$ is completely continuous from $L^\infty_{\rm per}$ into $L^\infty_{\rm per}$, and it holds*

$$\ker\big(I + \partial_x\mathcal{F}(1,x_0)\big) = \operatorname{span}\{\dot{x}_0\}, \tag{2.6}$$

$$\operatorname{im}\big(I + \partial_x\mathcal{F}(1,x_0)\big) = \left\{x \in L^\infty_{\rm per} : \int\limits_0^1 (\dot{x}_*(t) - x_*(t)) \cdot x(t)\, dt = 0\right\}. \tag{2.7}$$

Proof. We have $\big(\partial_x\mathcal{F}(1,x_0)x\big)(t) = \int_0^1 G(s,t)\big(I + f'(x_0(s))\big)x(s)\,ds$. Therefore (2.1) yields

$$\frac{d}{dt}\,\partial_x\mathcal{F}(1,x_0)x = \partial_x\mathcal{F}(1,x_0)x + \big(I + f'(x_0)\big)x. \tag{2.8}$$

Hence, $\partial_x\mathcal{F}(1,x_0)$ is a linear bounded operator from L^∞_{per} into the function space

$$W^{1,\infty}_{\text{per}} := \big\{x \in L^\infty_{\text{per}} :\ \dot{x} \in L^\infty_{\text{per}}\big\} \qquad \text{with norm} \quad \|x\|_\infty + \|\dot{x}\|_\infty.$$

But $W^{1,\infty}_{\text{per}}$ is compactly embedded into L^∞_{per}. Hence, $\partial_x\mathcal{F}(1,x_0)$ is completely continuous from L^∞_{per} into L^∞_{per}.

Because of $x_0 + \mathcal{F}(1,x_0) = 0$ and of (2.5) we have $S_\varphi x_0 + \mathcal{F}(1,S_\varphi x_0) = 0$ for all $\varphi \in \mathbb{R}$. Differentiating this identity with respect to φ in $\varphi = 0$ we get $\dot{x}_0 \in \ker\,(I + \partial_x\mathcal{F}(1,x_0))$.

Now, take $x \in \ker\big(I + \partial_x\mathcal{F}(1,x_0)\big)$, i.e., $x + \partial_x\mathcal{F}(1,x_0)x = 0$. Then $x \in W^{1,\infty}_{\text{per}}$, and (2.8) yields

$$\frac{d}{dt}\,\partial_x\mathcal{F}(1,x_0)x = \partial_x\mathcal{F}(1,x_0)x + \big(I + f'(x_0)\big)x = f'(x_0)x.$$

Hence, assumption (1.7) yields (2.6).

In order to prove (2.7) we consider the Hilbert space L^2_{per} with its scalar product $(x,y) := := \int_0^1 x(t)\cdot y(t)\,dt$. Because of the continuous and dense embedding $L^\infty_{\text{per}} \hookrightarrow L^2_{\text{per}}$ it follows that L^2_{per} is continuously and densily embedded into the dual space $(L^\infty_{\text{per}})^*$, where a function $x \in L^2_{\text{per}}$ has to be understood as an element of $(L^\infty_{\text{per}})^*$ by means of

$$\langle x,y\rangle := (x,y) \qquad \text{for all} \quad y \in L^\infty_{\text{per}},$$

where $\langle\cdot,\cdot\rangle : (L^\infty_{\text{per}})^* \times L^\infty_{\text{per}} \to \mathbb{R}$ is the dual pairing. The Fredholmness of $I + \partial_x\mathcal{F}(1,x_0)$ yields

$$\operatorname{im}\big(I + \partial_x\mathcal{F}(1,x_0)\big) = \Big\{x \in L^\infty_{\text{per}} : \langle\phi,y\rangle = 0 \ \text{ for all } \ \phi \in \ker\big(I + \partial_x\mathcal{F}(1,x_0)\big)^*\Big\},$$

$$\dim\ker\big(I + \partial_x F(1,x_0)\big)^* = 1.$$

Hence, in oder to prove (2.7) we have to prove that $\dot{x}_* - x_* \in \ker\big(I + \partial_x\mathcal{F}(1,x_0)\big)^*$. But this is easy to verify because for any smooth 1-periodic function $y : \mathbb{R} \to \mathbb{R}^n$ we have

$$\big\langle (I + \partial_x\mathcal{F}(1,x_0))^*(\dot{x}_* - x_*), y\big\rangle = \big\langle \dot{x}_* - x_*, (I + \partial_x\mathcal{F}(1,x_0)y\big\rangle =$$

$$= \big(\dot{x}_* - x_*, (I + \partial_x\mathcal{F}(1,x_0)y\big) =$$

$$= \int_0^1 \big(\dot{x}_*(t) - x_*(t)\big)\cdot\left(y(t) + \int_0^1 G(s,t)\big(y(s) + f'(x_0(s))y(s)\big)ds\right)dt =$$

$$= \int_0^1 x_*(t)\cdot\big(-\dot{y}(t) + f'(x_0(t))y(t)\big)\,dt = \int_0^1 y(t)\cdot\big(\dot{x}_*(t) + f'(x_0(s))^T x_*(t)\big)\,dt = 0.$$

Here we used (1.9) and (2.2).

