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A NOTE ON THE COEFFICIENT ESTIMATES FOR SOME CLASSES OF p-VALENT FUNCTIONS*

ЗАУВАЖЕННЯ ЩОДО КОЕФІЦІЄНТНИХ ОЦІНОК ДЛЯ ДЕЯКИХ КЛАСІВ p-ВАЛЕНТНИХ ФУНКЦІЙ

We obtain estimates of the Taylor – Maclaurin coefficients of some classes of p-valent functions. This problem was initially studied by Aouf in the paper "Coefficient estimates for some classes of p-valent functions" (Internat. J. Math. and Math. Sci. – 1988. – **11**. – P. 47 – 54). The proof given by Aouf was found to be partially erroneous. We propose the correct proof of this result.

Отримано оцінки для коефіцієнтів Тейлора – Маклорена для деяких класів p-валентних функцій. Ця задача була вперше розглянута Ауфом у роботі "Coefficient estimates for some classes of p-valent functions" (Internat. J. Math. and Math. Sci. – 1988. – **11**. – P. 47 – 54). Доведення, наведене Ауфом, виявилось частково помилковим. Ми пропонуємо коректне доведення цього результату.

1. Introduction. The concept of univalence has a natural extension as described in p-valent function theory. A functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

is said to be *p-valent* in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$. if it is analytic and assumes no value more than p times in $\mathbb{D}$ and there is some w such that $f(z) = w$ has exactly p solutions in $\mathbb{D}$, when roots are counted in accordance with their multiplicities. We let $\mathcal{S}_p$ denote the class of all functions that are analytic and p-valent in $\mathbb{D}$.

By definition, the function f is said to be *p-valent* (or *multivalent of order p*) in $\mathbb{D}$ if

$$f(z_1) = f(z_2) = \ldots = f(z_{p+1}), \quad z_1, z_2, \ldots, z_{p+1} \in \mathbb{D},$$

imply that $z_r = z_s$ for some pair such that $r \neq s$, and if there is some w such that the equation $f(z) = w$ has p roots (counted in accordance with their multiplicities) in $\mathbb{D}$. For example, $f(z) = z^2$ is a 2-valent in $\mathbb{D}$.

Let $\mathcal{S}_p^*$ denote the class of functions, which are analytic and *p-valent starlike* in $\mathbb{D}$. A function $f \in \mathcal{S}_p$ is said to be *p-valent starlike* functions in $\mathbb{D}$, if there exists a $\rho > 0$ such that for $\rho < |z| < 1$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{1.2}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi$$

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for $z = re^{i\theta}$. Goodman [10] has studied the class $\mathcal{S}_p^*$ and shown that a function in $\mathcal{S}_p^*$ has exactly p zeros and it is p-valent in $\mathbb{D}$.

Let $\mathcal{C}_p$ denote the class of functions, which are analytic and *p-valent convex* in $\mathbb{D}$. A function $f \in \mathcal{S}_p$ is said to be in $\mathcal{C}_p$, if there exists a $\rho > 0$ such that for $\rho < |z| < 1$,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \tag{1.3}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi$$

for $z = re^{i\theta}$. Goodman [10] proved that a function in $\mathcal{C}_p$ is at most p-valent and f' has exactly $p-1$ zeros in $\mathbb{D}$, multiple zeros being counted in accordance with their multiplicities. There is a closely analytic relation between $\mathcal{S}_p^*$ and $\mathcal{C}_p$ in the same way as Alexander theorem. Namely,

$$f \in \mathcal{C}_p \Longleftrightarrow \frac{zf'}{p} \in \mathcal{S}_p^*.$$

For $p = 1$, the classes $\mathcal{S}_p^*$ and $\mathcal{C}_p$ are the usually classes of univalent starlike and convex, respectively.

An analytic function f is said to be subordinate to an analytic function g if $f(z) = g(\phi(z))$, $z \in \mathbb{D}$, for some analytic function ϕ in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, $z \in \mathbb{D}$. We write this subordination relation by $f(z) \prec g(z)$ (see [7, 11, 18]). The relations (1.2) and (1.3) are respectively equivalent to

$$\frac{zf'(z)}{pf(z)} \prec \frac{1+z}{1-z} \quad \text{and} \quad \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z}.$$

In 1948, Goodman [9] has conjectured that if $f \in \mathcal{S}_p$, then

$$|a_n| \le \sum_{k=1}^{p} \frac{2k(p+n)\,!}{(p+k)\,!(p-k)\,!(n-p-1)\,!(n^2-k^2)} |a_k| \tag{1.4}$$

for $n > p$. For $p = 2$ and $n = 3$, this gives the conjecture that

$$|a_3| \le 5|a_1| + 4|a_2|. \tag{1.5}$$

For $p = 1$, inequality (1.4) reduces to the well-known Bieberbach conjecture $|a_n| \le n$. For instance, Goodman [10] showed that (1.5) is valid for f in $\mathcal{S}_2^*$ has the form (1.1) with all real coefficients a_n and this bound is sharp for all pairs $|a_1|, |a_2|$, not both zero. In the same paper, Goodman suggested the similar conjecture as (1.4) for $f \in \mathcal{C}_p$. For $n = p+1$, he proved the inequality (1.4) for the classes $\mathcal{S}_p^*$ and $\mathcal{C}_p$, respectively, when f has the form (1.1) with the conditions $a_1 = a_2 = \ldots = a_{p-2} = 0$ and all the coefficients a_n are real. Umezawa [31] obtained the coefficient bound $|a_n|$ for function belongs to the class of p-valent close-to-convex functions. In 1969, Livingston [17] proved inequality (1.4) for functions of the class p-valent close-to-convex, in case $a_1 = a_2 = \ldots = a_{p-2} = 0$ and the remaining the coefficients being complex.

