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EXISTENCE OF GLOBAL SOLUTIONS FOR SOME CLASSES OF INTEGRAL EQUATIONS

ІСНУВАННЯ ГЛОБАЛЬНИХ РОЗВ'ЯЗКІВ ДЕЯКИХ КЛАСІВ ІНТЕГРАЛЬНИХ РІВНЯНЬ

We study the existence of L^p-solutions for a class of Hammerstein integral equations and neutral functional differential equations involving abstract Volterra operators. Using compactness-type conditions, we establish the global existence of solutions. In addition, a global existence result for a class of nonlinear Fredholm functional integral equations involving abstract Volterra equations is given.

Вивчається існування L^p-розв'язків для класу інтегральних рівнянь Гаммерштейна та нейтральних функціональних диференціальних рівнянь з абстрактними операторами Вольтерра. Існування глобальних розв'язків встановлено за допомогою умов типу компактності. Крім того, наведено результат про глобальне існування розв'язку для класу нелінійних функціональних інтегральних рівнянь Фредгольма з абстрактними операторами Вольтерра.

1. Introduction. Many problems arising in modeling real world phenomena lead to mathematical models described by nonlinear integral equations in abstract spaces. The theory of nonlinear integral equations in abstract spaces, is a relatively old theory, but it is also current and has important applications in physics, engineering and biology. The concept of abstract Volterra operator (or causal operator), introduced by [47] and [46], plays an important role in physics and engineering [25, 42]. This concept arises naturally in classes of differential equations and integral equations such as ordinary differential equations, integro-differential equations, differential equations with finite or infinite delay, Volterra integral equations, neutral functional equations, and so on.

Let E be a real Banach space, $L^p([0,a],E)$ be the space of all (classes of) strongly measurable and Bochner integrable functions $u\colon [0,a]\to E$, and $\mathcal{L}(E)$ the space of all bounded linear operators from E into itself. In this paper, we consider the Hammerstein integral equation

$$u(t)=(\mathfrak{P}u)(t)+\lambda\int\limits_0^a K(t,s)(\mathfrak{Q}u)(s)ds, \qquad \text{a.e.} \quad t\in[0,a], \tag{1.1}$$

and the Volterra – Hammerstein integral equation

$$u(t)=(\mathfrak{P}u)(t)+\int\limits_0^t K(t,s)(\mathfrak{Q}u)(s)ds, \qquad \text{a.e.} \quad t\in[0,a], \tag{1.2}$$

where $\mathfrak{P},\mathfrak{Q}\colon L^p([0,a],E)\to L^p([0,a],E)$ are continuous abstract Volterra operators, $K\colon[0,a]\times\times[0,a]\to\mathcal{L}(E)$ is strongly measurable, $\lambda\in\mathbb{R}$, and we provide conditions under which these equations have solutions in $L^p([0,a],E)$. In addition, under suitable conditions we establish the existence of continuous solutions for the following nonlinear Fredholm functional-integral equation:

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$$x(t) = x_0(t) + \int_0^a F(t, s, (\mathfrak{Q}x)(s))\, ds, \quad t \in [0, a],$$

where $F(\cdot, \cdot, \cdot) \colon [0, a] \times [0, a] \times Y \to X$ is a Carathéodory function, $\mathfrak{Q} \colon C([0, a], X) \to L^\infty([0, a], Y)$ is a continuous causal operator, $x_0(\cdot) \in C([0, a], X)$, and X, Y are infinite dimensional spaces.

We recall that an operator $Q \colon L^p([0, a], E) \to L^p([0, a], E)$ is called an *abstract Volterra operators* (or a *causal operator*) if, for each $\tau \in [0, a)$ and for all $u, v \in L^p([0, a], E)$ with $u(t) = v(t)$ for every $t \in [0, \tau]$, we have $Qu(t) = Qv(t)$ for a.e. $t \in [0, \tau]$.

The study of differential equations involving abstract Volterra operators can be found in the monographs [10, 19, 32, 40], and also in the papers [1, 2, 4, 11, 12, 14, 24, 34, 35, 37, 38, 41, 48, 50, 51]. The existence of L^p-solutions for different classes of differential equations and integral equations were studied in [3, 6–9, 16, 26, 30, 31, 33, 36, 39, 43].

2. Preliminaries. Let E be a real Banach space endowed with the norm $\|\cdot\|$. If A is a nonempty subset in E, then $\overline{A}$, $\operatorname{conv}(A)$ and $\overline{\operatorname{conv}}(A)$ denote the closure of A, the convex hull of A and the closure of the convex hull of A, respectively. We denote by $C([0, a], E)$ the Banach space of continuous bounded functions from $[0, a]$ into E endowed with the norm $\|u(\cdot)\| = \sup_{0 \le t \le a} \|u(t)\|$. The space of all (classes of) strongly measurable functions $u \colon [0, a] \to E$ such that

$$\|u\|_p := \left(\int_0^a \|u(t)\|^p \right)^{1/p} < \infty$$

for $1 \le p < \infty$, will be denoted by $L^p([0, a], E)$. Then $L^p([0, a], E)$ is a Banach space with respect to the norm $\|u\|_p$. Also, we denote by $L^\infty([0, a], E)$ the space of all (classes of) strongly measurable functions $u(\cdot) \colon [0, a] \to E$ which are essentially bounded on $[0, a]$. Then $L^\infty([0, a], E)$ is a Banach space with respect to the norm

$$\|u\|_\infty := \operatorname*{ess\,sup}_{t \in [0,a]} \|u(t)\| = \inf\{M \ge 0; \|u(t)\| \le M \quad \text{for a.e.} \quad t \in [0, a]\}.$$

We recall that, if $1 \le p < q \le \infty$, then

$$L^q([0, a], E) \subset L^p([0, a], E)$$

and

$$\|u\|_p \le a^{1/p - 1/q} \|u\|_q \quad \text{for every} \quad u(\cdot) \in L^q([0, a], E).$$

In the following, for a given $p \ge 1$, we shall denote by $p' \ge 1$ its conjugate; that is, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We denote the space of all bounded linear operators acting on a Banach space E by $\mathcal{L}(E)$. Then $\mathcal{L}(E)$ is a Banach space with respect to the norm

$$\|T\| := \inf\{M \ge 0; \|Tu\| \le M\|u\| \ \text{ for all } \ u \in E\}, \quad T \in \mathcal{L}(E).$$

We denote by $\beta(A)$ the Hausdorff measure of non-compactness of a nonempty bounded set $A \subset E$, and it is defined by [27]:

$$\beta(A) = \inf\{\varepsilon > 0; \ A \text{ admits a finite cover by balls of radius} \le \varepsilon\}.$$

The Kuratowski measure of non-compactness of a nonempty bounded set $A \subset E$ is defined by [29]:

$$\alpha(A) = \inf\Big\{\delta > 0;\ \ A \ \text{ can be expressed as the union of a finite number of sets}$$
$$\text{such that the diameter of each set does not exceed } \ \delta\Big\},$$

where the diameter of a bounded set $A \subset E$ is defined by $\dim(A) = \sup\{\|x-y\|; x, y \in A\}$.