Lemma 2.2 is proved.

3. A parametrization of a neighbourhood of x_0. Let us define a closed codimension one subspace Y of L^∞_{per} by

$$Y := \left\{ y \in L^\infty_{\text{per}} : \int_0^1 y(t) \cdot x_*(t)\, dt = 0 \right\}.$$

In this section we show that any small neighbourhood of x_0 in L^∞_{per} can be smoothly parametrized by small $(\varphi, y) \in \mathbb{R} \times Y$ via $x = S_\varphi x_0 + y$.

Lemma 3.1. *The map $(\varphi, y) \in \mathbb{R} \times Y \mapsto f(\varphi, y) := S_\varphi x_0 + y \in L^\infty_{\text{per}}$ is a diffeomorphism of a neighbourhood of zero in $\mathbb{R} \times Y$ onto a neighbourhood of x_0 in L^∞_{per}.*

Proof. It holds $f(0,0) = x_0$ and $f'(0,0)(\varphi, y) = -\varphi \dot{x}_0 + y$ for all $(\varphi, y) \in \mathbb{R} \times Y$. On the other hand, assumption (1.9) yields that $\dot{x}_0 \notin Y$. Hence,

$$L^\infty_{\text{per}} = \text{span}\{\dot{x}_0\} \oplus Y. \tag{3.1}$$

Therefore $f'(0,0)$ is bijective from $\mathbb{R} \times Y$ onto L^∞_{per}. Hence, the local diffeomorphism theorem yields the claim.

Lemma 3.1 is proved.

4. An a priori estimate. If $x \approx \{S_\psi x_0 : \psi \in \mathbb{R}\}$ is a solution to (2.4), then there exists $\psi \in [0,1]$ with $x \approx S_\psi x_0$ and, hence, $S_{-\psi} x \approx x_0$. Therefore Lemma 3.1 yields that there exist $\varphi \approx 0$ and $y \in Y$ with $y \approx 0$ such that $S_{-\psi} x = S_\varphi x_0 + y$, i.e.,

$$x = S_\psi (S_\varphi x_0 + y).$$

Inserting this into (2.4) and using (2.5) we get

$$S_\varphi x_0 + y + \mathcal{F}(T, S_\varphi x_0 + y) + \varepsilon S_{-\psi} \mathcal{G}(T) = 0. \tag{4.1}$$

Lemma 4.1. *For all $\varphi, T \in \mathbb{R}$ and $y \in L^\infty_{\text{per}}$ it holds*

$$S_\varphi x_0 + y + \mathcal{F}(T, S_\varphi x_0 + y) = \left(I + \int_0^1 \partial_x \mathcal{F}(1, S_\varphi x_0 + ry) dr \right) y + (T-1) \partial_T \mathcal{F}(1, S_\varphi x_0 + y).$$

Proof. We have $S_\varphi x_0 + \mathcal{F}(T, S_\varphi x_0) = 0$ for all $\varphi \in \mathbb{R}$. Therefore

$$S_\varphi x_0 + y + \mathcal{F}(T, S_\varphi x_0 + y) = \left(I + \int_0^1 \partial_x \mathcal{F}(1, S_\varphi x_0 + ry) dr \right) y +$$

$$+ (T-1) \int_0^1 \partial_T \mathcal{F}(1 + r(T-1), S_\varphi x_0 + y) dr.$$

But $\mathcal{F}(\cdot, S_\varphi x_0 + y)$ is affine, hence

$$\int_0^1 \partial_T \mathcal{F}\big(1 + r(T-1), S_\varphi x_0 + y\big)\, dr = \partial_T \mathcal{F}(1, S_\varphi x_0 + y).$$

Lemma 4.1 is proved.