In addition, let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad p \in \mathbb{N}, \tag{1.6}$$

which are analytic and p-valent in $\mathbb{D}$. Denote by $A_1 := \mathcal{A}$, the class of all analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$ and $\mathcal{S}$ denotes the usual class of functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

For the special subclass $\mathcal{A}_p$ of $\mathcal{S}_p$, Hayman in [14] has showed that $|a_{p+1}| \le 2p$ and Jenkins in [15] has showed $|a_{p+2}| \le p(2p+1)$. Both of these results are consistent with (1.4). Both inequalities, for p-valent functions, are the analogues of the coefficient bounds $|a_2| \le 2$ and $|a_3| \le 3$, known for univalent functions. Goluzina [8], Patil and Thakare [22], Aouf [2], and several other authors also proved the coefficient bounds for certain subclasses of p-valent functions.

Recently the authors in [29] obtained the correct form of the coefficient bounds for the class

$$\mathcal{S}_p^*(A,B,\beta) := \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{f(z)} \prec \frac{p + \Big(pB + (A-B)(p-\beta)\Big)z}{1+Bz}, \quad z \in \mathbb{D} \right\},$$

where β, $0 \le \beta < p$ and $-1 \le B < A \le 1$. Here, we solve the coefficient bounds involving the Taylor–Maclaurin coefficients $|a_n|$ for $n \ge p+1$, for functions belonging to the classes $\mathcal{F}_p(\alpha,\beta,\lambda)$ and $\mathcal{C}_p(b,\lambda)$. These classes are defined below (see Definitions 1.2 and 1.3).

In [21], Padmanabhan introduced the class of starlike functions of order λ, $0 < \lambda \le 1$, defined as follows:

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in $\mathcal{T}(\lambda)$, if*

$$\left| \left(\frac{zf'(z)}{f(z)} - 1 \right) \Big/ \left(\frac{zf'(z)}{f(z)} + 1 \right) \right| < \lambda,$$

equivalently,

$$\frac{zf'(z)}{f(z)} \prec \frac{1+\lambda z}{1-\lambda z} \qquad \text{or} \qquad \frac{zf'(z)}{f(z)} \prec \frac{1-\lambda z}{1+\lambda z}$$

for all $z \in \mathbb{D}$ and $0 < \lambda \le 1$.

A function $f \in \mathcal{A}_p$ is said to be *p-valent α-spiral-like function of order β* in $\mathbb{D}$, if it is analytic and if there exists a $\rho > 0$ such that for $\rho < |z| < 1$

$$\operatorname{Re}\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > \beta \cos\alpha$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} d\theta = 2p\pi$$

for $z = re^{i\theta}$, $|\alpha| < \pi/2$ and $0 \le \beta < p$. The class of p-valent α-spiral-like of order β is denoted by $\mathcal{S}_{\alpha,p}(\beta)$. In [22], Patil and Thakare introduced the class $\mathcal{S}_{\alpha,p}(\beta)$. The subordination form of the definition of p-valent α-spiral-like function of order β defined follows: $f \in \mathcal{S}_{\alpha,p}(\beta)$ if and only if

$$e^{i\alpha}\frac{zf'(z)}{f(z)} \prec \left(\frac{p+(p-2\beta)z}{1-z}\right)\cos\alpha + ip\sin\alpha.$$

Two subclasses $\mathcal{F}_p(\alpha,\beta,\lambda)$ and $\mathcal{C}_p(b,\lambda)$ of p-valent functions in $\mathbb{D}$ were acquainted by Aouf in [2] which are defined as follows:

Definition 1.2. *A function $f \in \mathcal{A}_p$ is said to belong to the class $\mathcal{F}_p(\alpha,\beta,\lambda)$, if it satisfies the condition*

$$\left|\frac{H(f(z))-1}{H(f(z))+1}\right| < \lambda, \quad z \in \mathbb{D},$$

where

$$H(f(z)) = \frac{e^{i\alpha}\frac{zf'(z)}{f(z)} - \beta\cos\alpha - ip\sin\alpha}{(p-\beta)\cos\alpha}.$$

By subordination property, equivalently, it can be written as

$$e^{i\alpha}\frac{zf'(z)}{f(z)} \prec \left(\frac{p+(p-2\beta)\lambda z}{1-\lambda z}\right)\cos\alpha + ip\sin\alpha \tag{1.7}$$

for $0<\lambda\le 1$, $0\le\beta<p$, $p\in\mathbb{N}$ and $|\alpha|<\pi/2$.

Definition 1.3. *Let b be a non-zero complex number. For $0<\lambda\le 1$ and $p\in\mathbb{N}$, let $\mathcal{C}_p(b,\lambda)$ denote the class of functions $f(z)\in\mathcal{A}_p$ satisfying the relation*

$$\left|\frac{H(f(z))-1}{H(f(z))+1}\right| < \lambda \quad \text{for} \quad z \in \mathbb{D},$$

where

$$H(f(z)) = 1 + \frac{1}{pb}\left(1 + \frac{zf''(z)}{f'(z)} - p\right).$$

By subordination relation,

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{p(1+(2b-1)\lambda z)}{1-\lambda z}. \tag{1.8}$$

We note that a number of subclasses have been studied by several authors and the subclasses can be obtain by putting for different values of p,α,β,λ and b. We list some of them here.