Let $\gamma(\cdot)$ be either $\alpha(\cdot)$ or $\beta(\cdot)$. If A, B are bounded subsets of E, then (see [5, 27]):

(1) $\gamma(A) = 0$ if and only if $\overline{A}$ is compact;

(2) $\gamma(A) = \gamma(\overline{A}) = \gamma(\overline{\mathrm{conv}}(A))$;

(3) $\gamma(\lambda A) = |\lambda|\gamma(A)$ for every $\lambda \in \mathbb{R}$;

(4) $\gamma(A) \le \gamma(B)$ if $A \subset B$;

(5) $\gamma(A+B) \le \gamma(A) + \gamma(B)$;

(6) if $T : E \to E$ is a bounded linear operator, then $\gamma(TA) \le \|T\|\gamma(A)$;

(7) if $\{A_n\}_{n\ge 1}$ is a decreasing sequence of bounded closed nonempty subsets of E and $\lim\limits_{n\to\infty} \gamma(A_n) = 0$, then $\bigcap_{n=1}^{\infty} A_n$ is a nonempty and compact subset of E [29].

***Remark* 2.1.** In general, for any bounded set $A \subset E$, one has $\beta(A) \le \alpha(A) \le 2\beta(A)$ and both inequalities can be strict. Also, for any bounded set $A \subset E$, we have that $\gamma(A) \le \dim(A)$ and $\gamma(A) \le 2d$ if $\sup_{x\in A}\|x\| \le d$.

We recall the following lemma due to Heinz [21].

Lemma 2.1. *Let $\{u_n(\cdot); n \ge 1\}$ be a sequence in $L^1([0,a], E)$ such that there exists $m(\cdot) \in L^1([0,a], \mathbb{R}_+)$ with $\|u_n(t)\| \le m(t)$ for each $n \ge 1$ and for a.e. $t \in [0,a]$. Then the function $t \mapsto \psi(t) := \gamma(\{u_n(t); n \ge 1\})$ is integrable on $[0,a]$ and, for each $t \in [0,a]$, we have*

(a) (*Heinz* [21])

$$\alpha\left(\left\{\int_0^t u_n(s)ds; n \ge 1\right\}\right) \le 2\int_0^t \psi(s)ds,$$

(b) (*Kisielewicz* [28], Lemma 2.2)

$$\beta\left(\left\{\int_0^t u_n(s)ds; n \ge 1\right\}\right) \le \int_0^t \psi(s)ds,$$

provided that E is a separable banach space.

In the following, we let $\alpha_p(\cdot)$ denote the Kuratowski measures of noncompactness of sets in the space $L^p([0,a], E)$.

Lemma 2.2. *Let $1 \le p < \infty$ and let $V \subset L^p([0,a], E)$ be a countable set such that there exists $m(\cdot) \in L^1([0,b], \mathbb{R}_+)$ with $\|u(t)\| \le m(t)$ for each $u(\cdot) \in A$ and for a.e. $t \in [0,a]$.*

(a) [43, 44] *If*

$$\lim_{h\to 0}\sup_{u\in A}\int_0^a \big\|u(t+h) - u(t)\big\|^p dt = 0, \tag{2.1}$$

then

$$\alpha_p(A) \le 2 \left(\int_0^a [\alpha(V(t))]^p \, dt \right)^{1/p}.$$

(b) [20] (Theorem 1.2.8) *The set* V *is relatively compact in* $L^p([0,a],E)$ *if and only if* (2.1) *is satisfied and* $V(t)$ *is relatively compact in* E *for a.e.* $t \in [0,a]$.

3. A global existence results for Hammerstein integral equations. Let p and q be real numbers such that $q > p \ge 1$ and $p\left(1 - \dfrac{1}{q}\right) > 1$. We also assume that

(H_1) $\mathfrak{P}, \mathfrak{Q} \colon L^p([0,a],E) \to L^p([0,a],E)$ are continuous operators such that there exist $b(\cdot), c(\cdot) \in L^p([0,a],\mathbb{R}_+)$ and $d > 0$ with

$$\|(\mathfrak{P}u)(t)\| \le b(t) \qquad \text{and} \qquad \|(\mathfrak{Q}u)(t)\| \le c(t) + d\|u(t)\| \qquad \text{for a.e.} \quad t \in [0,a]$$

and for every $u(\cdot) \in L^p([0,a],E)$;

(H_2) K is a strongly measurable function from $[0,a] \times [0,a]$ into $\mathcal{L}(E)$ and

$$\operatorname*{ess\,sup}_{s \in [0,a]} \left(\int_0^a \|K(t,s)\|^q dt \right)^{1/q} := M < \infty.$$

Lemma 3.1. *If* (H_2) *holds, then*

$$\lim_{h \to 0} \int_0^a \left(\int_0^a \|K(t+h,s) - K(t,s)\|^q dt \right)^{1/q} ds = 0. \tag{3.1}$$

Proof. For a.e. $s \in [0,a]$, let us define the function $\psi_s(\cdot) \colon [0,a] \to \mathcal{L}(E)$ by $\psi_s(t) = K(t,s)$, $t \in [0,a]$. From (H_2) it follows that $\|\psi_s(\cdot)\|_q \in L^\infty([0,a],\mathbb{R}_+)$ and $\|\psi_s(\cdot)\|_q \le M < \infty$ for a.e. $s \in [0,a]$, so that $\psi_s(\cdot) \in L^q([0,a],\mathcal{L}(E))$ for a.e. $s \in [0,a]$. Let $\{h_n\}_{n \ge 1}$ be a sequence of real positive numbers such that $h_n \to 0$ as $n \to \infty$, and $t + h_n \in [0,a)$ for every $t \in [0,a)$ and $n \ge 1$. Also, for a.e. $s \in [0,a]$, let

$$\theta_n(s) := \left(\int_0^a \|\psi_s(t+h_n) - \psi_s(t)\|^q dt \right)^{1/q} =$$

$$= \left(\int_0^a \|K(t+h_n,s) - K(t,s)\|^q dt \right)^{1/q}, \quad n \ge 1.$$

Since $\psi_s(\cdot) \in L^q([0,a],\mathcal{L}(E))$ for a.e. $s \in [0,a]$, then from the fact that translations of L^p functions ($1 \le p < \infty$) are continuous in norm, we see that

$$\lim_{n \to \infty} \left(\int_0^a \|\psi_s(t+h_n) - \psi_s(t)\|^q dt \right)^{1/q} = 0 \qquad \text{for a.e.} \quad s \in [0,a],$$

so that $\lim_{n\to\infty} \theta_n(s) = 0$ for a.e. $s \in [0,a]$. On the other hand, since (H_2) implies

$$0 \le \theta_n(s) \le \|\theta_n\|_\infty \le 2M \qquad \text{for a.e.} \quad s \in [0,a] \quad \text{and all} \quad n \ge 1,$$

then, by the Dominated Convergence Theorem, we have $\lim_{n\to\infty} \int_0^a \theta_n(s)\, ds = 0$, so (3.1) is proved.

Lemma 3.2. *If* (H$_2$) *holds, then the function* $\xi(\cdot)\colon [0,a]\to\mathbb{R}_+$, *defined by*

$$\xi(t)=\left(\int\limits_0^a \|K(t,s)\|^{q'}ds\right)^{1/q'} \qquad \text{for a.e.} \quad t\in[0,a], \tag{3.2}$$

belongs to $L^q([0,a],\mathbb{R}_+)$. *Moreover,* $\|\xi\|_q\le Ma^{1/q'}$ *and* $\|\xi\|_p\le Ma^{1/p-1/q+1/q'}$.

