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ON THE RIGIDITY OF RANK GRADIENT
IN A GROUP OF INTERMEDIATE GROWTH

ПРО ЖОРСТКІСТЬ ГРАДІЄНТНОГО РАНГУ
В ГРУПІ ПРОМІЖНОГО ЗРОСТАННЯ

We introduce and investigate a rigidity property of rank gradient for an example of a group $\mathcal{G}$ of intermediate growth constructed by the first author in [*Grigorčuk R. I.* On Burnside's problem on periodic groups // Funktsional. Anal. i Prilozhen. – 1980. – **14**, № 1. – P. 53 – 54]. It is shown that $\mathcal{G}$ is normally (f, g)-RG rigid, where $f(n) = \log(n)$ and $g(n) = \log(\log(n))$.

Введено та вивчено властивість жорсткості градієнтного рангу для прикладу групи $\mathcal{G}$ проміжного зростання, що була введена першим автором в роботі [*Grigorčuk R. I.* On Burnside's problem on periodic groups // Funktsional. Anal. i Prilozhen. – 1980. – **14**, № 1. – P. 53 – 54]. Показано, що $\mathcal{G}$ є нормально (f, g)-RG жорсткою, де $f(n) = \log(n)$ та $g(n) = \log(\log(n))$.

1. Introduction. A group is said to be residually finite if it has sufficiently many subgroups of finite index so that the intersection of them is trivial. This is important class of groups studied throughout more than eight decades by various tools and means. Residually finite groups are the reach source of examples in group theory. In particular they are often used in the three main branches of modern group theory: geometric group theory, asymptotic group theory and measured group theory. Such groups have realization by actions on spherically homogeneous rooted trees as indicated in [6, 8], which, in many cases, gives a possibility to study them and their subgroup structure using the structure of the tree. They are also closely connected to the theory of profinite groups.

A very important invariant of residually finite group is a subgroup growth introduced by F. Grunewald, D. Segal and G. Smith [11] and studied by many researches (see a comprehensive book [13] and the literature therein on this subject). Recently another asymptotic characteristics of residually finite groups were introduced with the focus on the notion of rank gradient.

The rank gradient of a finitely generated residually finite group G is defined as

$$RG(G) = \inf \frac{d(H) - 1}{[G : H]}, \tag{1}$$

where the infimum is taken over all subgroups H in G of finite index and $d(H)$ is the rank of H (i.e., the minimal number of generators of H). It is a finite number because a subgroup of finite index in a finitely generated group is finitely generated, and the first question that arises is $RG(G) = 0$ or not.

This notion, as well as the notion of the rank gradient relative to the descending chain of subgroups (defined by (2)), were introduced for the first time by M. Lackenby [12] with motivation from 3-dimensional topology. Since that the rank gradient and its variations were intensively studied [1, 4, 7, 8, 12].

The definition (1) can be modified in various directions. Instead of $\inf$ one can consider $\sup$, instead of all subgroups one can consider only normal subgroups, or subgroups with index a power of a prime number p, etc. Another direction of modifications is to consider descending sequence H_n, $n = 1, 2, \ldots$, of subgroups of finite index and associated sequence $rg(n)$ of numbers defined as

$$rg(n) = RG(G, \{H_n\}) = \frac{d(H_n) - 1}{[G : H_n]} \tag{2}$$

and its limit

$$\lim_{n \to \infty} rg(n)$$

if it exists, or the upper and the lower limits

$$\limsup_{n \to \infty} rg(n),$$

$$\liminf_{n \to \infty} rg(n)$$

otherwise.

It is known [1, 12] that if G is amenable (the notion of amenable group was introduced by von Neumann [14] and by Bogolyubov in topological case [3], and $\{H_n\}$ is a sequence of subgroups of finite index satisfying some technical condition (sometimes called the Farber condition, it is equivalent to the essential freeness of the action of the group on the boundary of coset tree [7]), then the limit (2) exists and is equal to 0. The example of the lamplighter group $\mathcal{L} = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ show that $rg(n)$ may have arbitrary fast decay, just because the group has a subgroup of index 2 isomorphic to itself, so iterating this fact one gets a descending sequence of groups of growing index power of 2 but a fixed rank $= 2$ [4].

We suggest the following definition. Let $f(n)$, $g(n)$ be two increasing functions of natural argument n taking values in $\mathbb{N}$ and caring the limit ∞ when $n \to \infty$.

Definition 1.1.   (a) *A finitely generated group G is $(f, g) - RG$-rigid if there is $C \in \mathbb{N}$ such that for every subgroup $H < G$ of finite index*

$$g(d(H)) < Cf(C[G : H]) \qquad and \qquad f([G : H]) < Cg(Cd(H)).$$

(b) *G is normally $(f, g) - RG$-rigid if previous inequalities hold for each normal subgroup $H \lhd G$ of finite index.*

For instance the free group F_r of rank $r \geq 1$ is $(f, g) - RG$-rigid where $f(n) = g(n) = n$ as in this case the ratio $(d(H) - 1)/([F_r : H])$ is constant and equal to $r - 1$. Also $(n, n) - RG$-rigid are all groups with $RG(G) > 0$. For finitely presented groups this hold if and only if G is "large" in the sense of S. Pride, i.e., contains a subgroup of finite index that surjects onto a noncommutative free group, as shown in [12].