(1) $\mathcal{F}_p(0,0,1) =: \mathcal{S}_p^*$ and $\mathcal{C}_p(1,1) =: \mathcal{C}_p$ are respectively the classes of p-valent starlike and p-valent convex functions recognized by Goodman [10], and the class $\mathcal{F}_p(0,\beta,1) =: \mathcal{S}_p^*(\beta)$, p-valent starlike functions of order β was investigated by Goluzina [8]. $\mathcal{C}_p((1-\beta/p),1) =: \mathcal{C}_p(\beta)$, $0\le\beta<p$, the class of p-valent functions $g(z)$ for which $zg'(z)/p$ is in the class $\mathcal{S}_p^*(\beta)$.

(2) $\mathcal{F}_p(\alpha,0,1) =: \mathcal{S}_{\alpha,p}$ and $\mathcal{F}_p(\alpha,\beta,1) =: \mathcal{S}_{\alpha,p}(\beta)$, respectively define the class of p-valent α-spirallike functions and p-valent α-spirallike functions of order β.

(3) $\mathcal{C}_p(e^{-i\alpha}\cos\alpha,1)$ and $\mathcal{C}_p(e^{-i\alpha}(1-\beta/p)\cos\alpha,1)$, $0\le\beta<p$, $|\alpha|<\pi/2$, are the class of p-valent functions $g(z)$ for which $zg'(z)/p$ are p-valent α-spirallike functions and p-valent α-spirallike functions of order β respectively.

(4) The class $\mathcal{F}_1(\alpha,\beta,\lambda) =: \mathcal{F}(\alpha,\beta,\lambda)$ was studied by Gopalakrishna and Umarani [13].

(5) $\mathcal{C}_p(b,1)$ is the class of p-valent functions $g(z)\in\mathcal{A}_p$ satisfying

$$\operatorname{Re}\left\{p+\frac{1}{b}\left(1+\frac{zg''(z)}{g'(z)}-p\right)\right\}>0 \quad \text{for } z\in\mathbb{D}.$$

This class was considered by Aouf in [1].

(6) $\mathcal{F}_1(0,0,1)=:\mathcal{S}^*$ and $\mathcal{C}_1(1,1)=:\mathcal{C}$ are respectively the usual classes of starlike and convex functions; $\mathcal{F}_1(0,\beta,1)=:\mathcal{S}^*(\beta)$ and $\mathcal{C}_1(1-\beta,1)=:\mathcal{C}(\beta)$, $0\le\beta<1$, are respectively the classes of starlike and convex functions of order β were introduced by Robertson [24]; $\mathcal{F}_1(0,0,\lambda)=:\mathcal{T}(\lambda)$ (see Definition 1.1) and $\mathcal{C}_1(1,\lambda)=:\mathcal{C}(\lambda)$ is the class of functions $g(z)$ for which $zg'(z)\in\mathcal{S}(\lambda)$.

(7) $\mathcal{F}_1(\alpha,0,1)=:\mathcal{S}_\alpha$ and $\mathcal{C}_1(e^{-i\alpha}\cos\alpha,1)$, $|\alpha|<\pi/2$, respectively define the class of α-spirallike functions familiarized by Špaček [30] and the class of functions $g(z)$ for which $zg'(z)$ is α-spirallike introduced by Robertson [25]; $\mathcal{F}_1(\alpha,\beta,1)=:\mathcal{S}_\alpha(\beta)$ and $\mathcal{C}_1\big(e^{-i\alpha}(1-\beta)\cos\alpha,1\big)=:$ $=:\mathcal{C}_\alpha(\beta)$, $0\le\beta<1,|\alpha|<\pi/2$, are respectively the class of α-spirallike functions of order β introduced by Libra [16] and the class of functions $g(z)$ for which $zg'(z)$ is α-spirallike of order β studied by Chichra [4] and Sizuk [28].

(8) $\mathcal{C}_1(b,1)=:\mathcal{C}(b)$ is the class of functions $g(z)\in\mathcal{A}$ satisfying

$$\operatorname{Re}\left(1+\frac{1}{b}\frac{zg''(z)}{g'(z)}\right)>0 \quad \text{for } z\in\mathbb{D}$$

introduced by Wiatrowski [32] and studied in [19, 20].

2. Main results. Aouf evaluated the coefficient bounds for the functions from the classes $\mathcal{F}_p(\alpha,\beta,\lambda)$ and $\mathcal{C}_p(b,\lambda)$ in [2] in which the proofs are found to be incorrect. In the present paper, we provide their correct proofs. The following theorems were mistakenly proven by Aouf in [2].

Theorem A ([2], Theorem 2)**.** *Let $0<\lambda\le1$, $0\le\beta<p$, $p\in\mathbb{N}$ and $|\alpha|<\pi/2$. If $f(z)=$ $=z^p+\sum_{n=p+1}^{\infty}a_nz^n\in\mathcal{F}_p(\alpha,\beta,\lambda)$, then*

$$|a_n|\le\prod_{j=0}^{n-p-1}\frac{\lambda\left|j+2(p-\beta)e^{-i\alpha}\cos\alpha\right|}{j+1}$$

for $n\ge p+1$, and these bounds are sharp for all admissible α,β,λ and for each n.

