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LYAPUNOV-TYPE INEQUALITIES FOR TWO CLASSES OF NONLINEAR SYSTEMS WITH HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS

НЕРІВНОСТІ ТИПУ ЛЯПУНОВА ДЛЯ ДВОХ КЛАСІВ НЕЛІНІЙНИХ СИСТЕМ З ОДНОРІДНИМИ ГРАНИЧНИМИ УМОВАМИ ДІРІХЛЕ

We establish new Lyapunov-type inequalities for two classes of nonlinear systems with homogeneous Dirichlet boundary conditions, which generalize and improve some results known from the literature.

Встановлено нові нерівності типу Ляпунова для двох класів нелінійних систем з однорідними граничними умовами Діріхле, що узагальнюють та покращують деякі відомі результати.

1. Introduction. This paper is concerned with the problem of finding new Lyapunov-type inequalities for the cycled system

$$\left(r_i(x)\phi_{p_i}\left(u_i'\right)\right)' + f_i(x)\phi_{\alpha_i}\left(u_{i+1}\right) = 0 \tag{1.1}$$

and strongly coupled system

$$\left(r_1(x)\phi_{p_1}\left(u_i'\right)\right)' + f_i(x)\sum_{j=1}^{n}\phi_{p_1}\left(u_j\right) = 0 \tag{1.2}$$

for $i = 1, 2, \ldots, n$, where $n \in \mathbb{N}$, $u_{n+1}(x) := u_1(x)$ and $\phi_\gamma(u) = |u|^{\gamma-2}u$ with $\gamma > 1$ under the following hypotheses:

(H_1) $r_j(x) > 0$ and $f_i(x)$ for $i, j = 1, 2, \ldots, n$ are real-valued continuous functions defined on $\mathbb{R}$,

(H_2) the parameters $1 < \alpha_i,\ p_i < \infty$ for $i = 1, 2, \ldots, n$ satisfy $\prod_{i=1}^{n}\dfrac{\alpha_i - 1}{p_i - 1} = 1$.

The well-known Lyapunov inequality [16] for the second-order linear differential equation

$$u_1'' + f_1(x)\,u_1 = 0, \tag{1.3}$$

states that if $f_1(x)$ is continuous on $[a, b]$ and (1.3) has a real nontrivial solution $u_1(x)$ satisfying the Dirichlet boundary condition $u_1(a) = 0 = u_1(b)$, then the inequality

$$\frac{4}{b-a} \le \int_a^b |f_1(s)|\,ds \tag{1.4}$$

holds, and the constant 4 can not be replaced by a larger number. Since this result has proved to be a useful tool in oscillation theory, disconjugacy, eigenvalue problems, and many other applications in the study of various properties of solutions for differential and difference equations, there have been many proofs and generalizations or improvements of it in the literature. For some of the most recent works on Lyapunov-type inequalities, the reader is referred to [1 – 35].

Recently, Sim and Lee [23] obtained the following Lyapunov-type inequalities for systems (1.1) with $\alpha_i = p_i = p_1$, $r_i(x) = 1$, and $f_i(x) \geq 0$ for $i = 1, 2, \ldots, n$ and (1.2) with $r_1(x) = 1$ and $f_i(x) \geq 0$ for $i = 1, 2, \ldots, n$. Their results are as follows:

Theorem A. *Let* $f_i \in C([a, b], [0, \infty))$ *for* $i = 1, 2, \ldots, n$ *hold. If system* (1.1) *with* $\alpha_i = p_i = p_1$ *and* $r_i(x) = 1$ *for* $i = 1, 2, \ldots, n$ *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n$, *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 \leq \prod_{i=1}^{n} \int_a^b f_i(s) \left[2^{p_1 - 2} \left(\frac{(s-a)(b-s)}{b-a} \right)^{p_1 - 1} \right] ds \tag{1.5}$$

holds.