Proof. From (H$_2$) and Tonelli's theorem it is easy to see that the function $\xi(\cdot)\colon[0,a]\to\mathbb{R}_+$ is measurable on $[0,a]$. Now, from $q>p$ and $p\left(1-\dfrac{1}{q}\right)>1$ it follows that $q'<p<q$; that is $q/q'>1$. Then, from (H$_2$) and the integral version of Minkowski's inequality, we have

$$\int\limits_0^a \xi^q(t)dt=\int\limits_0^a\left(\int\limits_0^a\|K(t,s)\|^{q'}ds\right)^{q/q'}dt\le$$

$$\le\left[\int\limits_0^a\left(\int\limits_0^a\|K(t,s)\|^{q}dt\right)^{q'/q}ds\right]^{q/q'}\le M^q a^{q/q'},$$

so that $\xi(\cdot)\in L^q([0,a],\mathbb{R}_+)$ and $\|\xi\|_q\le Ma^{1/q'}$. Since $p<q$, $\|\xi\|_p\le a^{1/p-1/q}\|\xi\|_q\le$ $\le Ma^{1/p-1/q+1/q'}$.

Theorem 3.1. *Let conditions* (H$_1$), (H$_2$) *be satisfied. Suppose that there exist* $k_1\in[0,1)$ *and* $k_2>0$ *such that*

$$\alpha((\mathfrak{P}A)(t))\le k_1\alpha(A(t)) \qquad \text{and} \qquad \alpha((\mathfrak{Q}A)(t))\le k_2\alpha(A(t)) \tag{3.3}$$

for $t\in[0,a]$ *and for each bounded subset* $A\subset L^p([0,a],E)$.

Then there exists a positive number λ_0 *such that for every* $\lambda\in R$ *with* $|\lambda|<\lambda_0$, *the integral equation* (1.1) *has at least one solution in* $L^p([0,a],E)$.

Proof. First, we show that each solution of (1.1) is a priori bounded in $L^p([0,a],E)$. Indeed, since

$$\|u(t)\|\le b(t)+|\lambda|\int\limits_0^a\|K(t,s)\|\|(\mathfrak{Q}u)(s)\|ds, \quad t\in[0,a],$$

then, using the Minkowski's inequality and the integral version of Minkowski inequality, we obtain

$$\|u\|_p\le\left(\int\limits_0^a|b(t)|^pdt\right)^{1/p}+|\lambda|\left[\int\limits_0^a\left(\int\limits_0^a\|K(t,s)\|\|(\mathfrak{Q}u)(s)\|ds\right)^pdt\right]^{1/p}\le$$

$$\le\|b\|_p+|\lambda|\int\limits_0^a\left[\int\limits_0^a\left[\|K(t,s)\|\|(\mathfrak{Q}u)(s)\|\right]^pdt\right]^{1/p}ds\le$$

$$\le\|b\|_p+|\lambda|\int\limits_0^a\|(\mathfrak{Q}u)(s)\|\left(\int\limits_0^a\|K(t,s)\|^pdt\right)^{1/p}ds.$$

Since $q > p$ then, using (H_1), (H_2) and Hölder's inequality, we get

$$\int_0^a \|(\mathfrak{Q}u)(s)\| \left(\int_0^a \|K(t,s)\|^p dt\right)^{1/p} ds \le$$

$$\le a^{1/p-1/q} \int_0^a \|(\mathfrak{Q}u)(s)\| \left(\int_0^a \|K(t,s)\|^q dt\right)^{1/q} ds \le$$

$$\le M a^{1/p-1/q} \int_0^a \|(\mathfrak{Q}u)(s)\| ds \le M a^{1/p-1/q} a^{1/p'} \left(\int_0^a \|(\mathfrak{Q}u)(s)\|^p ds\right)^{1/p} \le$$

$$\le M a^{1/q'} \left(\|c\|_p + d\|u\|_p\right),$$

so that

$$\|u(\cdot)\|_p \le \|b(\cdot)\|_p + |\lambda| M a^{1/q'} \left(\|c(\cdot)\|_p + d\|u(\cdot)\|_p\right).$$

Put

$$\lambda_0 := \min\left\{\frac{1}{dMa^{1/q'}}, \frac{1-k_1}{2k_2 a^{1/p-1/q}\|\xi\|_p}\right\},$$

where the function $\xi(\cdot)$ is defined in (3.2). Then for each $|\lambda| < \lambda_0$, we have $\|u\|_p \le r$, where $r := \gamma(1-\rho)^{-1}$, $\rho := |\lambda| dMa^{1/q'} < 1$ and $\gamma := \|b\|_p + |\lambda| Ma^{1/q'}\|c\|_p$, so that u bounded in $L^p([0,a],E)$. Moreover, we remark that $\|\mathfrak{Q}u\|_p \le \|c\|_p + dr$ if $\|u\|_p \le r$. We also notice that

$$\|u(t)\| \le b(t) + |\lambda| a^{1/p-1/q}(\|c\|_p + dr)\xi(t) \quad \text{for a.e.} \quad t \in [0,a];$$

that is, for every $u \in B$, we have

$$\|u(t)\| \le \varphi(t) \qquad \text{for a.e.} \quad t \in [0,a], \tag{3.4}$$

where $\varphi(t) = b(t) + |\lambda| a^{1/p-1/q}(\|c\|_p + dr)\xi(t)$, $t \in [0,a]$ and $B := \{u(\cdot) \in L^p([0,a],E); \|u\|_p \le$ $\le r\}$. Moreover, from Lemma 3.2 it follows that $\varphi(\cdot) \in L^p([0,a],\mathbb{R}_+)$, and

$$\|\varphi\|_p \le \|b\|_p + |\lambda| M a^{2(1/p-1/q)+1/q'}(\|c\|_p + dr). \tag{3.5}$$

Now, define the operator $\mathfrak{T}\colon L^p([0,a],E) \to L^p([0,a],E)$ by

$$(\mathfrak{T}u)(t) = (\mathfrak{P}u)(t) + \lambda \int_0^a K(t,s)(\mathfrak{Q}u)(s)ds, \quad t \in [0,a]. \tag{3.6}$$

As above, we can show that

$$\|(\mathfrak{T}u)(t)\| \le \varphi(t) \qquad \text{for a.e.} \quad t \in [0,a],$$

and

$$\|\mathfrak{T}u\|_p \le \|b\|_p + |\lambda| M a^{1/q'} \left(\|c\|_p + d\|u\|_p\right),$$

for every $u(\cdot) \in L^p([0,a],E)$, so that $\mathfrak{T}$ is well defined. Moreover, it is easy to see that $\mathfrak{T}(B) \subset B$; that is, $\mathfrak{T}$ is an operator from B into itself. Next, we show that $\mathfrak{T}$ is a continuous operator. For this, let $\{u_n(\cdot)\}_{n\geq 1}$ be a convergent sequence in $L^p([0,a],E)$ such that $u_n(\cdot) \to u(\cdot)$ as $n \to \infty$. Since

$$\| (\mathfrak{T}u_n)(t) - (\mathfrak{T}u)(t)\| \leq \| (\mathfrak{P}u_n)(t) - (\mathfrak{P}u)(t)\| +$$
$$+|\lambda| \int_0^a \|K(t,s)\| \|(\mathfrak{Q}u_n)(s) - (\mathfrak{Q}u)(s)\| ds$$

for every $t \in [0,a]$, then using Minkowski's inequality we have

$$\|\mathfrak{T}u_n - \mathfrak{T}u\|_p \leq \left(\int_0^a \| (\mathfrak{P}u_n)(t) - (\mathfrak{P}u)(t)\|^p dt\right)^{1/p} +$$
$$+|\lambda| \left[\int_0^a \left(\int_0^a \|K(t,s)\| \|(\mathfrak{Q}u_n)(s) - (\mathfrak{Q}u)(s)\| ds\right)^p dt\right]^{1/p}. \tag{3.7}$$