Lemma 4.2. *There exist $\delta > 0$ and $c > 0$ such that for all solutions $(\varepsilon, T, \varphi, \psi, y) \in (0,\infty)^2 \times \times [0,1]^2 \times Y$ to* (4.1) *with $\varepsilon + |T-1| + \|y\|_\infty < \delta$ it holds*

$$|T-1| + \|y\|_\infty < c\varepsilon.$$

Proof. Suppose the contrary. Then there exist solutions $(\varepsilon_k, T_k, \varphi_k, \psi_k, y_k) \in (0,\infty)^2 \times [0,1]^2 \times \times Y,\ k = 1, 2, \ldots,$ to (4.1) with

$$\lim_{k\to\infty}\left(\varepsilon_k + |\varphi_k| + |T_k - 1| + \|y_k\|_\infty + \frac{\varepsilon_k}{|T_k-1| + \|y_k\|_\infty}\right) = 0. \tag{4.2}$$

Because of Lemma 4.1 it holds

$$\left(I + \int_0^1 \partial_x \mathcal{F}(1, S_{\varphi_k} x_0 + r y_k) dr\right) \frac{y_k}{|T_k - 1| + \|y_k\|_\infty} +$$

$$+\frac{T_k - 1}{|T_k - 1| + \|y_k\|_\infty} \partial_T \mathcal{F}(1, S_{\varphi_k} x_0 + y_k) = -\frac{\varepsilon_k}{|T_k - 1| + \|y_k\|_\infty} S_{-\psi_k} \mathcal{G}(T_k). \tag{4.3}$$

Without loss of generality we may assume that there exists $\tau \in \mathbb{R}$ with

$$\lim_{k\to\infty} \frac{T_k - 1}{|T_k - 1| + \|y_k\|_\infty} = \tau. \tag{4.4}$$

Moreover, because the operator $\partial_x \mathcal{F}(1, x_0)$ is completely continuous, without loss of generality we may assume that the sequence $\partial_x \mathcal{F}(1, x_0) y / \big(|T_k - 1| + \|y_k\|_\infty\big),\ k \in \mathbb{N}$, converges in L^∞_{per}. Hence, (4.2)–(4.4) yield that there exists $y \in Y$ with

$$\lim_{k\to\infty} \frac{y_k}{|T_k - 1| + \|y_k\|_\infty} = y \tag{4.5}$$

and

$$\left(I + \int_0^1 \partial_x \mathcal{F}(1, x_0) dr\right) y + \tau \partial_T \mathcal{F}(1, x_0) = 0. \tag{4.6}$$

Because of (1.9), (2.2) and (2.7) it follows

$$0 = \tau \int_0^1 \partial_T \mathcal{F}(1, x_0) \cdot (\dot{x}_* - x_*)\, dt =$$

$$= \tau \int_0^1 \left(\int_0^1 G(s,t) f(x_0(s))\, ds \right) \cdot \big(\dot{x}_*(t) - x_*(t)\big)\, dt =$$

$$= \tau \int_0^1 f(x_0(t)) \cdot x_*(t)\, dt = \tau \int_0^1 \dot{x}_0(t) \cdot x_*(t)\, dt = \tau. \tag{4.7}$$

Therefore (2.6) and (4.6) imply that $y \in Y \cap \ker(I + \partial_x \mathcal{F}(1, x_0)) = Y \cap \operatorname{span}\{\dot{x}_0\}$, and (3.1) yields $y = 0$.

Let us summarize: We got $\tau = 0$ and $y = 0$. But from (4.4) and (4.5) it follows $|\tau| + \|y\|_\infty = 1$, this is the needed contradiction.

Lemma 4.2 is proved.

5. Proof of Theorem 1.1. Suppose (1.4), (1.7) and (1.9), and take a sequence $(\varepsilon_k, T_k, x_k) \in \in (0,\infty)^2 \times L^\infty_{\text{per}},\ k \in \mathbb{N}$, with

$$x_k + \mathcal{F}(T_k, x_k) + \varepsilon_k \mathcal{G}(T_k) = 0 \tag{5.1}$$

and

$$\lim_{k\to\infty} \left(\varepsilon_k + |T_k - 1| + \inf_{\psi\in\mathbb{R}} \|x_k - S_\psi x_0\|_\infty \right) = 0. \tag{5.2}$$

For any k there exist $\psi_k \in [0,1]$ with

$$\inf_{\psi\in\mathbb{R}} \|x_k - S_\psi x_0\|_\infty = \|x_k - S_{\psi_k} x_0\|_\infty = \|S_{-\psi_k} x_k - x_0\|_\infty, \tag{5.3}$$

and without loss of generality we may assume that there exists $\varphi_0 \in [0,1]$ with

$$\lim_{k\to\infty} \psi_k = \varphi_0. \tag{5.4}$$

Hence, (5.3) yields $\|x_k - S_{\varphi_0}\|_\infty \to 0$ for $k \to \infty$, i.e., assertion (1.12) of Theorem 1.1 is proved.