Theorem B ([2], Theorem 3)**.** *Let $0<\lambda\le1$, $p\in\mathbb{N}$ and $b\ne0$ be any complex number. If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n\in\mathcal{C}_p(b,\lambda)$, then*

$$|a_n|\le\prod_{j=0}^{n-p-1}\frac{\lambda|j+2bp|}{j+1}$$

for $n\ge p+1$, and these bounds are sharp for all admissible α,β,λ and for each n.

First, we provide the correct form of the coefficients bounds for $f\in\mathcal{F}_p(\alpha,\beta,\lambda)$ as stated in Theorem A and its proof.

Theorem 2.1. *Let $0<\lambda\le1$, $0\le\beta<p$, $p\in\mathbb{N}$ and $|\alpha|<\pi/2$. If $f(z)\in\mathcal{F}_p(\alpha,\beta,\lambda)$ is in the form* (1.6), *then*

$$|a_{p+1}|\le2\lambda(p-\beta)\cos\alpha; \tag{2.1}$$

for $\lambda^2\big(2p-2\beta+(n-p-1)\big)^2 \le (n-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$,

$$|a_n| \le \frac{2\lambda(p-\beta)}{n-p}\cos\alpha, \quad n \ge p+2; \tag{2.2}$$

and for $\lambda^2\big(2p-2\beta+(n-p-1)\big)^2 > (n-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$,

$$|a_n| \le \prod_{j=1}^{n-p}\frac{\lambda\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|}{j}, \quad n \ge p+2. \tag{2.3}$$

The equality signs in (2.1), (2.2) *and* (2.3) *are attained.*

Proof. Let $f(z)\in\mathcal{F}_p(\alpha,\beta,\lambda)$. It follows from (1.7) that

$$e^{i\alpha}\frac{zf'(z)}{f(z)}=\left(\frac{p+(p-2\beta)\lambda\phi(z)}{1-\lambda\phi(z)}\right)\cos\alpha+ip\sin\alpha$$

for some analytic function $\phi(z)$ in $\mathbb{D}$ with $\phi(0)=0$ and $|\phi(z)|<1$. We divide the expansion by $\cos\alpha$ on both sides and get

$$e^{i\alpha}\sec\alpha zf'(z)-(p+ip\tan\alpha)f(z)=\lambda\Big(e^{i\alpha}\sec\alpha zf'(z)+(p-2\beta-ip\tan\alpha)f(z)\Big)\phi(z).$$

Substituting this in the series expansion (1.6), of $f(z)$, we find that

$$\sum_{k=0}^{\infty}\Big(e^{i\alpha}(k+p)\sec\alpha-p-ip\tan\alpha\Big)a_{k+p}z^{k+p}=$$

$$=\lambda\left(\sum_{k=0}^{\infty}\Big(e^{i\alpha}(k+p)\sec\alpha+p-2\beta-i\tan\alpha\Big)a_{k+p}z^{k+p}\right)\phi(z),$$

where $a_p=1$ and $\phi(z)=\sum_{k=0}^{\infty}w_{k+p}z^{k+p}$. Rewriting it, we obtain

$$\sum_{k=0}^{m}\Big(e^{i\alpha}(k+p)\sec\alpha-p-ip\tan\alpha\Big)a_{k+p}z^{k+p}+\sum_{k=m+1}^{\infty}C_kz^{k+p}=$$

$$=\lambda\left(\sum_{k=0}^{m-1}\Big(e^{i\alpha}(k+p)\sec\alpha+p-2\beta-i\tan\alpha\Big)a_{k+p}z^{k+p}\right)\phi(z)$$

for certain coefficients C_k. Since $|\phi(z)|<1$ in $\mathbb{D}$, then by Parseval – Gutzmer formula (see also Clunie's method [5] and [26, 27]), we get

$$\sum_{k=0}^{m}\Big|e^{i\alpha}(k+p)\sec\alpha-p-ip\tan\alpha\Big|^2|a_{k+p}|^2r^{2p+2k}+\sum_{k=m+1}^{\infty}|C_k|^2r^{2p+2k}\le$$

$$\le\lambda^2\left(\sum_{k=0}^{m-1}\Big|e^{i\alpha}(k+p)\sec\alpha+p-2\beta-i\tan\alpha\Big|^2|a_{k+p}|^2r^{2p+2k}\right).$$

Table 1

k	p	α	β	λ	T
all	1	all	all	1	positive
2	1	$\pm\pi/4$	0.9	0.9	-0.0236
3	2	$\pm\pi/3$	1	0.6	-5.92
3	2	$\pm\pi/3$	1	0.8	1.92

(This is the place where the incorrectness of Aouf's proof is found!)

Letting $r \to 1$, the above inequality can be written as

$$\left|e^{i\alpha}(m+p)\sec\alpha - p - ip\tan\alpha\right|^2 |a_{m+p}|^2 \le \sum_{k=0}^{m-1}\Bigg(\lambda^2\left|e^{i\alpha}(k+p)\sec\alpha + p - 2\beta - i\tan\alpha\right|^2 -$$
$$-\left|e^{i\alpha}(k+p)\sec\alpha - p - ip\tan\alpha\right|^2\Bigg)|a_{k+p}|^2.$$

Simplification of the above inequality leads

$$m^2\sec^2\alpha|a_{m+p}|^2 \le \sum_{k=0}^{m-1}\Big(\lambda^2(k+2p-2\beta)^2 - k^2(\sec^2\alpha - \lambda^2\tan^2\alpha)\Big)|a_{k+p}|^2$$

or

$$|a_{m+p}|^2 \le \frac{\cos^2\alpha}{m^2}\Bigg(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{m}\Big(\lambda^2(k-1+2p-2\beta)^2 -$$
$$-(k-1)^2(\sec^2\alpha - \lambda^2\tan^2\alpha)\Big)\Bigg)|a_{k+p-1}|^2.$$