We present the following two results. Let $\mathcal{G} = \langle a, b, c, d \rangle$ be an infinite 2-group constructed by the first author in [10]. Recall that it has intermediate growth between polynomial and exponential and has many other interesting properties [5, 6, 9]. $\mathcal{G}$ has many other ways to be defined but for us it will be important that it has a natural action by automorphisms of a rooted binary tree $\mathcal{T}$ shown by Fig. 1, as explained, for instance, in [6].
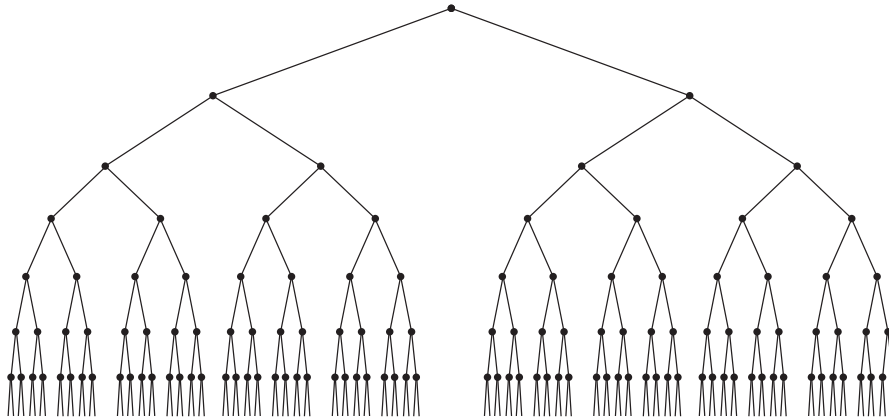
Fig. 1. Binary tree.

Theorem 1.1. *The group $\mathcal{G}$ is normally $(f, g) - RG$-rigid with*

$$f(n) = \log\log(n), \qquad g(n) = \log(n).$$

Moreover, there there is a constant $D > 1$ such that

$$\frac{1}{D}\log(d(H)) \leq \log\log([\mathcal{G} : H]) \leq D\log(d(H)) \tag{3}$$

hold for every nontrivial normal subgroup $H \lhd \mathcal{G}$.

Conjecture 1.1. *The group $\mathcal{G}$ is $(f, g) - RG$-rigid with the same functions f, g as in previous theorem.*

If the conjecture is true then we have interesting rigidity property concerning the rank and index of subgroups of finite index. At least we have this property for normal subgroups as shows the above theorem. Recall that all normal subgroups in $\mathcal{G}$ have finite index because $\mathcal{G}$ is just-infinite (i.e., infinite, but every proper quotients finite) as shown in [6]. Interestingly, the group $\mathcal{G}$ was used by M. Lackenby in [12] to demonstrate some phenomenon that may hold for the rank gradient. The present article develops the observation made by M. Lackenby concerning the rank gradient in $\mathcal{G}$.

At the moment we are able only to confirm the conjecture for important subclass of subgroups of finite index in $\mathcal{G}$, namely for stabilizers of vertices of the binary rooted tree $\mathcal{T}$ on which the group $\mathcal{G}$ acts. The vertices of $\mathcal{T}$ are in bijection with finite words over binary alphabet $\{0, 1\}$. Let v be a vertex and $\text{st}_{\mathcal{G}}(v)$ be its stabilizer which has index 2^n in $\mathcal{G}$ if v is a vertex of level n.

Theorem 1.2. *There is a constant D such that the inequalities* (3) *hold for all subgroups $H = \text{st}_{\mathcal{G}}(v)$, where v run over the set of vertices in $\mathcal{T}$. In fact*

$$\frac{d(H) - 1}{[\mathcal{G} : H]} = \frac{n + 3}{2^n}$$

if v is a vertex of level $n \geq 2$.

Observe that this result is announced in [4].

2. The group $\mathcal{G}$. We recall some basic facts about the group $\mathcal{G}$ and its subgroups. $\mathcal{G}$ can be defined as a group of automorphisms of a rooted binary tree $\mathcal{T}$ shown in Fig. 1 (the root, corresponding to the empty word, is a fixed point for the action). The generators a, b, c, d of $\mathcal{G}$ are involutions, the elements b, c, d commute and together with the identity element they constitute the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The stabilizer of the first level $H = \mathrm{st}_{\mathcal{G}}(1)$ is a subgroup of index 2 in $\mathcal{G}$ generated by elements b, c, d, b^a, c^a, d^a (where $x^y = x^{-1}yx$), and the restrictions of H on the left and right subtrees $\mathcal{T}_0$, $\mathcal{T}_1$ with the roots at vertices 0, 1 determine surjective homomorphisms $\varphi_0, \varphi_1 : H \to \mathcal{G}$. The direct product $\psi = \varphi_0 \times \varphi_1$ of them determines the embedding $H \to \mathcal{G} \times \mathcal{G}$ and acts on generators as:

$$b \to (a,c), \qquad c \to (a,d), \qquad d \to (1,b), \qquad b^a \to (c,a), \qquad c^a \to (d,a), \qquad d^a \to (b,1).$$
$$(4)$$

Together with the information that the generator a permute the subtrees $\mathcal{T}_0$, $\mathcal{T}_1$ (without extra action inside them), this uniquely determines the group $\mathcal{G}$.