Theorem B. *Let* $f_i \in C([a, b], [0, \infty))$ *for* $i = 1, 2, \ldots, n$ *hold. If system* (1.2) *with* $r_1(x) = 1$ *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n$, *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 \leq n \sum_{i=1}^{n} \int_a^b f_i(s) \left[2^{p_1 - 2} \left(\frac{(s-a)(b-s)}{b-a} \right)^{p_1 - 1} \right] ds \tag{1.6}$$

holds.

More recently, Rodrigues [22] generalizes and improves Theorems A and B, respectively, as follows:

Theorem C. *Let* $f_i \in C([a, b], [0, \infty))$ *for* $i = 1, 2, \ldots, n$ *hold. If system* (1.1) *with* $\alpha_i = p_i = p_1$ *and* $r_i(x) \geq k_i > 0$ *for* $i = 1, 2, \ldots, n$ *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n$, *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 \leq \left(\prod_{i=1}^{n} k_i^2 \right)^{-1} \prod_{i=1}^{n} \int_a^b f_i(s) \left[2^{p_1 - 2} \left(\frac{(s-a)(b-s)}{b-a} \right)^{p_1 - 1} \right] ds \tag{1.7}$$

holds.

Theorem D. *Let* $f_i \in C([a, b], [0, \infty))$ *for* $i = 1, 2, \ldots, n$ *hold. If system*

$$\left(r_i(x) \phi_{p_1}(u_i') \right)' + f_i(x) \sum_{j=1}^{n} \phi_{p_1}(u_j) = 0, \tag{1.8}$$

where $r_i(x) \geq k_i > 0$ *for* $i = 1, 2, \ldots, n$, *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n$, *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 \leq \left(M + (n-1)\, M^2\right) \sum_{i=1}^{n} \int_a^b f_i\,(s) \left[2^{p_1-2} \left(\frac{(s-a)\,(b-s)}{b-a}\right)^{p_1-1}\right] ds \tag{1.9}$$

holds, where $M = \max_{i=1,\ldots,n} \left\{\dfrac{1}{k_i}\right\}$.

It is easy to see that if we take $M \neq 1$, then the inequality (1.9) is better than (1.6) when $M + (n-1)\, M^2 < n$ in the sense that (1.6) follows from (1.9), but not conversely. If $M = 1$, then Theorem D with $r_i\,(x) = 1$ for $i = 1, 2, \ldots, n$ coincides with Theorem B. Moreover, if we take $1 < \prod_{i=1}^{n} k_i^2$, then the inequality (1.7) is better than (1.5) in the sense that (1.5) follows from (1.7), but not conversely. Similarly, if $\prod_{i=1}^{n} k_i^2 < 1$, then Theorem A gives a better result than Theorem C with $r_i\,(x) = 1$ for $i = 1, 2, \ldots, n$ in the sense that (1.7) follows from (1.5), but not conversely. If $\prod_{i=1}^{n} k_i^2 = 1$, then Theorem C with $r_i\,(x) = 1$ for $i = 1, 2, \ldots, n$ coincides with Theorem A.

Throughout the paper, for the sake of brevity, we denote

$$D_i\,(x) = \frac{1}{\xi_i^{1-p_i}(x) + \eta_i^{1-p_i}(x)}, \quad E_i\,(x) = 2^{p_i-2} \left(\frac{1}{\xi_i(x)} + \frac{1}{\eta_i(x)}\right)^{1-p_i}$$

and

$$F_i = 2^{-p_i} \left(\xi_i\,(x) + \eta_i\,(x)\right)^{p_i-1} = 2^{-p_i} \left(\int_a^b r_i^{1/(1-p_i)}(s) ds\right)^{p_i-1},$$

where

$$\xi_i(x) = \int_a^x r_i^{1/(1-p_i)}(s) ds, \quad \eta_i(x) = \int_x^b r_i^{1/(1-p_i)}(s) ds$$

for $i = 1, 2, \ldots, n$.