Above inequality can be rewritten by replacing $m+p$ by n as

$$|a_n|^2 \le \frac{\cos^2\alpha}{(n-p)^2}\Bigg(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{n-p}\Big(\lambda^2(k-1+2p-2\beta)^2 -$$
$$-(k-1)^2(\sec^2\alpha - \lambda^2\tan^2\alpha)\Big)\Bigg)|a_{k+p-1}|^2 \quad \text{for } n \ge p+1. \tag{2.4}$$

Note that the terms under the summation in the right-hand side of (2.4) may be positive as well as negative. We verify it by including here a table (see Table 1) for values of

$$T := \lambda^2(k-1+2p-2\beta)^2 - (k-1)^2(\sec^2\alpha - \lambda^2\tan^2\alpha)$$

for various choices for k, p, α, β and λ. So, we can not apply direct principle of mathematical induction in (2.4) to establish the desired bounds for $|a_n|$. Therefore, we are considering different cases for this.

First, for $n = p + 1$, we readily see that (2.4) reduces to

$$|a_{p+1}| \le 2\lambda(p-\beta)\cos\alpha,$$

which is equivalent to (2.1).

Secondly, $\lambda^2\big(2p-2\beta+(n-p-1)\big)^2 \le (n-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$ for $n \ge p+2$. Since all the terms under the summation in (2.4) are negative, we get

$$|a_n| \le \frac{2\lambda(p-\beta)}{n-p}\cos\alpha.$$

This gives the bound for $|a_n|$ as asserted in (2.2). The equality holds in (2.1) and (2.2) for the rotation of the functions

$$k_{n,p,\alpha,\beta,\lambda,}(z) = \frac{z^p}{\big(1+\lambda z^{n-1}\big)^{\zeta_n}}.$$

Here $\zeta_n := 2(p-\beta)e^{-i\alpha}\cos\alpha/(n-1)$.

Finally, we consider the case $\lambda^2\big(2p-2\beta+(n-p-1)\big)^2 > (n-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$ for $n \ge p+2$ and obtain bound for $|a_n|$ stated in (2.3). We see that all the terms under the summation in (2.4) are nonnegative. We prove the inequality by the usual induction principle. Fix n, $n \ge p+2$ and suppose that (2.3) holds for $k = 3, 4, \ldots, n-p$. Then by (2.4), we obtain

$$|a_n|^2 \le \frac{\cos^2\alpha}{(n-p)^2}\left(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{n-p}\Big(\lambda^2(2p-2\beta+k-1)^2 - \right.$$
$$\left. -(k-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Big)\right)\prod_{j=1}^{k-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2}. \qquad (2.5)$$

It is now sufficient to prove that the square of the right-hand side of (2.3) is equal to the right-hand side of (2.5), that is to show

$$\prod_{j=1}^{m-p}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2} = \frac{\cos^2\alpha}{(m-p)^2}\left(4\lambda^2(p-\beta)^2 + \sum_{k=2}^{m-p}\Big(\lambda^2(2p-2\beta+k-1)^2 - \right.$$
$$\left. -(k-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Big)\right)\prod_{j=1}^{k-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2}, \qquad (2.6)$$

when $\lambda^2\big(2p-2\beta+(m-p-1)\big)^2 > (m-p-1)^2\big(\sec^2\alpha-\lambda^2\tan^2\alpha\big)$ for $m \ge p+2$.

The equation (2.6) is valid for $m = p+2$. Suppose that (2.6) is true for all m, $p+2 < m \le n-p$. Then by (2.5), we have

$$|a_n|^2 \le \frac{\cos^2\alpha}{(n-p)^2}\left\{4\lambda^2(p-\beta)^2 + \sum_{k=2}^{n-p-1}\Big(\lambda^2(2p-2\beta+k-1)^2 - \right.$$
$$-(k-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\Big)\prod_{j=1}^{k-1}\frac{\lambda^2\big|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\big|^2}{j^2} +$$

$$+\left(\lambda^2(2p-2\beta+n-p-1)^2-(n-p-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\right)\times$$
$$\left.\times\prod_{j=1}^{n-p-1}\frac{\lambda^2\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|^2}{j^2}\right\}.$$

By induction hypothesis for $m=n-1$, we get

$$|a_n|^2\le\frac{\cos^2\alpha}{(n-p)^2}\left\{\frac{(n-p-1)^2}{\cos^2\alpha}\prod_{j=1}^{n-p-1}\frac{\lambda^2\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|^2}{j^2}+\right.$$
$$+\left(\lambda^2(2p-2\beta+n-p-1)^2-(n-p-1)^2(\sec^2\alpha-\lambda^2\tan^2\alpha)\right)\times$$
$$\left.\times\prod_{j=1}^{n-p-1}\frac{\lambda^2\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|^2}{j^2}\right\},$$

i.e.,

$$|a_n|^2\le\frac{\lambda^2}{(n-p)^2}\left((2p-2\beta+n-p-1)^2\cos^2\alpha+(n-p-1)^2\sin^2\alpha\right)\times$$
$$\times\prod_{j=1}^{n-p-1}\frac{\lambda^2\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|^2}{j^2}.$$