The action of $\mathcal{G}$ on $\mathcal{T}$ also can be described by the following recursive rules:

$$a(0w) = 1w, \qquad a(1w) = 0w,$$
$$b(0w) = 0a(w), \qquad b(1w) = 1c(w),$$
$$c(0w) = 0a(w), \qquad c(1w) = 1d(w),$$
$$d(0w) = 0w, \qquad d(1w) = 1b(w),$$

where $w \in \{0,1\}^*$, and $\{0,1\}^*$ denotes the set of all finite words over the binary alphabet. The important property of the action of $\mathcal{G}$ on $\mathcal{T}$ is level transitivity, i.e., transitivity of the action on each level $V_n = \{0,1\}^n$.

Additionally to the stabilizers $\mathrm{st}_{\mathcal{G}}(n)$ of levels $n = 1, 2, \ldots$, an important descending series of normal subgroups is the series of rigid stabilizers $\mathrm{rist}_{\mathcal{G}}(n)$ which are subgroups generated by rigid stabilizers $\mathrm{rist}_{\mathcal{G}}(v)$, $v \in \{0,1\}^n$, of vertices of the nth level and $\mathrm{rist}_{\mathcal{G}}(v)$ is a subgroup in $\mathcal{G}$ fixing vertex v and consisting of elements acting trivially outside the subtree $\mathcal{T}_v$ in $\mathcal{T}$ with a root at v. The rigid stabilizers of distinct vertices of the same level commute and are conjugate (because of the level transitivity). Thus algebraically the $\mathrm{rist}_{\mathcal{G}}(n)$ is a direct product of copies of the same group (which may depend on the level n in general case). Observe that $\{\mathrm{st}_{\mathcal{G}}(n)\}_{n=1}^{\infty}$ and $\{\mathrm{rist}_{\mathcal{G}}(n)\}_{n=1}^{\infty}$ are descending chains of normal subgroups of finite index with trivial intersection. The structure of groups $\mathrm{st}_{\mathcal{G}}(n)$ and $\mathrm{rist}_{\mathcal{G}}(n)$ is well understood and described in [2].

Let $B = \langle b \rangle^{\mathcal{G}}$ be a normal closure of generator b and $K = \langle (ab)^2 \rangle^{\mathcal{G}}$. For each n there is a natural embedding $\psi_n : \mathrm{st}_{\mathcal{G}}(n) \to \mathcal{G} \times \ldots \times \mathcal{G}$ into a direct product of 2^n copies of $\mathcal{G}$ which is the nth iteration of the embedding ψ and has a geometric meaning of the attaching to the element $g \in \mathrm{st}_{\mathcal{G}}(n)$ the 2^n-tuple $(g_1, \ldots, g_{2^n})$ of its restrictions on the subtrees with the roots at the nth level. Instead writing

$$\psi_n(g) = (g_1, \ldots, g_{2^n})$$

we will write

$$g = (g_1, \ldots, g_{2^n}).$$

In particular the relations (4) can be rewritten as $b = (a,c)$, $c = (a,d)$, $d = (1,b)$, $b^a = (c,a)$, $c^a = (d,a)$, $d^a = (b,1)$.
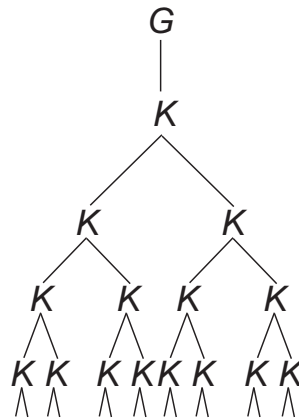
Fig. 2. Structure of branching subgroups.

The important facts about groups $\mathrm{st}_{\mathcal{G}}(n)$ and $\mathrm{rist}_{\mathcal{G}}(n)$ are that their ψ_{n-3} images have the decompositions:

$$\mathrm{st}_{\mathcal{G}}(3) \times \mathrm{st}_{\mathcal{G}}(3) \times \ldots \times \mathrm{st}_{\mathcal{G}}(3) \tag{5}$$

(products of 2^{n-3} copies of $\mathrm{st}_{\mathcal{G}}(3)$) when $n \geq 4$, and

$$K \times K \times \ldots \times K \tag{6}$$

(product of 2^n copies of K) when $n \geq 2$, respectively [2]. We will use later the notations K_n for $\mathrm{rist}_{\mathcal{G}}(n)$ when $n \geq 2$ and keep in mind their decompositions (5) and (6). We also denote by K_1 a subgroup in $\mathrm{st}_{\mathcal{G}}(n)$ whose ψ-image is $K \times K$. The group $\mathcal{G}$ is regularly branched over K as K_1 is a subgroup of K, and $[\mathcal{G} : K_1] < \infty$. Thus $\mathcal{G}$ contains subgroups shown by Fig. 2. To each level n corresponds a group K_n that fixes each vertex v of this level and whose restriction to the subtree $\mathcal{T}_v$ with the root v is K (if to identify $\mathcal{T}_v$ with $\mathcal{T}$). Moreover K_n is a direct product of this projections.