Now, we give some properties of concave and convex functions which are useful in the comparison of our results. We know that since the function $h(x) = x^{p_i-1}$ is concave for $x > 0$ and $1 < p_i < 2$, Jensen's inequality $h\left(\dfrac{\omega+v}{2}\right) \geq \dfrac{1}{2}\left(h(\omega) + h(v)\right)$ with $\omega = \dfrac{1}{\xi_i(x)}$ and $v = \dfrac{1}{\eta_i(x)}$ implies

$$D_i\,(x) \geq E_i\,(x) \tag{1.10}$$

for $1 < p_i < 2$, $i = 1, 2, \ldots, n$. If $p_i > 2$, then $h(x) = x^{p_i-1}$ is convex for $x > 0$. So, the inequality (1.10) is reversed, i.e.,

$$D_i\,(x) \leq E_i\,(x) \tag{1.11}$$

for $p_i > 2$, $i = 1, 2, \ldots, n$. In addition, since the function $l(x) = x^{1-p_i}$ is convex for $x > 0$ and $p_i > 1$ for $i = 1, 2, \ldots, n$, Jensen's inequality $l\left(\dfrac{\omega+v}{2}\right) \leq \dfrac{1}{2}\left(l(\omega) + l(v)\right)$ with $\omega = \xi_i(x)$ and $v = \eta_i(x)$ implies

$$D_i\,(x) \leq F_i. \tag{1.12}$$

Moreover, by using the inequality

$$4AB \leq (A+B)^2$$

with $A = \xi_i\,(x) > 0$ and $B = \eta_i\,(x) > 0$ for $i = 1, 2, \ldots, n$, we obtain the inequality

$$E_i\,(x) \le F_i. \tag{1.13}$$

In this paper, we state and prove several new generalized Lyapunov-type inequalities for systems (1.1) and (1.2). Our motivation comes from the recent papers of Rodrigues [22], Sim and Lee [23] and Tang and He [24]. Our aim is to remove some restrictions on the functions $f_i\,(x)$ and $r_i\,(x)$ for $i = 1, 2, \ldots, n$ in [22]. In fact, we generalize and improve some known results in the literature.

Since our attention is restricted to the Lyapunov-type inequalities for the systems of differential equations, we shall assume the existence of the nontrivial solution of the system (1.1) or (1.2). For readers who contributed to the existence of the solution of these type systems, we refer to the paper by Lee et al. [14].

2. Lyapunov-type inequalities for system (1.1). One of the main results of this section is the following theorem.

Theorem 2.1. *Let the hypotheses* (H_1) *and* (H_2) *hold. If system* (1.1) *has a real nontrivial solution* $(u_1(x),\ u_2(x), \ldots, u_n\,(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n,$ *where* a, $b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 < \prod_{i=1}^{n} \left(\int_a^b |f_i\,(s)|\, D_i^{\frac{1}{p_i}}\,(s)\, D_{i+1}^{\frac{\alpha_i - 1}{p_{i+1}}}\,(s)\, ds \right)^{\frac{1}{\alpha_i - 1}} \tag{2.1}$$

holds, where $D_{n+1}\,(x) := D_1\,(x)$.

Proof. Let $u_i(a) = 0 = u_i(b)$, where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u_i for $i = 1, 2, \ldots, n$ are not identically zero on $[a, b]$. It follows from $u_i(a) = 0 = u_i(b)$ and the Hölder's inequality that

$$|u_i\,(x)|^{p_i} \le \left(\int_a^x r_i^{1/(1-p_i)}(s) ds \right)^{p_i - 1} \int_a^x r_i\,(s) \left|u_i'\,(s)\right|^{p_i} ds = \xi_i^{p_i - 1}(x) \int_a^x r_i\,(s) \left|u_i'\,(s)\right|^{p_i} ds \tag{2.2}$$

and

$$|u_i\,(x)|^{p_i} \le \left(\int_x^b r_i^{1/(1-p_i)}(s) ds \right)^{p_i - 1} \int_x^b r_i\,(s) \left|u_i'\,(s)\right|^{p_i} ds = \eta_i^{p_i - 1}(x) \int_x^b r_i\,(s) \left|u_i'\,(s)\right|^{p_i} ds \tag{2.3}$$