Because of Lemma 3.1 and of (5.3) for large k there exist $\varphi_k \in [0,1]$ and $y_k \in Y$ with

$$S_{-\psi_k} x_k = S_{\varphi_k} x_0 + y_k \qquad \text{and} \qquad \lim_{k\to\infty} = \big(|\varphi_k| + \|y_k\|_\infty\big) = 0.$$

Inserting $x_k = S_{\psi_k}(S_{\varphi_k} x_0 + y_k)$ into (5.1) and using (2.5) we get

$$S_{\varphi_k} x_0 + y_k + \mathcal{F}(T_k, S_{\varphi_k} x_0 + y_k) + \varepsilon_k S_{-\psi_k} \mathcal{G}(T_k) = 0. \tag{5.5}$$

Further, from Lemma 4.2 it follows that there exists a bounded sequence $(\tau_k, z_k) \in \mathbb{R} \times Y,\ k \in \mathbb{N}$, such that $T_k = 1 + \varepsilon_k \tau_k$ and $y_k = \varepsilon_k z_k$. Inserting this into (5.5), dividing by ε_k and using Lemma 4.1 we get

$$\left(I + \int_0^1 \partial_x \mathcal{F}(1, S_{\varphi_k} x_0 + r\varepsilon_k z_k) dr \right) z_k + \tau_k \partial_T \mathcal{F}(1, S_{\varphi_k} x_0 + \varepsilon_k z_k) = -S_{-\psi_k} \mathcal{G}(1 + \varepsilon_k \tau_k). \tag{5.6}$$

Without loss of generality we may assume that there exists $\tau_0 \in \mathbb{R}$ such that $\tau_k \to \tau_0$ for $k \to \infty$. Moreover, (2.7) yields

$$\lim_{k\to\infty} \int_0^1 (\dot{x}_* - x_*) \cdot \left(z_k + \int_0^1 \partial_x \mathcal{F}(1, S_{\varphi_k} x_0 + r\varepsilon_k z_k)\, dr\, z_k \right) dt = 0,$$

and (4.7) implies

$$\lim_{k\to\infty} \int_0^1 (\dot{x}_* - x_*) \cdot \partial_T \mathcal{F}(1, S_{\varphi_k} x_0 + \varepsilon_k z_k)\, dt = 1.$$

Hence, from (1.10), (2.2), (5.4) and (5.6) it follows

$$\tau_0 = -\lim_{k\to\infty} \int_0^1 S_{-\psi_k} \mathcal{G}(1 + \varepsilon_k \tau_k) \cdot (\dot{x}_* - x_*)\, dt =$$

$$= -\lim_{k\to\infty}\int_0^1 \mathcal{G}(1+\varepsilon_k\tau_k)\cdot S_{\psi_k}(\dot{x}_* - x_*)\,dt = -\int_0^1 \mathcal{G}(1)\cdot S_{\varphi_0}(\dot{x}_* - x_*)\,dt =$$

$$= -\int_0^1 \left(\int_0^1 G(s,t)h(s)\,ds + G(0,t)d\right)\cdot\big(\dot{x}_*(t+\varphi_0) - x_*(t*\varphi_0)\big)\,dt =$$

$$= -\int_0^1 h(t)\cdot x_*(t)\,dt - \int_0^1 \frac{e}{1-e}e^{-t}d\cdot\big(\dot{x}_*(t+\varphi_0) - x_*(t+\varphi_0)\big)\,dt =$$

$$= -\int_0^1 h(t)\cdot x_*(t)\,dt - d\cdot x_*(\varphi_0) = \Phi(\varphi_0). \tag{5.7}$$

Therefore, assertion (1.11) of Theorem 1.1 is proved.

6. Proof of Theorem 1.2. Suppose (1.4), (1.7) and (1.9), and let $\varphi_0 \in [0,1]$ and $\tau_0 \in \mathbb{R}$ be given such that (1.13) is true.

We have to determine all solutions $x \approx S_{\varphi_0}x_0$ to (2.4) with $(\varepsilon, T) \in K(\varepsilon_0, \tau_0)$ and $\varepsilon_0 \approx 0$. Because of Lemma 3.1 we are allowed to make the ansatz

$$x = S_{\varphi_0}(S_\varphi x_0 + y) \qquad \text{with} \quad \varphi \approx 0, \quad y \approx 0, \quad y \in Y.$$

Inserting this ansatz into (2.4) we get

$$S_\varphi x_0 + y + \mathcal{F}(T, S_\varphi x_0 + y) + \varepsilon S_{-\varphi_0}\mathcal{G}(T) = 0. \tag{6.1}$$

Further, because of Lemma 4.2 we are allowed to make the ansatz

$$T = 1 + \varepsilon\tau, \qquad y = \varepsilon z,$$

and we get, after deviding by ε and using Lemma 4.1,

$$\left(I + \int_0^1 \partial_x\mathcal{F}(1, S_\varphi x_0 + r\varepsilon z)dr\right)z + \tau\partial_T\mathcal{F}(1, S_\varphi x_0 + \varepsilon z) + S_{-\varphi_0}\mathcal{G}(1+\varepsilon\tau) = 0. \tag{6.2}$$

We are going to solve equation (6.2) with respect to $(\varphi, z) \approx (\varphi_0, z_0)$ ($z_0 \in Y$ is defined below, see (6.6)) for given $(\varepsilon, \tau) \approx (0, \tau_0)$ by means of the implicit function theorem. Remark that φ_0 and τ_0 are given by assumption (1.13).