3. Proof of Theorem 1.1. We begin this section with reminding that the group $\mathcal{G}$ is branch group as defined in [2], because it acts level transitive on the tree $\mathcal{T}$ and rigid stabilizers $\mathrm{rist}_{\mathcal{G}}(n)$, $n = 1, 2, \ldots$, have finite index in $\mathcal{G}$. The branch structure of $\mathcal{G}$ that we will use is given by the Fig. 2 and was mentioned in previous section. The proof of the Theorem 1.1 is based on the following proposition.

Proposition 3.1. *Let $N \lhd \mathcal{G}$ be a nontrivial normal subgroup. Let n be a smallest nonnegative integer such that $N < \mathrm{st}_{\mathcal{G}}(n)$ but N is not a subgroup of $\mathrm{st}_{\mathcal{G}}(n+1)$. Then*
    (a) $\mathrm{st}_{\mathcal{G}}(n+5) < N$.
    (b) *If $n \geq 4$, then*

$$\mathrm{rist}_{\mathcal{G}}(n+3) < N < \mathrm{rist}_{\mathcal{G}}(n-3).$$

Proof. For the part (a) we first address the reader to the proof of the Theorem 4 given in [6] as we will follow the same line in our arguments. First, let us make a comparison of notations used in [6] and here. The branch structure for the group $\mathcal{G}$ is given in our case by the pair $\big(\{L_n\}, \{H_n\}\big)$, where $L_n = K$ and $H_n = K_n$ for all n. The reader should keep in mind the picture given by the Fig. 2.

Also, the normal subgroup in the statement of Theorem 4 is denoted by P while in the proposition under consideration it is denoted N. The proof of Theorem 4 from [6] (modulo of the change of notations) show that if $N < \operatorname{st}_{\mathcal{G}}(n)$ but N is not a subgroup of $\operatorname{st}_{\mathcal{G}}(n+1)$, then N contains commutator subgroup $K'_{n+1} = [K_{n+1}, K_{n+1}]$ (recall that $\operatorname{rist}_{\mathcal{G}}(n) = K_n$ if $n > 1$). By Proposition 9 from [6] $K' = K_2$, hence $K'_{n+1} = K \times K \times \ldots \times K$ (2^{n+3} factors) so $K'_{n+1} = K_{n+3}$, and $N > K_{n+3}$.

Also by Proposition 9 from [6] we have the inclusion $K > \operatorname{st}_{\mathcal{G}}(3)$ from which, together with the fact presented by factorization (5), we conclude that $N > \operatorname{st}_{\mathcal{G}}(n+5)$. The part (a) has been proven as well as the lower bound in part (b). To get part (b) under assumption that $n > 3$ we observe that the inclusion $K > \operatorname{st}_{\mathcal{G}}(3)$ and factorization (5) imply that $K_{n-3} > \operatorname{st}_{\mathcal{G}}(n)$, so we are done.

Proposition 3.1 is proved.

The group $\mathcal{G}$ is not virtually cyclic (by many reasons, for instance because it is finitely generated infinite torsion group). Let $N \triangleleft G$ be a nontrivial normal subgroup. It is automatically of finite index, as $\mathcal{G}$ is just-infinite group [6] (i.e., infinite group with every proper quotient finite). In fact we can assume from the beginning that N is a normal subgroup of finite index when proving Theorem 1.1 (so the just-infiniteness property is not needed). If n in the statement of Proposition 3.1 is less than 4, then there are only finitely many subgroups in $\mathcal{G}$ containing $\operatorname{st}_{\mathcal{G}}(5)$, their ranks are ≥ 2, and so there is a constant D satisfying the condition of the Theorem 1.1. Therefore we can assume that $n \geq 4$. Now apply part (b) of the proposition.

The quotient $\operatorname{rist}_{\mathcal{G}}(n-3) / \operatorname{rist}_{\mathcal{G}}(n+3) = K_{n-3}/K_{n+3}$ is isomorphic to

$$A := (K/K_6)^{2^{n-3}} = K/K_6 \times \ldots \times K/K_6$$

(2^{n-3} factors). The group K/K_6 is a finite 2-group of certain nilpotency class l (for us it is not important the exact value of l). Therefore A is nilpotent of the class l as well and it is generated by not more than $3 \cdot 2^{n-3}$ elements as K is 3-generated group [6]. It is well known that a subgroup of finitely generated nilpotent group is finitely generated and there is a universal upper bound on the ranks of subgroups of nilpotent group G in terms of $d(G)$ and class of nilpotency c of G.