Now, using (H$_2$) and the integral version of Minkowski inequality, we obtain

$$\left[\int_0^a \left(\int_0^a \|K(t,s)\| \|(\mathfrak{Q}u_n)(s) - (\mathfrak{Q}u)(s)\| ds\right)^p dt\right]^{1/p} \leq$$
$$\leq \int_0^a \left[\int_0^a [\|K(t,s)\| \|(\mathfrak{Q}u_n)(s) - (\mathfrak{Q}u)(s)\|]^p dt\right]^{1/p} ds \leq$$
$$\leq \int_0^a \|(\mathfrak{Q}u_n)(s) - (\mathfrak{Q}u)(s)\| \left(\int_0^a \|K(t,s)\|^p dt\right)^{1/p} ds \leq$$
$$\leq a^{1/p-1/q} \int_0^a \|(\mathfrak{Q}u_n)(s) - (\mathfrak{Q}u)(s)\| \left(\int_0^a \|K(t,s)\|^q dt\right)^{1/q} ds \leq$$
$$\leq M a^{1/p-1/q} a^{1/p'} \left(\int_0^a \|(\mathfrak{Q}u_n)(s) - (\mathfrak{Q}u)(s)\|^p ds\right)^{1/p} =$$
$$= M a^{1/q'} \|\mathfrak{Q}u_n - \mathfrak{Q}u\|_p,$$

so that (3.7) become

$$\|\mathfrak{T}u_n - \mathfrak{T}u\|_p \leq \|\mathfrak{P}u_n - \mathfrak{P}u\|_p + M|\lambda| a^{1/q'} \|\mathfrak{Q}u_n - \mathfrak{Q}u\|_p.$$

Since $\mathfrak{P}$ and $\mathfrak{Q}$ are continuous operators, from the above inequality it follows that $\|\mathfrak{T}u_n - \mathfrak{T}u\|_p \to 0$ as $n \to \infty$, and so $\mathfrak{T}$ is a continuous operator. In the next step, we will show that

$$\lim_{h\to 0}\sup_{u\in B}\int\limits_0^a \big\|(\mathfrak{T}u)(t+h)-(\mathfrak{T}u)(t)\big\|^p dt = 0. \tag{3.8}$$

If $t \in [0,a]$ and $t+h\in[0,a]$, then for every $u(\cdot)\in B$ we have

$$\big\|(\mathfrak{T}u)(t+h)-(\mathfrak{T}u)(t)\big\| \le \big\|(Pu)(t+h)-(Pu)(t)\big\| +$$
$$+|\lambda|\int\limits_0^a \big\|K(t+h,s)-K(t,s)\big\|\|(\mathfrak{Q}u)(s)\|ds.$$

Using Minkowski's inequality, we obtain

$$J := \left(\int\limits_0^a \|(\mathfrak{T}u)(t+h)-(\mathfrak{T}u)(t)\|^p dt\right)^{1/p} \le \left(\int\limits_0^a \|(\mathfrak{P}u)(t+h)-(\mathfrak{P}u)(t)\|^p dt\right)^{1/p} +$$
$$+|\lambda|\left[\int\limits_0^a\left(\int\limits_0^a \|K(t+h,s)-K(t,s)\|\|(\mathfrak{Q}u)(s)\|ds\right)^p dt\right]^{1/p} = J_1+J_2. \tag{3.9}$$

Since $\mathfrak{P}u\in L^p([0,a],E)$, then from the fact that translations of L^p-functions $(1\le p<\infty)$ are continuous in norm, we see that $J_1\to 0$ as $h\to 0$.

Next, using the integral version of Minkowski inequality, we get

$$J_2 \le \int\limits_0^a\left(\int\limits_0^a \left[\|K(t+h,s)-K(t,s)\|\|(\mathfrak{Q}u)(s)\|\right]^p dt\right)^{1/p} ds =$$
$$= \int\limits_0^a \|(\mathfrak{Q}u)(s)\|\left(\int\limits_0^a \|K(t+h,s)-K(t,s)\|^p dt\right)^{1/p} ds \le$$
$$\le a^{1/p-1/q}\int\limits_0^a \|(\mathfrak{Q}u)(s)\|\left(\int\limits_0^a \|K(t+h,s)-K(t,s)\|^q dt\right)^{1/q} ds \le$$
$$\le a^{1/p-1/q}\left(\int\limits_0^a \|(\mathfrak{Q}u)(s)\|^{q'} ds\right)^{1/q'}\left[\int\limits_0^a\left(\int\limits_0^a \|K(t+h,s)-K(t,s)\|^q dt\right) ds\right]^{1/q} \le$$
$$\le a^{1/q'-1/q}\|\mathfrak{Q}u\|_p\left[\int\limits_0^a\left(\int\limits_0^a \|K(t+h,s)-K(t,s)\|^q dt\right) ds\right]^{1/q},$$

so that

$$J_2^q \le a^{q/q'-1}\left(\|c\|_p+dr\right)^q\int\limits_0^a\left(\int\limits_0^a \|K(t+h,s)-K(t,s)\|^q dt\right) ds.$$

Then, from Lemma 3.1, we have $J_2 \to 0$ as $h \to 0$. Therefore, from (3.9) it follows that

$$J^p = \int_0^a \|(\mathfrak{T}u)(t+h) - (\mathfrak{T}u)(t)\|^p dt \to 0 \qquad \text{as} \quad h \to 0,$$

uniformly with respect to $u \in B$, so that (3.8) is proved. Next, let A be a countable subset of B such that $A \subset \overline{\text{co}}((\mathfrak{T}A) \cup \{0\})$. We will use the compactness criteria from Lemma 2.2 to show that A is a relatively compact set in $L^p([0,a],E)$. First, from (3.8) we have

$$\lim_{h\to 0} \sup_{u\in A} \int_0^a \|u(t+h) - u(t)\|^p \, dt = 0. \tag{3.10}$$

Since A is a bounded set in $L^p([0,a],E)$ then, from (3.10) and Lemma 2.2, we have

$$\alpha_p(A) \le 2 \left(\int_0^a [\alpha(A(t))]^p \, dt \right)^{1/p}. \tag{3.11}$$