Let us define $x_1 \in L^\infty_{\text{per}}$ by $x_1(t) := \int_0^1 G(s,t)\dot{x}_0(s)\,ds$. Then (1.9) and (2.2) imply

$$\int_0^1 x_1(t)\cdot\big(\dot{x}_*(t) - x_*(t)\big)\,dt = \int_0^1 \dot{x}_0\cdot x_*(t)\,dt = 1.$$

Hence, (2.7) yields

$$L_{\rm per}^\infty = \text{span}\,\{x_1\} \oplus \text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big), \tag{6.3}$$

and $P = P^2 \in \mathcal{L}(L_{\rm per}^\infty)$, defined by $Px := \int_0^1 x(t) \cdot (\dot{x}_*(t) - x_*(t))\, dt x_1$, is the projection corresponding to the topological sum (6.3), i.e.,

$$\ker P = \text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big), \qquad \text{im}\, P = \text{span}\,\{x_1\}.$$

Finally, for $\varphi \in \mathbb{R}$ we define $L_\varphi \in \mathcal{L}\big(L_{\rm per}^\infty; \mathbb{R} \times \text{im}(I + \partial_x \mathcal{F}(1, x_0))\big)$ by

$$L_\varphi x := \left(\int_0^1 x \cdot S_\varphi(\dot{x}_* - x_*)\, dt, (I - P)x \right).$$

Lemma 6.1. (i) *The operator* $I + \partial_x \mathcal{F}(1, x_0)$ *is bijective from* Y *onto* $\text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big)$.

(ii) *There exists* $\delta > 0$ *such that for all* $\varphi \in \mathbb{R}$ *with* $|\varphi| < \delta$ *the operator* L_φ *is bijective from* $L_{\rm per}^\infty$ *onto* $\mathbb{R} \times \text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big)$.

Proof. (i) If for $y \in Y$ we have $\big(I + \partial_x \mathcal{F}(1, x_0)\big)y = 0$, i.e., $y \in Y \cap \ker\big(I + \partial_x \mathcal{F}(1, x_0)\big) = Y \cap \text{span}\,\{\dot{x}_0\}$, then (3.1) yields $y = 0$. Hence, $I + \partial_x \mathcal{F}(1, x_0)$ is injective.

If $\tilde{x} \in \text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big)$ is given, then there exists $x \in L_{\rm per}^\infty$ with $\tilde{x} = \big(I + \partial_x \mathcal{F}(1, x_0)\big)x$. Moreover, because of (3.1) there exist $\xi \in \mathbb{R}$ and $y \in Y$ with $x = \xi \dot{x}_0 + y$. Because of (2.6) it follows

$$\tilde{x} = \big(I + \partial_x \mathcal{F}(1, x_0)\big)(\xi \dot{x}_0 + y) = \big(I + \partial_x \mathcal{F}(1, x_0)\big)y.$$

Hence, $I + \partial_x \mathcal{F}(1, x_0)$ is surjective from Y onto $\text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big)$.

(ii) If $L_0 x = 0$, then $Px = (I - P)x = 0$, i.e., $x = 0$. Hence, L_0 is injective. Moreover, for arbitrary $(\xi, \tilde{x}) \in \mathbb{R} \times \text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big)$ it holds

$$L_0(\xi x_1 + \tilde{x}) = \left(\int_0^1 (\xi x_1 + \tilde{x}) \cdot (\dot{x}_* - x_*)\, dt, (I - P)(\xi x_1 + \tilde{x}) \right) = (\xi, \tilde{x}).$$

Hence, L_0 is surjective from $L_{\rm per}^\infty$ onto $\mathbb{R} \times \text{im}\,\big(I + \partial_x \mathcal{F}(1, x_0)\big)$.

But the map $\varphi \in \mathbb{R} \mapsto L_\varphi \in \mathcal{L}\big((L_{\rm per}^\infty; \mathbb{R} \times \text{im}(I + \partial_x \mathcal{F}(1, x_0))\big)$ is continuous (even C^1-smooth) with respect to the operator norm, and the set of all bijective maps is open in $\mathcal{L}\big((L_{\rm per}^\infty; \mathbb{R} \times \text{im}(I + \partial_x \mathcal{F}(1, x_0))\big)$. Hence, assertion (ii) is proved.

Lemma 6.1 is proved.