We use the most simple upper bound given by the following lemma.

Lemma 3.1. *Let G be a finitely generated nilpotent group of class c. Then for every subgroup $H < G$ the upper bound*

$$d(H) < d(G)^c$$

holds.

Proof. If $c = 1$ then G is Abelian and thus $d(H) \leq d(G)$. For G of class c let $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$, $i = 1, 2, \ldots$, be the elements of the lower central series. Suppose G is generated by set S. Each factor $\gamma_i(G)/\gamma_{i+1}(G)$ is generated by the iterated commutators $[s_1, [s_2, \ldots]]$ of length $i+1$, where $s_j \in S$. Thus $d(\gamma_i(G)/\gamma_{i+1}(G)) \leq d(G)^{i+1}$. Denote $H_i = H \cap \gamma_i(G)$. Then $H_i\gamma_{i+1}(G)/\gamma_{i+1}(G)$ is an Abelian group, and thus $d(H_i\gamma_{i+1}(G)/\gamma_{i+1}(G)) \leq d(\gamma_i(G)/\gamma_{i+1}(G)) \leq$ $\leq d(G)^{i+1}$. Note that

$$d(H) \leq d(H\gamma_1(G)/\gamma_1(G)) + d(H \cap \gamma_1(G)) \leq d(G) + d(H_1).$$

Applying this iteratively we obtain

$$d(H) \leq d(G) + d(H_1) \leq d(G) + d(H_1\gamma_2(G)/\gamma_2(G)) + d(H_2) \leq$$

$$\leq d(G) + d(\gamma_1(G)/\gamma_2(G)) + d(H_2) \leq$$

$$\leq d(G) + d(G)^2 + d(H_2) \leq \ldots \leq d(G) + \ldots + d(G)^{c-1} \leq d(G)^c.$$

Lemma 3.1 is proved.

Using this lemma we get

$$d(N) < (3 \cdot 2^{n-3})^{l+2} = 3^{l+2} 2^{(n-3)(l+2)} < a_1 2^{a_2 n}$$

for some positive constants a_1, a_2.

Now we are going to give a lower bound for $d(N)$. We factorize the inclusions $K_{n+3} < N < < K_{n-3}$ by K'_{n+3} getting

$$K_{n+3}/K'_{n+3} < N/K'_{n+3} < K_{n-3}/K'_{n+3}.$$

The group K_{n-3}/K'_{n+3} is a direct product of 2^{n-3} copies of the group K/K_8 as $K = K_2$. K/K_8 is a finite 2-group. Let s be its nilpotency class, so K_{n-3}/K'_{n+3} also has nilpotency class s. As $\bar{N} := N/K'_{n+3}$ is a subgroup of K_{n-3}/K'_{n+3} its class of nilpotency is $\leq s$.

The group K_{n+3}/K'_{n+3} is a direct product of 2^{n+3} copies of the group $K/K' = K/K_2$. Let $t = d(K/K')$ (in fact $t = 3$). Then $d(K_{n+3}/K'_{n+3}) = t 2^{n+3}$. Using Lemma 3.1 we conclude

$$d\big(K_{n+3}/K'_{n+3}\big) = t 2^{n+3} \leq \big(d(N)\big)^{s+2}$$

from which we conclude that there are positive constants a_3, a_4 such that

$$a_3 2^{a_4 n} \leq d(N).$$

Now using the part (a) of the Proposition 3.1 we provide an upper and lower bounds for the index $[\mathcal{G} : N]$. For this purpose we use the fact that

$$\big[\mathcal{G} : \mathrm{st}_{\mathcal{G}}(n)\big] = 2^{5 \cdot 2^{n-3}+2}$$

as shown at the end of the proof of Theorem 14 from [6]. Hence part (a) of the proposition lead us to the existence of positive constants a_5, a_6 and constants a_7, a_8 such that

$$2^{a_5 2^n + a_7} \leq [\mathcal{G} : N] \leq 2^{a_6 2^n + a_8}.$$

Taking the double logarithm with base 2 of this inequalities and applying the same logarithm to the previously obtained inequalities

$$a_3 2^{a_4 n} \leq d(N) \leq a_1 2^{a_2 n}$$

a simple calculus lead us to the proving of the theorem.

4. Proof of Theorem 1.2. We begin the proof of Theorem 1.2. The groups R_n, Q_n and P_n which are defined below are the finite index analogs of groups R, Q, P introduced and studied in [2]. The recursive relations between them are analogous to the corresponding relations between R, Q, P given in [2] by Theorems 4.4 and 4.5. The proof is based on a number of computations that we split in propositions and lemmas.