for $x \in [a, b]$ and $i = 1, 2, \ldots, n$. Adding the inequalities (2.2) and (2.3), we have

$$|u_i\,(x)|^{p_i} \le R_i D_i\,(x)\,, \quad x \in [a, b]\,, \tag{2.4}$$

where $R_i = \displaystyle\int_a^b r_i\,(s) \left|u_i'\,(s)\right|^{p_i} ds$ for $i = 1, 2, \ldots, n$. After that by using similar technique to the proof of Theorem 3.1 in Tang and He [24], it can be showed that the equality case in (2.4) does not hold. Thus, we can write

$$|u_i\,(x)|^{p_i} < R_i D_i\,(x) \tag{2.5}$$

for $x \in (a, b)$ and $i = 1, 2, \ldots, n$. In addition, we can rewrite the inequality (2.5) as follows:

$$|u_{i+1}(x)|^{p_{i+1}} < R_{i+1} D_{i+1}(x) \tag{2.6}$$

for $x \in (a,b)$ and $i = 0,1,2,\ldots,n-1$. If we take the $1/p_i$ and $(1/p_{i+1})$th powers of both side of inequalities (2.5) and (2.6), we obtain

$$|u_i(x)| < R_i^{1/p_i} D_i^{1/p_i}(x) \tag{2.7}$$

for $i = 1,2,\ldots,n$ and

$$|u_{i+1}(x)| < R_{i+1}^{1/p_{i+1}} D_{i+1}^{1/p_{i+1}}(x) \tag{2.8}$$

for $i = 0,1,2,\ldots,n-1$, respectively. On the other hand, for $i = 1,2,\ldots,n-1$, multiplying the ith equation of system (1.1) by u_i, integrating from a to b and taking into account that $u_i(a) = 0 = u_i(b)$, we have from the inequalities (2.7) and (2.8)

$$R_i \le \int_a^b |f_i(s)|\, |u_i(s)|\, |u_{i+1}(s)|^{\alpha_i - 1}\, ds <$$

$$< R_i^{1/p_i} R_{i+1}^{(\alpha_i-1)/p_{i+1}} \int_a^b |f_i(s)|\, D_i^{1/p_i}(s)\, D_{i+1}^{(\alpha_i-1)/p_{i+1}}(s)\, ds$$

and hence

$$R_i^{(p_i-1)/p_i} < R_{i+1}^{(\alpha_i-1)/p_{i+1}} \int_a^b |f_i(s)|\, D_i^{1/p_i}(s)\, D_{i+1}^{(\alpha_i-1)/p_{i+1}}(s)\, ds \tag{2.9}$$

for $i = 1,2,\ldots,n-1$. Now, for $i = n$, multiplying the nth equation of system (1.1) by u_n, integrating from a to b and taking into account that $u_n(a) = 0 = u_n(b)$, we have from $u_{n+1}(x) = u_1(x)$ and the inequality (2.7) with $i = 1$ and $i = n$

$$R_n \le \int_a^b |f_n(s)|\, |u_n(s)|\, |u_{n+1}(s)|^{\alpha_n - 1}\, ds = \int_a^b |f_n(s)|\, |u_n(s)|\, |u_1(s)|^{\alpha_n - 1}\, ds <$$

$$< R_n^{1/p_n} R_1^{(\alpha_n-1)/p_1} \int_a^b |f_n(s)|\, D_n^{1/p_n}(s)\, D_1^{(\alpha_n-1)/p_1}(s)\, ds$$

and hence

$$R_n^{(p_n-1)/p_n} < R_1^{(\alpha_n-1)/p_1} \int_a^b |f_n(s)|\, D_n^{1/p_n}(s)\, D_1^{(\alpha_n-1)/p_1}(s)\, ds \tag{2.10}$$

for $n \in \mathbb{N}$. Raising the both sides of inequalities (2.9) and (2.10), to the power e_i for each $i = 1,2,\ldots,n-1$ and e_n, respectively, and multiplying the resulting inequalities side by side, we obtain