Define by $\mathcal{H}(\varepsilon, \tau, \varphi, z)$ the left-hand side of (6.2). Because of Lemma 6.1 (ii) the equation (6.2) with $\varphi \approx 0$ is equivalent to

$$L_\varphi \mathcal{H}(\varepsilon, \tau, \varphi, z) =: \big(\mathcal{H}_1(\varepsilon, \tau, \varphi, z), \mathcal{H}_2(\varepsilon, \tau, \varphi, z)\big) = 0. \tag{6.4}$$

The maps $\mathcal{H}_1 \in C^1\big((0, \infty) \times \mathbb{R}^2 \times Y; \mathbb{R}\big)$ and $\mathcal{H}_2 \in C^1\big((0, \infty) \times \mathbb{R}^2 \times Y; \text{im}(I + \partial_x \mathcal{F}(1, x_0))\big)$, which are defined in (6.4), satisfy

$$\mathcal{H}_1(0, \tau, \varphi, z) = \int_0^1 \Big(\big(I + \partial_x \mathcal{F}(1, S_\varphi x_0)\big)z + \tau \partial_T \mathcal{F}(1, S_\varphi x_0) + S_{-\varphi_0} \mathcal{G}(1) \Big) \cdot S_\varphi(\dot{x}_* - x_*)\, dt,$$

$$\mathcal{H}_2(0,\tau,\varphi,z) = (I-P)\Big(\big(I+\partial_x\mathcal{F}(1,S_\varphi x_0)\big)z + \tau\partial_T\mathcal{F}(1,S_\varphi x_0) + S_{-\varphi_0}\mathcal{G}(1)\Big).$$

But (2.5) yields $S_\varphi\partial_x\mathcal{F}(1,x_0) = \partial_x\mathcal{F}(1,S_\varphi x_0)S_\varphi$ and $\partial_T\mathcal{F}(1,S_\varphi x_0) = S_\varphi\partial_T\mathcal{F}(1,x_0)$. Therefore

$$\int_0^1 \big(I+\partial_x\mathcal{F}(1,S_\varphi x_0)\big)z\cdot S_\varphi(\dot{x}_* - x_*)\,dt =$$

$$= \int_0^1 S_{-\varphi}\big(I+\partial_x\mathcal{F}(1,S_\varphi x_0)\big)z\cdot(\dot{x}_* - x_*)\,dt =$$

$$= \int_0^1 \big(I+\partial_x\mathcal{F}(1,x_0)\big)S_\varphi z\cdot(\dot{x}_* - x_*)\,dt = 0$$

and (cf. (4.7))

$$\int_0^1 \partial_T\mathcal{F}(1,S_\varphi x_0)\cdot S_\varphi(\dot{x}_* - x_*)\,dt =$$

$$= \int_0^1 S_{-\varphi}\partial_T\mathcal{F}(1,S_\varphi x_0)\cdot(\dot{x}_* - x_*)\,dt =$$

$$= \int_0^1 \partial_T\mathcal{F}(1,x_0)\cdot(\dot{x}_* - x_*)\,dt = 1$$

and (cf. (5.7))

$$\int_0^1 S_{-\varphi_0}\mathcal{G}(1)\cdot S_\varphi(\dot{x}_* - x_*)\,dt = -\Phi(\varphi_0+\varphi).$$

Hence,

$$\mathcal{H}_1(0,\tau,\varphi,z) = \tau - \Phi(\varphi_0+\varphi). \tag{6.5}$$

Therefore $\varepsilon=0,\ \tau=\tau_0,\ \varphi=0,\ z=z_0$ is a solution to (6.4), i.e., to (6.2) with

$$z_0 := -\big(I+\partial_x\mathcal{F}(1,x_0)\big)^{-1}(I-P)\big(\tau_0\partial_T\mathcal{F}(1,x_0) + S_{-\varphi_0}\mathcal{G}(1)\big). \tag{6.6}$$

Here $\big(I+\partial_x\mathcal{F}(1,x_0)\big)^{-1} \in \mathcal{L}\big(\mathrm{im}\,\big(I+\partial_x\mathcal{F}(1,x_0)\big);Y\big)$ is the inverse operator to the operator $I+\partial_x\mathcal{F}(1,x_0) \in \mathcal{L}\big(Y;\mathrm{im}(I+\partial_x\mathcal{F}(1,x_0))\big)$, cf. Lemma 6.1 (i).