Let us introduce the elements $t = (ab)^2$, $u = (bada)^2 = (t, 1)$, $v = (abad)^2 = (1, t)$. The direct computation shows the result of their conjugation and of element $(ac)^4$ by generators as shown in the next subsections:

4.1. Conjugates of t, u, v:

$$t^a = t^{-1},$$
$$t^b = t^{-1},$$
$$t^c = t^{-1}v, \tag{7}$$
$$t^d = v^{-1}t,$$
$$t^{dd^a} = v^{-1}tu;$$

$$u^a = v,$$
$$u^b = u^{-1},$$
$$u^c = u^{-1}, \tag{8}$$
$$u^d = u,$$
$$u^{dd^a} = u^{-1};$$

$$v^a = u,$$
$$v^b = t^{-1}v^{-1}t,$$
$$v^c = t^{-1}vt, \tag{9}$$
$$v^d = v^{-1},$$
$$v^{dd^a} = v^{-1}.$$

4.2. Conjugates of $(ac)^4$. Denote

$$x_0 = (ac)^4,$$
$$x_0^a = x_0,$$
$$x_0^b = (1, u)x_0,$$
$$x_0^c = x_0,$$
$$x_0^d = (1, u)x_0,$$
$$x_0^{dd^a} = (u, u)x_0.$$

Now let us introduce more elements and show the result of their conjugation.

4.3. Definition and conjugates of x_m, u_m, v_m. Let $x_m \in \text{rist}(1^m)$, such that $x_m(1^mw) = 1^m(ac)^4(w)$. Note that $x_0 = (ac)^4$. Analogously, $u_m, v_m \in \text{rist}(1^m)$, $u_m(1^mw) = 1^mu(w)$, $v_m(1^mw) = 1^mv(w)$. Then (proof by induction since $x_m = (1, x_{m-1})$, $u_m = (1, u_{m-1})$)

$$x_m^b = \begin{cases} u_{m+1}x_m, & \text{if } 3|m, \\ x_m, & \text{if } 3|(m-1), \\ u_{m+1}x_m, & \text{if } 3|(m-2), \end{cases} \tag{10}$$

$$x_m^c = \begin{cases} x_m, & \text{if } 3|m, \\ u_{m+1}x_m, & \text{if } 3|(m-1), \\ u_{m+1}x_m, & \text{if } 3|(m-2), \end{cases}$$

$$x_m^d = \begin{cases} u_{m+1}x_m, & \text{if } 3|m, \\ u_{m+1}x_m, & \text{if } 3|(m-1), \\ x_m, & \text{if } 3|(m-2), \end{cases} \tag{11}$$

$$x_0^{dd^a} = (u, u)x_0, \tag{12}$$

and if $m > 0$, then

$$x_m^{dd^a} = (1, x_{m-1}^b) = \begin{cases} u_{m+1}x_m, & \text{if } 3|m, \\ u_{m+1}x_m, & \text{if } 3|(m-1), \\ x_m, & \text{if } 3|(m-2), \end{cases} \tag{13}$$

$$x_1^{x_0} = (1, x_0^{dd^a}) = (1, 1, u, u)(1, x_0) = (1, 1, u, u)x_1. \tag{14}$$

And if $n + 1 < m$, then

$$x_m^{x_n} = (\ldots, x_{m-n-2}^b) = \begin{cases} x_m, & \text{if } 3|(m-n), \\ u_{m+1}x_m, & \text{if } 3 \nmid (m-n). \end{cases} \tag{15}$$

Here is the list of conjugates of u_m and v_m:

$$u_m^b = \begin{cases} u_m^{-1}, & \text{if } 3|m, \\ u_m^{-1}, & \text{if } 3|(m-1), \\ u_m, & \text{if } 3|(m-2), \end{cases} \tag{16}$$

$$u_m^{dd^a} = \begin{cases} u_m, & \text{if } 3|m, \\ u_m^{-1}, & \text{if } 3|(m-1), \\ u_m^{-1}, & \text{if } 3|(m-2), \end{cases} \tag{17}$$

$$v_m^b = \begin{cases} v_{m-1}^{-1}v_m^{-1}v_{m-1}, & \text{if } 3|m, \\ v_{m-1}^{-1}v_mv_{m-1}, & \text{if } 3|(m-1), \\ v_m^{-1}, & \text{if } 3|(m-2), \end{cases}$$

$$v_m^{dd^a} = \begin{cases} v_m^{-1}, & \text{if } 3|m, \\ v_{m-1}^{-1}v_m^{-1}v_{m-1}, & \text{if } 3|(m-1), \\ v_{m-1}^{-1}v_m v_{m-1}, & \text{if } 3|(m-2). \end{cases} \tag{18}$$

In the next two subsections we introduce sequences of subgroups R_n and Q_n, $n = 1, 2, \ldots,$ and prove some structural results about them as well as ranks (or dimension) of associated quotients (spaces).

4.4. Groups R_n. Let $R_1 = K = \langle t, u, v \rangle$. Let $R_n = (K \times R_{n-1})\{1, (ac)^4\}$ for $n \geq 2$. Then the following proposition is true.

Proposition 4.1.

$$R_2 = \big\langle x_0, u_0, u_1, v_0, (u, u) \big\rangle,$$

$$R_n = \big\langle x_0, \ldots, x_{n-2}, u_0, u_1, u_2, v_{n-2}, (u, u) \big\rangle$$

for $n \geq 3$.