On simplification, the above inequality leads to

$$|a_n|\le\prod_{j=1}^{n-p}\frac{\lambda\left|2(p-\beta)e^{-i\alpha}\cos\alpha+j-1\right|}{j}.$$

It is easy to prove that the bounds are sharp as can be seen by the rotation of the function

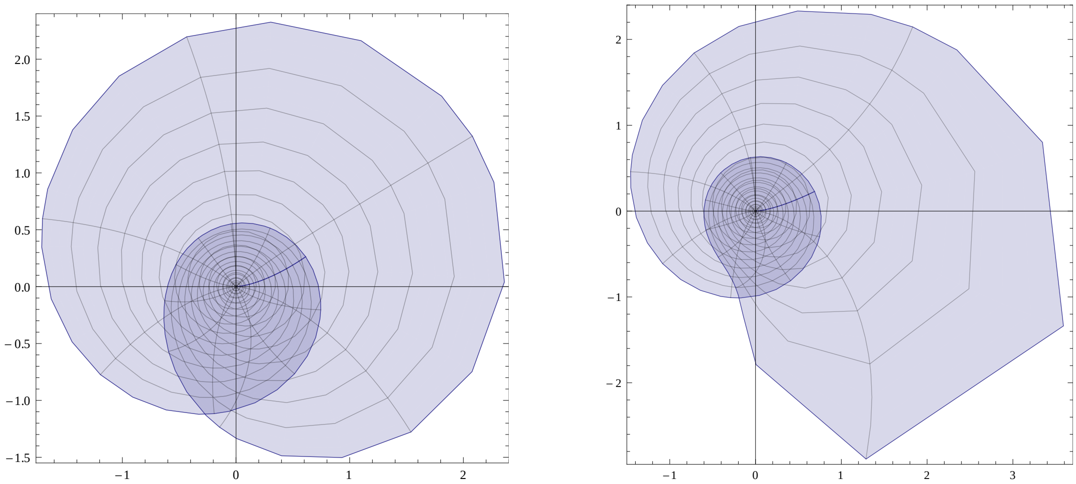
$$k_{p,\alpha,\beta,\lambda}(z)=\frac{z^p}{(1+\lambda z)^{\zeta}}.$$

Here $\zeta:=2(p-\beta)e^{-i\alpha}\cos\alpha$.

Theorem 2.1 is proved.

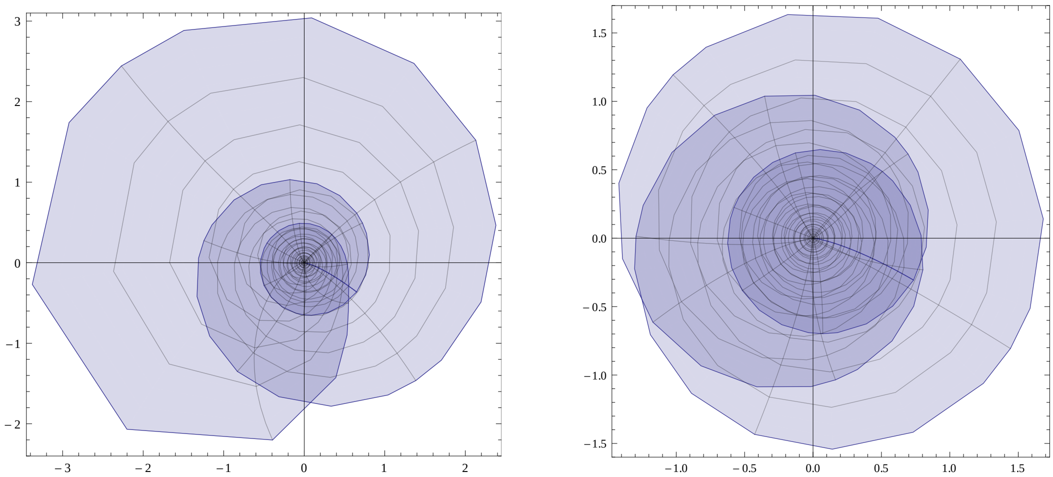
Table 2

k_n	p	α	β	λ
k_1	2	$\pi/4$	1	0.5
k_2	2	$\pi/4$	1.5	0.9
k_3	3	$-\pi/3$	2	0.8
k_4	3	$-\pi/3$	0.5	0.2



(*a*) (*b*)

Fig. 1. Images of the unit disk under k_1 (*a*) and k_2 (*b*).



(*a*) (*b*)

Fig. 2. Images of the unit disk under k_3 (*a*) and k_4 (*b*).

***Remark* 2.1.** Letting the different values of p, α, β and λ in Theorem 2.1, we obtain results which were proved in [8 – 10, 13, 16, 22 – 24, 33].

For different values of p, α, β and λ (see Table 2), the images of the unit disk under the extremal functions $k_n := k_{p,\alpha,\beta,\lambda}(z)$ are described in Figures 1 and 2.

We now give the correct form of the statement stated in Theorem B and its proof.