On the other hand, using the properties of the Kuratowski measures of noncompactness and (3.3), we have

$$\begin{aligned} \alpha(A(t)) &\le \alpha\left(\overline{\text{co}}((\mathfrak{T}A)(t) \cup \{0\})\right) = \alpha\left((\mathfrak{T}A)(t)\right) \le \\ &\le \alpha\left((\mathfrak{P}A)(t) + \lambda \int_0^a K(t,s)(\mathfrak{Q}A)(s)ds \right) \le \\ &\le \alpha\left((\mathfrak{P}A)(t)\right) + |\lambda| \alpha \left(\int_0^a K(t,s)(\mathfrak{Q}A)(s)ds \right) \le \\ &\le k_1 \alpha(A(t)) + |\lambda| \alpha \left(\int_0^a K(t,s)(\mathfrak{Q}A)(s)ds \right). \end{aligned} \tag{3.12}$$

Next, for each $u(\cdot) \in A$, the function $s \mapsto \|K(t,s)(\mathfrak{Q}u)(s)\|$ is measurable on $[0,t]$ for a.e. $t \in [0,a]$. From (3.4) it follows that

$$\big\|K(t,s)(\mathfrak{Q}u)(s)\big\| \le \|K(t,s)\| \big(c(s) + d\|u(t)\|\big) \le \|K(t,s)\| \left(c(s) + d\varphi(t)\right),$$

and consequently

$$\begin{aligned} \int_0^a \|K(t,s)(\mathfrak{Q}u)(s)\| ds \le \left(\int_0^a \|K(t,s)\|^{q'} ds \right)^{1/q'} \left(\int_0^a \left(c(s) + d\varphi(t)\right)^q ds \right)^{1/q} \le \\ \le a^{1/p - 1/q} \big(\|c\|_p + d\|\varphi\|_p\big) \xi(t), \end{aligned}$$

so that $s \mapsto \big\|K(t,s)(Qu)(s)\big\|$ belong to $L^1([0,a],\mathbb{R}_+)$ for a.e. $t \in [0,a]$. Hence, from Lemma 3.1, Hölder's inequality and Lemma 3.2, we have

$$\alpha\left(\int_0^a K(t,s)(\mathfrak{Q}A)(s)ds\right) \le 2\int_0^a \alpha\left(K(t,s)(\mathfrak{Q}A)(s)\right)ds \le$$

$$\le 2k_2\int_0^a \|K(t,s)\|\alpha\left((A)(s)\right)ds \le$$

$$\le 2k_2\left(\int_0^a \|K(t,s)\|^{q'}ds\right)^{1/q'}\left(\int_0^a [\alpha\left((A)(s)\right)]^q\,ds\right)^{1/q} \le$$

$$\le 2k_2a^{1/p-1/q}\left(\int_0^a [\alpha\left((A)(s)\right)]^p\,ds\right)^{1/p}\xi(t), \tag{3.13}$$

so that, from (3.12) and Lemma 2.2, we obtain

$$\left(\int_0^a [\alpha\left((A)(s)\right)]^p\,ds\right)^{1/p} \le k_1\left(\int_0^a [\alpha\left((A)(s)\right)]^p\,ds\right)^{1/p} +$$

$$+2k_2|\lambda|a^{1/p-1/q}\|\xi\|_p\left(\int_0^a [\alpha\left((A)(s)\right)]^p\,ds\right)^{1/p} \le$$

$$\le (k_1+2|\lambda|k_2a^{1/p-1/q}\|\xi\|_p)\left(\int_0^a [\alpha\left((A)(s)\right)]^p\,ds\right)^{1/p}. \tag{3.14}$$

Since $k_1+2|\lambda|k_2a^{1/p-1/q}\|\xi\|_p<1$, from the last inequality we obtain

$$\left(\int_0^a [\alpha\left((A)(s)\right)]^p\,ds\right)^{1/p} = 0$$

and thus, from (3.11) it follows that $\alpha_p(A)=0$; that is, A is a relatively compact set in $L^p([0,a],E)$. Summarizing, we have shown that $\mathfrak{T}\colon B\to B$ is a continuous operator with the property that for a countable subset A of B such that $A\subset\overline{\mathrm{co}}\big((\mathfrak{T}A)\cup\{0\}\big)$ we have that A is relatively compact. Since B is a closed and convex set in $L^p([0,a],E)$ then, by the Mönch fixed point theorem, it follows that there exists $u(\cdot)\in B$ such that $u=\mathfrak{T}u$; that is, the integral equation (1.1) has a least one solution $u(\cdot)\in B$.

Theorem 3.1 is proved.

Remark **3.1.** Suppose that $\lambda=1$ and the conditions (H_1), (H_2) are satisfied. If (3.3) holds for some $k_1,k_2\ge 0$ with $k_1+2k_2a^{1/p-1/q}\|\xi\|_p<1$, then from the above proof it is easy to see that the integral equation (1.1) has at least one solution in $L^p([0,a],E)$.

Theorem 3.2. *Let conditions* (H_1), (H_2) *be satisfied and suppose that* (3.3) *holds for some* $k_1,k_2\ge 0$ *with* $k_1+2k_2a^{1/p-1/q}\|\xi\|_p<1$. *Then the integral equation* (1.2) *has at least one solution in* $L^p([0,a],E)$.

Proof. If we put

$$K^*(t,s) := \begin{cases} K(t,s) & \text{if} \quad 0 \le s \le t \le a, \\ 0 & \text{otherwise,} \end{cases}$$

and $\lambda = 1$, then the integral equation (1.2) is equivalent to

$$u(t) = (\mathfrak{P}u)(t) + \int_0^a K^*(t,s)(\mathfrak{Q}u)(s)ds \qquad \text{a.e.} \quad t \in [0,a]. \tag{3.15}$$

Since K satisfies (H_2), it follows that K^* is a strongly measurable function from $[0,a] \times [0,a]$ into $\mathcal{L}(E)$,

$$\operatorname*{ess\,sup}_{s \in [0,a]} \left(\int_0^a \|K^*(t,s)\|^q dt \right)^{1/q} := M < \infty,$$

and

$$\lim_{h \to 0} \int_0^a \left(\int_0^a \|K^*(t+h,s) - K^*(t,s)\|^q dt \right)^{1/q} ds = 0.$$

Also, it is easy to check that the function $\xi^* : [0,a] \to \mathbb{R}_+$, defined by

$$\xi^*(t) = \left(\int_0^a \|K^*(t,s)\|^{q'} ds \right)^{1/q'} \qquad \text{for a.e.} \quad t \in [0,a],$$

belongs to $L^q([0,a], \mathbb{R}_+)$, $\|\xi^*\|_q \le M a^{1/q'}$ and $\|\xi^*\|_p \le M a^{1/p - 1/q + 1/q'}$. Then, by Remark 3.1, it follows that the integral equation (3.15) has at least one solution in $L^p([0,a], E)$, so that the integral equation (1.2) has at least one solution in $L^p([0,a], E)$.

Theorem 3.2 is proved.