Further, we have

$$\partial_\varphi\mathcal{H}_1(0,\tau_0,0,z_0) = -\Phi'(\varphi_0),$$

$$\partial_z\mathcal{H}_1(0,\tau_0,0,z_0) = 0,$$

$$\partial_z \mathcal{H}_2(0,\tau_0,0,z_0) = I + \partial_x \mathcal{F}(1,x_0).$$

Hence, assumption $\Phi'(\varphi_0) \neq 0$ (cf. (1.13)) and Lemma 6.1 (i) imply that the partial derivative with respect to (φ, z) of $L_\varphi \mathcal{H}(\varepsilon,\tau,\varphi,z)$ in the point $\varepsilon = 0, \tau = \tau_0, \varphi = 0, z = z_0$ is bijective from $\mathbb{R} \times \operatorname{im}\big(I + \partial_x \mathcal{F}(1,x_0)\big)$ onto L^∞_{per}. Therefore the implicit function theorem yields that there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in [0,\varepsilon_0]$ and $\tau \in [\tau_0 - \varepsilon_0, [\tau_0 + \varepsilon_0]$ there exists exactly one solution

$$\varphi = \tilde{\varphi}(\varepsilon,\tau), \qquad z = \tilde{z}(\varepsilon,\tau)$$

to (6.4) with$|\varphi| + \|z\|_\infty < \delta$. Moreover, the data-to-solution maps $\tilde{\varphi}$ and $\tilde{z}$ are C^1-smooth. Hence, for all $(\varepsilon, T) \in K(\varepsilon_0,\tau_0)$ we have a solution $x = \hat{x}(\varepsilon,T)$ to (2.4), where the data-to-solution map $\hat{x}$ is defined as

$$\hat{x}(\varepsilon, 1+\varepsilon\tau) := S_{\varphi_0}\big(S_{\tilde{\varphi}(\varepsilon,\tau)}x_0 + \varepsilon \tilde{z}(\varepsilon,\tau)\big).$$

The map $\hat{x}$ is C^1-smooth because the maps maps $\tilde{\varphi}$ and $\tilde{z}$ are C^1-smooth. Moreover,

$$\inf_{\psi\in\mathbb{R}} \|\hat{x}(\varepsilon,1+\varepsilon\tau) - S_\psi x_0\|_\infty \leq \inf_{\psi\in\mathbb{R}} \|(S_{\varphi_0 + \tilde{\varphi}(\varepsilon,\tau)} - S_\psi)x_0\|_\infty + \varepsilon\|\tilde{z}(\varepsilon,\tau)\|_\infty = \varepsilon\|\tilde{z}(\varepsilon,\tau)\|_\infty,$$

i.e., assertion (1.14) of Theorem 1.2 is proved.

Let us fix $\tau \in [\tau_0 - \varepsilon_0, \tau_0 + \varepsilon_0]$ and define $\hat{\varphi}(\tau) := \varphi_0 + \tilde{\varphi}(0,\tau)$. Then $\Phi(\hat{\varphi}(\tau)) = \tau$ (cf. (6.5)), and

$$\big\|\hat{x}(\varepsilon,1+\varepsilon\tau) - S_{\hat{\varphi}(\tau)}x_0\big\|_\infty = \Big\|\big(S_{\varphi_0+\tilde{\varphi}(\varepsilon,\tau)} - S_{\varphi_0+\tilde{\varphi}(0,\tau)}\big)x_0 + \varepsilon\tilde{z}(\varepsilon,\tau)\Big\|_\infty \to 0 \quad \text{for} \quad \varepsilon \to 0,$$

i.e., assertion (1.15) of Theorem 1.2 is proved also.

Finally, let us prove the uniqueness assertion of Theorem 1.2 (i). We have to show that for any solution $(\varepsilon, T, x) \in (0,\infty)^2 \times L^\infty_{\text{per}}$ to (2.4) with $T = 1 + \varepsilon\tau$ and $\varepsilon \approx 0$, $\tau \approx \tau_0$, $x \approx S_{\varphi_0}x_0$ it holds $x = \hat{x}(\varepsilon, T)$.

Let $(\varepsilon_k, T_k, x_k) \in (0,\infty)^2 \times L^\infty_{\text{per}}$, $k \in \mathbb{N}$, be a sequence of solutions to (2.4) with $T_k = 1 + \varepsilon_k\tau_k$ and

$$\lim_{k\to\infty}\big(\varepsilon_k + |\tau_k - \tau_0| + \|x_k + S_{\varphi_0}x_0\|_\infty\big) = 0.$$

Then for large k we have $x_k = S_{\varphi_0}\left(S_{\varphi_k}x_0 + \varepsilon_k z_k\right)$ with $z_k \in Y$ and $\|z_k\|_\infty \leq$ const and $\varphi_k \to 0$ (cf. Lemmas 3.1 and 4.2). Hence, for large k we get $\mathcal{H}(\varepsilon_k,\tau_k,\varphi_k,z_k) = 0$ (cf. (6.2)), i.e.,

$$L_{\varphi_k}\mathcal{H}(\varepsilon_k,\tau_k,\varphi_k,z_k) = 0.$$

In particular, the equation $\mathcal{H}_2(\varepsilon_k,\tau_k,\varphi_k,z_k) = 0$ yields

$$\left((I-P)\left(I + \int_0^1 \partial_x\mathcal{F}(1, S_{\varphi_k}x_0 + r\varepsilon_k z_k)dr\right)\right)z_k =$$