Theorem 2.2. *Let* $0 < \lambda \leq 1$, $p \in \mathbb{N}$ *and* $b \neq 0$ *be any complex number. If* $f(z) \in \mathcal{C}_p(b, \lambda)$ *is of the form* (1.6), *then*

$$|a_{p+1}| \leq \frac{2\lambda p^2 |b|}{1 + p}; \tag{2.7}$$

for $|2bp+n-p-1| \le n-p-1$ (*equivalently* $|1+2bp| \le 1$),

$$|a_n| \le \frac{2\lambda p^2|b|}{n(n-p)}, \quad n \ge p+2; \tag{2.8}$$

and for $|2bp+n-p-1| > n-p-1$,

$$|a_n| \le \frac{p}{n} \prod_{j=0}^{n-p-1} \frac{\lambda|j+2bp|}{j+1}, \quad n \ge p+2. \tag{2.9}$$

The equality signs in (2.7), (2.8) *and* (2.9) *are attained.*

Proof. Let $f(z) \in \mathcal{C}_p(b,\lambda)$. By the equation (1.8), we see that there is an analytic function $\phi\colon \mathbb{D} \to \overline{\mathbb{D}}$ with $\phi(0)=0$ such that

$$1+\frac{zf''(z)}{f'(z)} = \frac{p(1+(2b-1)\lambda\phi(z))}{1-\lambda\phi(z)},$$

or

$$zf''(z)-(p-1)f'(z) = -\lambda\Big((p-2bp-1)f'(z)-zf''(z)\Big)\phi(z).$$

Using the representation (1.6), we observe that

$$\sum_{k=1}^{\infty} k(k+p)a_{k+p}z^k = \lambda\left(2p^2b+\sum_{k=1}^{\infty}(k+p)(k+2bp)a_{k+p}z^k\right)\phi(z).$$

We apply Clunie's method [5] for $m \in \mathbb{N}$ (see also [26, 27]) and obtain

$$\sum_{k=1}^{m} k^2(k+p)^2|a_{k+p}|^2 \le \lambda^2\left(4p^4|b|^2+\sum_{k=1}^{m-1}(k+p)^2|k+2bp|^2|a_{k+p}|^2\right).$$

The above inequality yields

$$|a_{m+p}|^2 \le \frac{1}{m^2(m+p)^2}\left(4\lambda^2p^4|b|^2+\sum_{k=1}^{m-1}(k+p)^2\big(\lambda^2|k+2bp|^2-k^2\big)|a_{k+p}|^2\right).$$

Replacing $m+p$ by n, we get

$$|a_n|^2 \le \frac{1}{n^2(n-p)^2}\left(4\lambda^2p^4|b|^2+\sum_{k=1}^{n-p-1}(k+p)^2\big(\lambda^2|k+2bp|^2-k^2\big)|a_{k+p}|^2\right) \tag{2.10}$$

for $n \ge p+1$.

Note that the terms under the summation in the right-hand side of (2.10) may be positive as well as negative. We inspect it by including here a table (see Table 3) for values of

$$U := \lambda^2|k+2bp|^2-k^2$$

for different choices of k, p, b and λ. So, we can not apply direct mathematical induction in (2.10) to prove the required coefficients bounds for $f \in \mathcal{C}_p(b,\lambda)$. Therefore, we are taking different cases for this.

Table 3

k	p	b	λ	V
2	1	1	0.1	-3.998
2	1	1	0.6	1.76
4	2	$3-2i$	0.2	-3.2
4	2	$3-2i$	0.3	12.8

(This is the place where the in correctness of Aouf's proof is found!)

First, for $n = p+1$, (2.10) reduces to

$$|a_{p+1}| \le \frac{2\lambda p^2|b|}{1+p}.$$

This proves (2.7).

Secondly, we consider the case $|2bp+n-p-1| \le n-p-1$ (equivalently $|1+2bp| \le 1$) for $n \ge p+2$. Since all the terms under the summation in (2.10) are nonpositive, we get

$$|a_n| \le \frac{2\lambda p^2|b|}{n(n-p)},$$

which establishes (2.8). The equality holds in (2.7) and (2.8) for the rotation of the functions $k_{n,p,b,\lambda}(z) \in \mathcal{C}_p(b,\lambda)$ given by

$$k'_{n,p,b,\lambda}(z) = \frac{pz^{p-1}}{(1+\lambda z^{n-1})^{2bp/(n-1)}}.$$

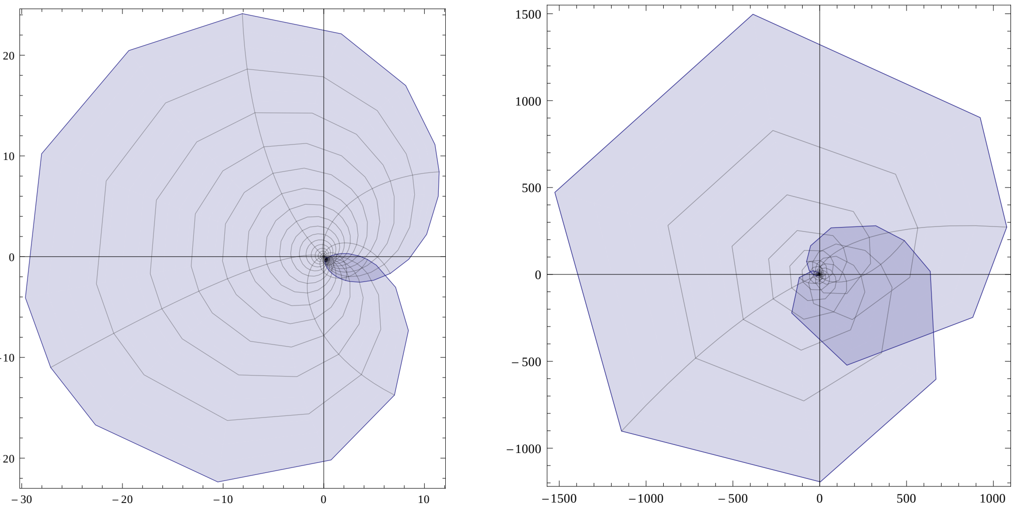
Finally, we prove (2.9) when $|1+2bp| \ge |2bp+n-p-1| > n-p-1$ for $n \ge p+2$. We see that all the terms under the summation in (2.10) are positive. We prove the inequality by the mathematical induction. We consider that (2.9) holds for $k = 3,4,\ldots,n-p$. Then from (2.10), we obtain

$$|a_n|^2 \le \frac{1}{n^2(p-n)^2}\left(4\lambda^2p^4|b|^2 + \sum_{k=1}^{n-p-1} p^2\Big(\lambda^2|k+2bp|^2 - k^2\Big)\prod_{j=0}^{k-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}\right). \tag{2.11}$$