***Remark* 3.2.** Suppose that there exist $m_0 > 0$, $k_2 > 0$ such that

$$\alpha((\mathfrak{P}A)(t)) \le m_0 \left(\int_0^t [\alpha(A(s))]^p \, ds \right)^{1/p} \qquad \text{and} \qquad \alpha((\mathfrak{Q}A)(t)) \le k_2 \alpha(A(t)) \tag{3.16}$$

for $t \in [0,a]$ and for each bounded subset $A \subset L^p([0,a], E)$. We notice that if there exists $m_1 > 0$ such that

$$\alpha((\mathfrak{P}A)(t)) \le m_1 \int_0^t \alpha(A(s))ds, \quad t \in [0,a],$$

then

$$\alpha((\mathfrak{P}A)(t)) \le m_1 a^{1/p'} \left(\int_0^t [\alpha(A(s))]^p \, ds \right)^{1/p},$$

so that $\mathfrak{P}$ satisfies (3.16) with $m_0 := m_1 a^{1/p'}$. Now, let A be a countable subset of B such that $A \subset \overline{\text{co}}((\mathfrak{T}A) \cup \{0\})$, where $\mathfrak{T}$ is defined by (3.6) and $B := \{u \in L^p([0,a], E); \|u\|_p \le r\}$. Then (3.12) becomes

$$\alpha(A(t)) \leq m_0 \left(\int_0^t [\alpha(A(s))]^p \, ds \right)^{1/p} + |\lambda| \alpha \left(\int_0^a K(t,s)(\mathfrak{Q}A)(s) ds \right) \tag{3.17}$$

for all $t \in [0,a]$. Then, by (3.13), (3.14) and (3.17), we obtain

$$\left(\int_0^a [\alpha\left((A)(s)\right)]^p \, ds \right)^{1/p} \leq m_0 \left(\int_0^a [\alpha\left((A)(s)\right)]^p \, ds \right)^{1/p} +$$

$$+2k_2 |\lambda| a^{1/p-1/q} \|\xi\|_p \left(\int_0^a [\alpha\left((A)(s)\right)]^p \, ds \right)^{1/p} \leq$$

$$\leq (m_0 + 2|\lambda| k_2 a^{1/p-1/q} \|\xi\|_p) \left(\int_0^a [\alpha\left((A)(s)\right)]^p \, ds \right)^{1/p}.$$

If $m_0 + 2|\lambda| k_2 a^{1/p-1/q} \|\xi\|_p < 1$, then the last inequality implies

$$\left(\int_0^a [\alpha\left((A)(s)\right)]^p \, ds \right)^{1/p} = 0.$$

Therefore, under conditions (H_1), (H_2) the result of Theorem 3.1 remains true if (3.16) holds for some $m_0 > 0$. Consequently, the result of Theorem 3.2 remains also true if (3.16) holds for some $m_0 > 0$ with $m_0 + 2k_2 a^{1/p-1/q} \|\xi\|_p < 1$.

4. Neutral functional differential equation. The aim of this section is to apply Theorem 3.2 to a class of neutral functional differential equations involving abstract Volterra equations. Some interesting results about neutral differential equations can be found in [17, 18, 22, 23]. In the following, we consider the neutral functional differential equation

$$\frac{d}{dt} [u(t) - (\mathfrak{C}u)(t)] = (\mathfrak{Q}u)(t) \qquad \text{for a.e.} \quad t \in [0,a], \tag{4.1}$$

together the initial conditions $u(0) = u_0$, where $\mathfrak{C}, \mathfrak{Q} \colon L^p([0,a],E) \to L^p([0,a],E)$ are continuous causal operators such that $(\mathfrak{C}u)(0) = \theta$ for every $u(\cdot) \in L^p([0,a],E)$.

A function $u(\cdot) \in L^p([0,a],E)$ is said to be a solution of (4.1) with initial condition $u(0) = u_0$ if $t \mapsto u(t) - (\mathfrak{C}u)(t)$ is an absolutely continuous function and satisfies (4.1) for a.e. $t \in [0,a]$. Note that $u(\cdot)$ itself may not be differentiable on the interval of existence. It is easy to see that if $u(\cdot)$ is a solution of equation (4.1), then it satisfies the integral equation

$$u(t) = (\mathfrak{P}u)(t) + \int_0^t (\mathfrak{Q}u)(s) \, ds \qquad \text{for a.e.} \quad t \in [0,a], \tag{4.2}$$

where $(\mathfrak{P}u)(t) := u_0 + (\mathfrak{C}u)(t),\ t \in [0,a]$. Conversely, if $u(\cdot) \in L^p([0,a],E)$ satisfies the integral equation (4.2), then $u(\cdot)$ is a solution of equation (4.1) with initial value $u(0) = u_0$. Let condition (H_1) be satisfied and suppose that there exists $k_1 \in [0,1)$ and $k_2 > 0$ such that $\mathfrak{C}$ and $\mathfrak{Q}$ satisfy (3.3).

Taking $K(t,s) = I$ for all $(t,s) \in \Delta := \{(t,s); 0 \le s \le t \le a\}$, by Theorem 3.2 it follows that the integral equation (4.2) has at least one solution in $L^p([0,a],E)$. A similar result was obtained by Corduneanu [10] (Section 6.4) in the finite dimensional case. For instance, the above result can be applied to the neutral functional differential equation

$$\frac{d}{dt}\left[u(t) - \int_0^t K(t,s)u(s)ds\right] = g(t,u(t)) \qquad \text{for a.e.} \quad t \in [0,a], \tag{4.3}$$

with the initial conditions $u(0) = u_0$, where $K\colon \Delta \to \mathcal{L}(E)$ satisfies (H_2) and $g\colon [0,a] \times E \to E$ is a Carathéodory function; that is,

(a) $g(t,\cdot) \in C(E,E)$ for each $t \in [0,a]$;

(b) $g(\cdot,u)$ is strongly measurable for each $u \in E$;

(c) There exist $m_g(\cdot) \in L^p([0,a],\mathbb{R}_+)$ and $d \ge 0$ such that

$$\|g(t,u)\| \le m_g(t) + d\|u\| \qquad \text{for every} \quad t \in [0,a] \qquad \text{and} \qquad u \in E.$$

Also, we assume that the following condition holds:

(H_3) $t \mapsto u(t) - \int_0^t K(t,s)u(s)\,ds$ is an absolutely continuous function on $[0,a]$.

Now, it is easy to see that if $u(\cdot)$ is a solution of equation (4.3), then it satisfies the following integral equation:

$$u(t) = (\mathfrak{P}u)(t) + \int_0^t (\mathfrak{Q}u)(s)\,ds \qquad \text{for a.e.} \quad t \in [0,a], \tag{4.4}$$

where

$$(\mathfrak{P}u)(t) := u_0 + \int_0^t K(t,s)u(s)ds, \quad t \in [0,a],$$

is a Volterra operator and

$$(\mathfrak{Q}u)(t) := g(t,u(t)), \quad t \in [0,a],$$

is the Nemitskii operator. Conversely, if $u(\cdot) \in L^p([0,a],E)$ satisfies the integral equation (4.4), then $u(\cdot)$ is a solution of equation (4.3) with initial value $u(0) = u_0$.

Theorem 4.1. *Suppose that* $K\colon \Delta \to \mathcal{L}(E)$ *satisfies* (H_2) *and* $g(\cdot,\cdot)\colon [0,a] \times E \to E$ *is a Carathéodory function such that there exists* $k_2 > 0$ *such that* $Ma^{1/p'+1/q'} + 2k_2 a^{1/p-1/q}\|\xi\|_p < 1$ *and*

$$\alpha(g(t,A)) \le k_2\alpha(A) \tag{4.5}$$

for $t \in [0,a]$ *and for each bounded subset* $A \subset E$. *If* (H_3) *hold, then the neutral functional differential equation* (4.3) *has at least one solution in* $L^p([0,a],E)$ *satisfying the initial condition* $u(0) = u_0$.