$$= -(I-P)\big(\tau_k\partial_T\mathcal{F}(1, S_{\varphi_k}x_0 + \varepsilon_k z_k) + S_{-\varphi_0}\mathcal{G}(1+\varepsilon_k\tau_k)\big),$$

i.e., $\|z_k - z_0\|_\infty \to 0$ (cf. (6.6)). Here we used that

$$\lim_{k\to\infty}\left\|(I-P)\left(I+\int_0^1 \partial_x\mathcal{F}(1,S_{\varphi_k}x_0+r\varepsilon_k z_k)dr\right)-I-\partial_x\mathcal{F}(1,x_0)\right\|_{\mathcal{L}(L^\infty_{\mathrm{per}})}=0,$$

and, hence, for large k the operator $(I-P)\left(I+\int_0^1 \partial_x\mathcal{F}(1,S_{\varphi_k}x_0+r\varepsilon_k z_k)dr\right)$ is bijective from Y onto $\operatorname{im}\left(I+\partial_x\mathcal{F}(1,x_0)\right)$, and its inverse is bounded with respect to the operator norm (uniformly with respect to k).

Let us summarize: We got that $(\varepsilon_k,\tau_k,\varphi_k,z_k)$ is a solution to (6.4), which is, for large k, close to the solution $(0,\tau_0,0,z_0)$. Hence, the uniqueness assertion of the implicit function theorem yields $\varphi_k=\tilde{\varphi}(\varepsilon_k,\tau_k)$, $z_k=\tilde{z}(\varepsilon_k,\tau_k)$, i.e., $x_k=\hat{x}(\varepsilon_k,T_k)$.

References

1. *Chillingworth D.* Bifurcation from an orbit of symmetry // Singularities and Dynamical Systems / Ed. S. N. Pnevmatikos. – Amsterdam: North-Holland Math. Stud., 1985. – P. 285 – 294.
2. *Chillingworth D.* Generic multiparameter bifurcation from a manifold // Dyn. Stab. Syst. – 2000. – **15**. – P. 101 – 137.
3. *Dancer E. N.* Breaking of symmetries for forced equations // Math. Ann. – 1983. – **262**. – P. 472 – 485.
4. *Hale J. K.* Bifurcation near families of solutions // Different. Equat. Proc. Uppsala Int. Conf. / Eds G. Berg, M. Essen and A. Pleijl. – Stockholm: Acta Univ. Upsaliensis, 1977. – P. 91 – 100.
5. *Hale J. K., Táboas P.* Interaction of damping and forcing in a second order equation // Nonlinear Anal. TMA. – 1978. – **2**. – P. 77 – 84.
6. *Levinson N.* Small periodic perturbations of an autonomous system with a stable orbit // Ann. Math. – 1950. – **52**. – P. 727 – 738.
7. *Loud W. S.* Periodic solutions of a perturbed autonomous system // Ann. Math. – 1959. – **70**. – P. 490 – 529.
8. *Novicenko V., Pyragas K.* Phase reduction of weakly perturbed limit cycle oscillations in time-delay systems // Phys. D. – 2012. – **241**. – P. 1090 – 1098.
9. *Recke L.* Forced frequency locking of rotating waves // Ukr. Math. J. – 1998. – **50**, № 1. – P. 94 – 101.
10. *Recke L., Peterhof D.* Abstract forced symmetry breaking and forced frequency locking of modulated waves // J. Different. Equat. – 1998. – **144**. – P. 233 – 262.
11. *Recke L., Samoilenko A. M., Teplinsky A., Tkachenko V., Yanchuk S.* Frequency locking by external forcing in systems with rotational symmetry // Discrete Contin. Dyn. Syst. – 2011. – **31**. – P. 847 – 875.
12. *Recke L., Samoilenko A. M., Tkachenko V., Yanchuk S.* Frequency locking of modulated waves // SIAM J. Appl. Dyn. Syst. – 2012. – **11**. – P. 771 – 800.
13. *Samoilenko A. M., Recke L.* Conditions for synchronization of an oscillatory system // Ukr. Math. J. – 2005. – **57**, № 8. – P. 1089 – 1119.
14. *Táboas P.* Periodic solutions for a forced Lotka – Volterra equation // J. Math. Anal. and Appl. – 1987. – **124**. – P. 82 – 97.
15. *Vanderbauwhede A.* Symmetry and bifurcation near families of solutions // J. Different. Equat. – 1980. – **36**. – P. 173 – 187.
16. *Vanderbauwhede A.* Local bifurcation and symmetry // Res. Notes Math. – Boston: Pitman, 1982. – **75**.
17. *Vanderbauwhede A.* Note on symmetry and bifurcation near families of solutions // J. Different. Equat. – 1983. – **47**. – P. 99 – 106.

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