Proof. Since $R_2 = (K \times K)\{1, x_0\}$, it follows that

$$R_2 = \big\langle x_0, (t, 1), (u, 1), (v, 1), (1, t), (1, u), (1, v) \big\rangle.$$

Now, by (7) $t^{dd^a} = v^{-1}tu$, and since $x_0 = (dd^a, dd^a)$, we have that $(t, 1)^{x_0} = (v, 1)^{-1}(t, 1)(u, 1)$, and the same for $(1, t)$. So we can discard $(v, 1)$, $(1, v)$ from the list. Notice also that multiplying $(u, 1)$ by $(1, u)$ we can replace $(u, 1)$ with (u, u). Since $u_0 = u = (t, 1)$, $u_1 = (1, u)$, and $v_0 = v = (1, t)$, we obtain the generators for R_2.

We have $R_3 = (K \times R_2)\{1, x_0\}$, hence R_3 is generated by

$$R_3 = \big\langle x_0, (t, 1), (u, 1), (v, 1), x_1 = (1, x_0), u_1 = (1, u_0), u_2 = (1, u_1), v_1 = (1, v_0), (1, 1, u, u) \big\rangle.$$

In the same way as for R_2 we can discard $(v, 1)$ and replace $(u, 1)$ by (u, u). By (14) $x_1^{x_0} = (1, 1, u, u)x_1$, and so we can discard $(1, 1, u, u)$, and we are done.

Now suppose

$$R_n = \big\langle x_0, \ldots, x_{n-2}, u_0, u_1, u_2, v_{n-2}, (u, u) \big\rangle.$$

Then from the formula $R_{n+1} = (K \times R_n)\{1, (ac)^4\}$ we obtain that R_{n+1} is generated by

$$\big\langle x_0, (t, 1), (u, 1), (v, 1), x_1, \ldots, x_{n-1}, u_1, u_2, u_3, v_{n-1}, (1, 1, u, u) \big\rangle.$$

As above, we can discard $(v, 1)$ and $(1, 1, u, u)$ and replace $(u, 1)$ with (u, u). It is left to note that $x_2^{x_0} = u_3 x_2$ by (15), so $u_3 = [x_0, x_2]$ and hence we can discard it from the list.

Proposition 4.1 is proved.

For any group G define $G^{(2)} = G/G(X^2)$, where $G(X^2)$ is the subgroup generated by all squares of elements in G. Then $G^{(2)}$ is elementary abelian 2-group, and so $\dim_{\mathbb{F}_2} G^{(2)}$ is defined.

Proposition 4.2. $\dim_{\mathbb{F}_2} R_n^{(2)} = n + 4$ for $n \geq 3$, $\dim_{\mathbb{F}_2} R_2^{(2)} = 5$, $\dim_{\mathbb{F}_2} K^{(2)} = 3$.

Proof. The equality for K follows from the theory of the group $\langle a, b, c, d \rangle$.

To prove the rest, we need the following obvious lemma.

Lemma 4.1. *Suppose $G = H \rtimes (\mathbb{Z}/2\mathbb{Z})$. Let $\alpha \colon H^{(2)} \to H^{(2)}$ be the operator on $H^{(2)}$ induced by the action of $\mathbb{Z}/2\mathbb{Z}$. Then $G^{(2)} = H^{(2)}/(1 + \alpha)(H^{(2)}) \oplus \mathbb{Z}/2\mathbb{Z}$.*

Note that by (7), (8), (9), dd^a induces the following action on $K^{(2)}$: $[t, u, v] \mapsto [t + u + v, u, v]$. Hence $(1 + dd^a)(K^{(2)}) = \{0, u + v\}$. Thus $\dim_{\mathbb{F}_2} R_2^{(2)} = 2 + 2 + 1 = 5$.

Now, it follows from (12), (13) that $(1 + dd^a)(x_0) = (u, u)$, $(1 + dd^a)(x_1) = u_2$ and $(1 + dd^a)(x_m) = 0$ for $m > 1$, since from (15) we have for $m \geq 3$ that $u_m = [x_{m-3}, x_{m-1}]$.

Also, from (17), (18), $(1 + dd^a)(u_m) = 0$, $(1 + dd^a)(v_m) = 0$ for all $m \geq 0$, and $(1 + dd^a)((u, u)) = 0$.

Thus $(1 + dd^a)(R_2^{(2)}) = \{0, (u, u)\}$, and so $\dim_{\mathbb{F}_2} R_3^{(2)} = 2 + 5 - 1 + 1 = 7$. Analogously, $(1 + dd^a)(R_n^{(2)}) = \langle (u, u), u_2 \rangle$ for $n \geq 3$, and so by induction $\dim_{\mathbb{F}_2} R_{n+1}^{(2)} = 2 + \dim_{\mathbb{F}_2} R_n^{(2)} - 2 + 1 = n + 5$.

Lemma 4.1 is proved.

Corollary 4.1. $d(R_1) = 3$, $d(R_2) = 5$, $d(R_n) = n + 4$ *for* $n \geq 3$.