Proof. From (H_2) and Theorem 9.5.1 in [15] it follows that $\mathfrak{P}$ is a continuous operator from $L^p([0,a],E)$ into itself. If V is a bounded countable set in $L^p([0,a],E)$, then we have

$$\alpha((\mathfrak{P}V)(t)) \le \alpha\left(\int_0^t K(t,s)V(s)ds\right) \le \int_0^t \alpha\left(K(t,s)V(s)\right)ds \le$$

$$\le \int_0^t \|K(t,s)\|\alpha\left(V(s)\right)ds \le$$

$$\le \left(\int_0^t \|K(t,s)\|^q ds\right)^{1/q} \left(\int_0^t [\alpha\left(V(s)\right)]^{q'} ds\right)^{1/q'} \le$$

$$\le M a^{1/q'-1/p} \left(\int_0^t [\alpha\left(V(s)\right)]^p ds\right)^{1/p},$$

so that $\mathfrak{P}$ satisfies (3.8). Also, by (c) it follows that the Nemitskii operator $\mathfrak{Q}$ is a continuous operator from $L^p([0,a],E)$ into itself. Next, using (4.5), for any bounded and countable set in $L^p([0,a],E)$ we have $\alpha\left((\mathfrak{Q}V)(t)\right) = \alpha\left(g(t,V(t))\right) \le k_2\alpha\big(V(t)\big)$ for $t \in [0,a]$, so that $\mathfrak{Q}$ also satisfies (3.3). Consequently, (H_1), (H_2) and (3.3) are satisfied so that, by Remark 3.2, the neutral functional differential equation (4.3) has at least one solution in $L^p([0,a],E)$ satisfying the initial condition $u(0) = u_0$.

Theorem 4.1 is proved

5. A global existence result for nonlinear Fredholm functional integral equations. In this section we obtain a result on the global existence of solutions for a nonlinear Fredholm functional integral equation involving an abstract Volterra operator. A similar result was obtained by Warga [49] ([Theorem II.5.1]) in the finite dimensional case. If X, Y are given real separable Banach spaces, we denote by $C(Y,X)$ the Banach space of all continuous and bounded functions from Y into X endowed with the norm $\|f(\cdot)\|_{C(Y,X)} = \sup_{y\in Y}\|f(y)\|$. We shall identify two functions $g(\cdot,\cdot), h(\cdot,\cdot)\colon [0,a]\times Y \to X$ if $g(t,\cdot) = h(t,\cdot)$ a.e. on$[0,a]$, and we will denote by $\Omega := \Omega([0,a]\times$ $\times\, Y, X)$ the vector space of (equivalence classes of) all functions $g(\cdot,\cdot)\colon [0,a]\times Y \to X$ such that:

(c_1) $g(t,\cdot) \in C(Y,X)$ for each $t \in [0,a]$;

(c_2) $g(\cdot,y)$ is strongly measurable for each $y \in Y$;

(c_3) there exists a function $m_g(\cdot) \in L^p([0,a],\mathbb{R}_+)$ such that

$$\|g(t,\cdot)\|_{C(Y,X)} \le m_g(t) \qquad \text{for every} \quad t \in [0,a].$$

An element of Ω is called a Carathéodory function.

***Remark* 5.1.** It is easy to see that the function $t \mapsto \|g(t,\cdot)\|_{C(Y,X)}$ is Lebesgue integrable on $[0,a]$ for every $g(\cdot,\cdot) \in \Omega$. Moreover, the function $g \mapsto \|g\|_\Omega \colon \Omega \to \mathbb{R}_+$, given by

$$\|g\|_\Omega := \int_0^a \|g(t,\cdot)\|_{C(Y,X)}\,dt$$

is a norm on Ω. Also, for every $g(\cdot,\cdot) \in \Omega$ and for every strongly measurable functions $y(\cdot)\colon [0,a] \to Y$, the function $t \mapsto g(t, y(t))$ is Bochner integrable on $[0,a]$.

In the following, if $F(\cdot) \in C([0,a],\Omega)$ is a given function, then we will write $F(t,s,y)$ instead of $F(t)(s,y)$ for $(s,y) \in [0,a] \times Y$.

Consider the nonlinear Fredholm functional-integral equation

$$u(t) = u_0(t) + \int\limits_0^a F(t,s,(\mathfrak{Q}u)(s))ds, \quad t \in [0,a], \tag{5.1}$$

where $F(\cdot) \in C([0,a],\Omega)$, $\mathfrak{Q}\colon C([0,a],X) \to L^\infty([0,a],Y)$, and $u_0(\cdot) \in C([0,a],X)$ are assumed to satisfy the following assumptions:

(A_1) $\mathfrak{Q}\colon C([0,a],X) \to L^\infty([0,a],Y)$ is continuous and there exists $b>0$ and $0<c<1$ such that

$$\|\mathfrak{Q}u\|_\infty \le b(1+\|u(\cdot)\|)^c, \quad u(\cdot) \in C([0,a],X);$$

(A_2) there exist $0<d<1$ and an integrable function $h(\cdot,\cdot)\colon [0,a]\times[0,a] \to \mathbb{R}_+$ such that

$$\gamma := \sup_{0\le t\le a} \int\limits_0^a h(t,s)ds < \infty$$

and

$$\|F(t,s,y)\| \le h(t,s)(1+\|y\|)^d \qquad \text{for} \quad t,s \in [0,a] \qquad \text{and} \qquad y \in Y;$$

(A_3) there exist $k, k_0 > 0$ and $\psi(\cdot) \in L^1([0,a],\mathbb{R}_+)$ such that

$$\beta(F(t,s,B)) \le k\beta(B)$$

for all $t,s \in [0,a]$ and any bounded set $B \subset Y$, and

$$\beta((\mathfrak{Q}V)(t)) \le k_0\beta\big(V(t)\big)$$

for every $t \in [0,a]$ and every bounded set $V \subset C([0,a],X)$.

Theorem 5.1. *If assumptions* (A_1)–(A_3) *are satisfied,* $u_0(\cdot) \in C([0,a],X)$ *and* $kk_0 < 1$, *then the integral equation* (5.1) *has at least one solution in* $C([0,a],X)$.