We now prove that the square of the right-hand side of (2.9) is equal to the right-hand side of (2.11), that is

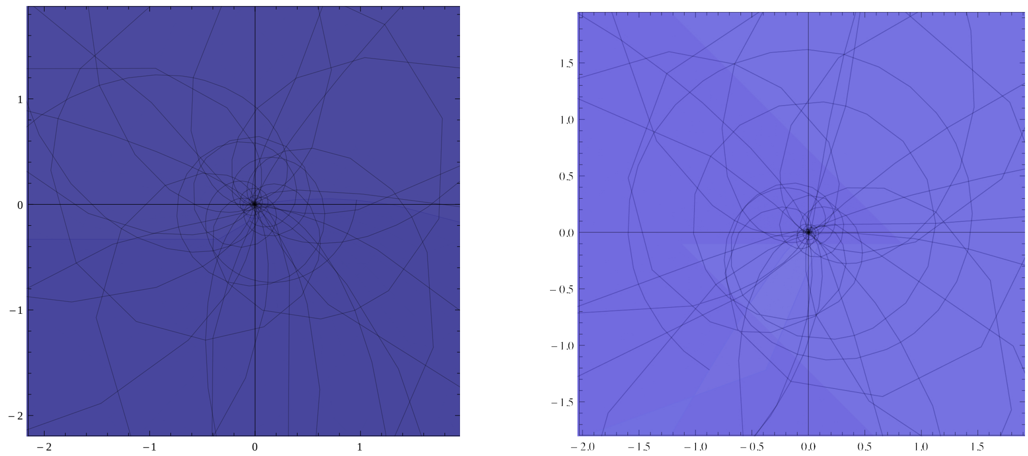
$$\prod_{j=0}^{m-p-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2} = \frac{1}{(p-m)^2}\left(4\lambda^2p^2|b|^2 + \sum_{k=1}^{n-p-1}\Big(\lambda^2|k+2bp|^2 - k^2\Big)\times\right.$$
$$\left.\times\prod_{j=0}^{k-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}\right) \tag{2.12}$$

when $|2bm+p-p-1| > m-p-1$, $m \ge p+2$.



(a) (b)

Fig. 3. Images of the unit disk under g_1 (a) and g_2 (b).



(a) (b)

Fig. 4. Images of the unit disk under g_3 (a) and g_4 (b).

For $m = p + 2$, the equation (2.12) is recognized. Suppose that (2.12) is true for all m, $p + 2 <$ $< m \leq n - p$. Then from (2.11), we have

$$|a_n|^2 \leq \frac{1}{n^2(p-n)^2}\left(4\lambda^2 p^4 |b|^2 + \sum_{k=1}^{n-p-2} p^2\left(\lambda^2|k+2bp|^2 - k^2\right)\prod_{j=0}^{k-1}\frac{\lambda^2|j+2bp|^2}{(j+1)^2} + \right.$$

$$\left. + p^2\left(\lambda^2|n-p-1+2bp|^2 - (n-p-1)^2\right)\prod_{j=0}^{n-p-2}\frac{\lambda^2|j+2bp|^2}{(j+1)^2}\right).$$

Table 4

g_n	p	b	λ
g_1	2	$1+i$	0.4
g_2	2	$2-3i$	0.4
g_3	3	$1-2i$	0.7
g_4	3	$3-2i$	0.7

Using the relation (2.12) for $m = n-1$, we find that

$$|a_n|^2 \leq \frac{1}{n^2(p-n)^2}\left(p^2(p-n+1)^2 \prod_{j=0}^{n-p-2} \frac{\lambda^2|j+2bp|^2}{(j+1)^2} + \right.$$

$$\left. + p^2\left(\lambda^2|n-p-1+2bp|^2 - (n-p-1)^2\right) \prod_{j=0}^{n-p-2} \frac{\lambda^2|j+2bp|^2}{(j+1)^2}\right).$$

It is equivalent to

$$|a_n| \leq \frac{p\lambda|j+2bp|}{n(p-n)} \prod_{j=0}^{n-p-2} \frac{\lambda|j+2bp|}{(j+1)}$$

which establishes (2.9).

The bounds are sharp for the rotation of the function $k_{p,b,\lambda}(z) \in \mathcal{C}_p(b,\lambda)$ given by

$$k'_{p,b,\lambda}(z) = \frac{pz^{p-1}}{(1+\lambda z)^{2bp}}.$$

Theorem 2.2 is proved.

***Remark* 2.2.** Letting the different values of p, b and λ in Theorem 2.2, we obtain results which were proved in [1, 10, 24, 32].

For different values of p, b and λ (see Table 4), the images of the unit disk under the extremal functions $g_n := k'_{p,b,\lambda}(z)$ are described in Figures 3 and 4.

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