4.5. Groups Q_n. Let $Q_1 = B = \langle b, t, u, v \rangle = K \rtimes \{1, b\}$ and $Q_n = (K \times R_{n-1})\langle b, (ac)^4 \rangle$ for $n > 1$.

Proposition 4.3. $Q_n = R_n \rtimes \{1, b\}$,

$$Q_2 = \langle b, x_0, u_0, v_0, (u, u) \rangle,$$

$$Q_n = \langle b, x_0, x_1, \ldots, x_{n-2}, u_0, u_2, v_{n-2}, (u, u) \rangle$$

for $n \geq 3$.

Proof. The proof is analogous to the proof of Proposition 4.1, using lists of generators for R_n from the statement of Proposition 4.1, and the additional fact that $x_0^b = u_1 x_0$, hence $u_1 = x_0^b x_0$.

Proposition 4.4. $\dim_{\mathbb{F}_2} Q_n^{(2)} = n + 4$ *for* $n \geq 3$, $\dim_{\mathbb{F}_2} Q_2^{(2)} = 5$, *and* $\dim_{\mathbb{F}_2} Q_1^{(2)} = 4$.

Proof. Note that to compute $\dim_{\mathbb{F}_2} Q_n^{(2)}$ we may use the Lemma 4.1, since $Q_n = R_n \rtimes \{1, b\}$. Thus we need to compute the induced action of b on $R_n^{(2)}$.

Note that $(1 + b)(K^{(2)}) = 0$, by (7), (8), (9). Also, $(1 + b)(x_0) = u_1$, $(1 + b)(x_m) = 0$ for $m \geq 1$ by (10). $(1 + b)(u_m) = 0$ for $m \geq 0$ by (16). Finally, $(1 + b)((u, u)) = (u, u) + (v, 1) + (1, u)$, and since $(t, 1)^{x_0} = (v, 1)^{-1}(t, 1)(u, 1)$, it follows that $(v, 1) + (u, 1) = 0$, and thus $(1 + b)((u, u)) = (u, u) + (v, 1) + (1, u) = 0$.

Hence $(1 + b)(R_n^{(2)}) = \langle u_1 \rangle$, and thus $\dim_{\mathbb{F}_2} Q_2^{(2)} = 5 - 1 + 1 = 5$, and $\dim_{\mathbb{F}_2} Q_n^{(2)} = n + 4 - 1 + 1 = n + 4$.

Corollary 4.2. $d(Q_1) = 4$, $d(Q_2) = 5$, $d(Q_n) = n + 4$ *for* $n \geq 3$.

Finally we introduce groups P_n.

4.6. Groups P_n. Let $P_n = \operatorname{st}_{\mathcal{G}}(1^n)$. Then $P_1 = \operatorname{st}_{\mathcal{G}}(1)$, and $P_n = (K \times Q_{n-1})\langle c, (ac)^4 \rangle$ for $n > 1$.

Proposition 4.5. $P_n = R_n \rtimes \{1, b, c, d\}$ *and*

$$P_n = \langle c, d, x_0, x_1, \ldots, x_{n-2}, u_0, v_{n-2}, (u, u) \rangle$$

for $n \geq 2$.

Proof. Analogous to the proofs of Propositions 4.1 and 4.3, using the additional fact that $x_1^d = u_2 x_1$ by (11).

Proposition 4.6.

$$\dim_{\mathbb{F}_2} P_n^{(2)} = n + 4 \tag{19}$$

for $n \geq 2$, *and* $\dim_{\mathbb{F}_2} P_1^{(2)} = 4$.

Proof. For $n = 1$ it follows from $P_1 = \mathrm{st}(1) = \langle d, c, d^a, c^a \rangle$. For $n \geq 2$ it follows from $P_n = R_n \rtimes \{1, b, c, d\}$ and a slight generalization of the Lemma 4.1: $P_n^{(2)} = \left(R_n^{(2)}/V_n \right) \oplus \mathbb{F}_2^2$, where $V_n = (1 + b)R_n^{(n)} + (1 + c)R_n^{(n)}$.

Now we are ready to prove Theorem 1.2. As $\mathcal{G}$ acts level transitive on $\mathcal{T}$, for each vertex v of the level n the group $\mathrm{st}_{\mathcal{G}}(v)$ is conjugate in $\mathcal{G}$ to P_n. From Proposition 4.5 it follows that the minimum number of generators of P_n is $\leq n + 4$. The last proposition shows that it is exactly $n + 4$ when $n \geq 2$. As index of P_n in $\mathcal{G}$ is 2^n (because of the level transitivity of $\mathcal{G}$) the ratio

$$\frac{d(H) - 1}{[\mathcal{G} : H]} = \frac{n + 3}{2^n}$$

for $H = \mathrm{st}_{\mathcal{G}}(v)$, where vertex v belongs to the level $n \geq 2$. If v is a vertex of the first level then $H = \mathrm{st}_{\mathcal{G}}(1)$ is a subgroup of index 2 and is 4-generated (by elements b, c, b^a, c^a). Taking all this into account we get the conclusion of the Theorem 1.2.

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