Proof. Since $F(t,\cdot,\cdot) \in \Omega$ and the function $s \mapsto (\mathfrak{Q}u)(s)$ is strongly measurable on $[0,a]$ for each $u(\cdot) \in C([0,a],X)$, by Remark 5.1 it follows that the function $s \mapsto F(t,s,(\mathfrak{Q}u)(s))$ is Bochner integrable on $[0,a]$ for every $t \in [0,a]$, so that the operator

$$(Ku)(t) := u_0(t) + \int\limits_0^a F(t,s,(\mathfrak{Q}u)(s))ds, \quad t \in [0,a],$$

is well defined for every $u(\cdot) \in C([0,a],X)$. Since $0<c,d<1$, it is easy to check that, for a given $\overline{r} \ge \max\{1, \gamma(1+2b)^c\}$, we have $\gamma\big[1+b(1+\overline{r})^d\big]^c \le \overline{r}_0$. Let

$$W_r := \big\{u(\cdot) \in C([0,a],X); \|u(\cdot)\| \le r\big\},$$

where $r := \overline{r} + \|u_0(\cdot)\|$. First, we remark that, for every $u(\cdot) \in W_r$, we obtain

$$\|(\mathfrak{Q}u)(s)\| \leq b(1 + \|u(s)\|)^c \leq r_0 := b(1 + r)^c$$

for a.e. $s \in [0, a]$, so that $\mathfrak{Q}W_r \subset \{y(\cdot) \in L^\infty([0, a], Y); \|y(\cdot)\| \leq r_0\}$. From ($A_1$) and ($A_2$) it follows that, for each $u(\cdot) \in W_r$, we get

$$\|(Ku)(t)\| \leq \|u_0(\cdot)\| + \int_0^a \|F(t, s, (\mathfrak{Q}u)(s))\| ds \leq$$

$$\leq \|u_0(\cdot)\| + \int_0^a h(t, s)(1 + \|(\mathfrak{Q}u)(s)\|)^d ds \leq$$

$$\leq \|u_0(\cdot)\| + (1 + \|\mathfrak{Q}u\|_\infty)^d \sup_{0 \leq t \leq a} \int_0^a h(t, s) ds \leq$$

$$\leq \|u_0(\cdot)\| + \gamma \Big[1 + b(1 + \overline{r})^d\Big]^c \leq \|u_0(\cdot)\| + \overline{r} = r,$$

so that $Ku \in W_r$ for every $u(\cdot) \in W_r$. Since W_r is bounded and $KW_r \subset W_r$, KW_r is also bounded. Now, we show that K is a continuous operator on W_r. For this, let $\{u_n(\cdot)\}_{n \geq 1}$ be a sequence in W_r converging to some $u(\cdot) \in W_r$. Then by (A_1) we have that $\lim_{n \to \infty} (\mathfrak{Q}u_n)(s) = (\mathfrak{Q}u)(s)$ for a.e. $s \in [0, a]$. Also, since $F(t, \cdot, \cdot) \in \Omega$,

$$\lim_{n \to \infty} F(t, s, (\mathfrak{Q}u_n)(s)) = F(t, s, (\mathfrak{Q}u)(s))$$

and

$$\big\|F(t, s, (\mathfrak{Q}u_n)(s))\big\| \leq \sup_{\|y\| \leq r_0} \|F(t, s, y\| \leq (1 + r_0)^d h(t, s)$$

for each $t \in [0, a]$ and for a.e. $s \in [0, a]$, by the Lebesgue dominated convergence theorem we obtain

$$\lim_{n \to \infty} \int_0^a F(t, s, (\mathfrak{Q}u_n)(s) ds = \int_0^a F(t, s, (\mathfrak{Q}u)(s) ds$$

for each $t \in [0, a]$. Consequently, K is a continuous operator. Next, for every $t, s \in [0, a]$ and every $u(\cdot) \in W_r$, we have

$$\big\|(Ku)(t) - (Ku)(s)\big\| \leq \int_0^a \|F(t, \tau, (\mathfrak{Q}u)(\tau)) - F(s, \tau, (\mathfrak{Q}u)(\tau))\| d\tau \leq$$

$$\leq \int_0^a \sup_{y \in Y} \big\|F(t, \tau, y) - F(s, \tau, y)\big\| d\tau =$$

$$= \int_0^a \big\|F(t, \tau, \cdot) - F(s, \tau, \cdot)\big\|_{C(Y,X)} d\tau = \big\|F(t, \cdot, \cdot) - F(s, \cdot, \cdot)\big\|_\Omega.$$

Since $F(\cdot) \in C([0, a], \Omega)$, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left\|(Ku)(t)-(Ku)(s)\right\| \leq \left\|F(t,\cdot,\cdot)-F(s,\cdot,\cdot)\right\|_{\Omega} \leq \varepsilon$$

for all $t,s \in [0,a]$ with $|t-s| \leq \delta$ and for every $u(\cdot) \in W_r$, so that KW_r is equicontinuous. Next, put $W_0 := W_r$ and define $W_{n+1} = \overline{\operatorname{conv}}(KW_n)$, $n = 0,1,2,\ldots$.

Now, from $KW_0 \subset W_0$, it follows that

$$W_1 = \overline{\operatorname{conv}}(KW_0) \subset \overline{\operatorname{conv}}(W_0) = W_0,$$

and thus, $W_1 \subset C([0,a],X)$ is bounded, closed, convex and equicontinuous. By Mathematical induction it is easy to see that $W_{n+1} \subset W_n$ and $W_n \subset C([0,a],X)$ are bounded, closed, convex and equicontinuous for $n = 0,1,2,\ldots$. Next, since $C([0,a],X)$ is separable, then for each $n = 0,1,2,\ldots,$ there exists a countable set $V^n = \{v_k^n; k = 1,2,\ldots\} \subset C([0,a],X)$ such that $\overline{V^n} = W_n$. Then, by Lemma 3.1, the properties of the measure of noncompactness and (A_3), we have

$$\beta(W_{n+1}(t)) = \beta\big(\overline{\operatorname{conv}}((KW_n)(t))\big) = \beta\big((KW_n)(t)\big) = \beta\big((K\overline{V^n})(t)\big) \leq$$

$$\leq \beta\left(\int_0^a F(t,s,(\mathfrak{Q}\overline{V^n})(s))ds\right) \leq$$

$$\leq k\int_0^a \beta\left((\mathfrak{Q}\overline{V^n})(s)\right)ds \leq kk_0\int_0^a \beta\left((\overline{V^n}(s))\right)ds,$$

that is,

$$\beta\big(W_{n+1}(t)\big) \leq kk_0\int_0^a \beta\left(W_n(s)\right)ds, \quad t\in[0,b].$$

From a finite number of steps, we obtain

$$\beta(W_n(t)) \leq (kk_0)^n\int_0^a \beta\left(W_0(s)\right)ds, \quad t\in[0,b], \quad n\geq 1.$$

From $W_{n+1} \subset W_n$, $n = 0,1,2,\ldots,$ and property (4) of the measure of noncompactness, it follows that, for each $t \in [0,b]$, the sequence $\{\beta\left(W_n(t)\right)\}_{n\geq 0}$ is bounded and decreasing. Hence, there exists $h(t) := \lim_{n\to\infty}\beta\left(W_n(t)\right)$, $t\in[0,b]$. Taking $n\to\infty$ on both sides of the last inequality we get

$$h(t) = \lim_{n\to\infty}\beta\left(W_n(t)\right) \leq \lim_{n\to\infty}(kk_0)^n\int_0^a \beta\left(W_0(s)\right)ds = 0, \quad t\in[0,b],$$

and thus, $h(t) = \lim_{n\to\infty}\beta\left(W_n(t)\right) = 0$, $t\in[0,b]$. Since W_n, $n = 0,1,\ldots,$ are bounded and equicontiuous, it follows that $\lim_{n\to\infty}\beta_c\left(W_n\right) = 0$. By property (7) of the measure of noncompactness, it follows that $W := \bigcap_{n=0}^{\infty} W_n$ is a compact set of $C([0,a],X)$ and $KW \subset W$. Consequently, by the Schauder fixed point theorem, it follows that the operator K has at least one fixed point $u(\cdot)\in W$, which is a solution of (5.1).

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