$$\prod_{i=1}^{n} R_i^{\frac{p_i-1}{p_i} e_i} < \prod_{i=1}^{n} R_{i+1}^{\frac{\alpha_i-1}{p_{i+1}} e_i} \prod_{i=1}^{n} \left(\int_a^b |f_i(s)|\, D_i^{\frac{1}{p_i}}(s)\, D_{i+1}^{\frac{\alpha_i-1}{p_{i+1}}}(s)\, ds \right)^{e_i}$$

and hence

$$R_1^{\frac{p_1-1}{p_1}e_1-\frac{\alpha_n-1}{p_1}e_n}\prod_{i=2}^{n}R_i^{\frac{p_i-1}{p_i}e_i-\frac{\alpha_{i-1}-1}{p_i}e_{i-1}}<\prod_{i=1}^{n}\left(\int_a^b|f_i(s)|\,D_i^{\frac{1}{p_i}}(s)\,D_{i+1}^{\frac{\alpha_i-1}{p_{i+1}}}(s)\,ds\right)^{e_i}. \tag{2.11}$$

Next, we prove that $R_i>0$ for $i=1,2,\ldots,n$. If the inequality $R_i>0$ is not true, then $R_i=0$ for $i=1,2,\ldots,n$. If $R_i=\displaystyle\int_a^b r_i(s)\left|u_i'(s)\right|^{p_i}ds=0$, then it follows that

$$u_i'(x)\equiv 0 \tag{2.12}$$

for $a\le x\le b$ and $i=1,2,\ldots,n$. Combining (2.2) with (2.12), we obtain that $u_i(x)\equiv 0$ for $a\le x\le b$, which contradicts $u_i(x)\not\equiv 0$ for $a\le x\le b$ and $i=1,2,\ldots,n$. Therefore, $R_i>0$ for $i=1,2,\ldots,n$ holds. Now, we find a relation between α_i and p_i for $i=1,2,\ldots,n$ such that $R_i>0$ for $i=1,2,\ldots,n$ cancel out in the inequality (2.11), i.e., solve the homogeneous linear system

$$(p_1-1)\,e_1-(\alpha_n-1)\,e_n=0,$$

$$(p_i-1)\,e_i-(\alpha_{i-1}-1)\,e_{i-1}=0$$

for $i=2,3,\ldots,n$. We observe that by hypothesis $\prod_{i=1}^{n}\dfrac{\alpha_i-1}{p_i-1}=1$, this system admits a nontrivial solution. Hence, we may take $e_i=e_1\dfrac{p_1-1}{\alpha_i-1}$ where $e_1>0$ and $i=1,2,\ldots,n$. Therefore, we obtain the inequality (2.1).

Theorem 2.1 is proved.

Another main result of this section is the following theorem.

Theorem 2.2. *Let the hypotheses (H_1) and (H_2) hold. If system* (1.1) *has a real nontrivial solution $(u_1(x),u_2(x),\ldots,\,u_n(x))$ such that $u_i(a)=0=u_i(b)$ for $i=1,2,\ldots,n$, where $a,b\in\mathbb{R}$ with $a<b$ are consecutive zeros and u_i for $i=1,2,\ldots,n$ are not identically zero on $[a,b]$, then the inequality*

$$1<\prod_{i=1}^{n}\left(\int_a^b|f_i(s)|\,E_i^{\frac{1}{p_i}}(s)\,E_{i+1}^{\frac{\alpha_i-1}{p_{i+1}}}(s)\,ds\right)^{\frac{1}{\alpha_i-1}} \tag{2.13}$$

holds, where $E_{n+1}(x):=E_1(x)$.

Proof. Let $u_i(a)=0=u_i(b)$, where $a,b\in\mathbb{R}$ with $a<b$ are consecutive zeros and u_i for $i=1,2,\ldots,n$ are not identically zero on $[a,b]$. As in the proof of Theorem 2.1, we have the inequalities (2.2) and (2.3). Multiplying the inequalities (2.2) and (2.3) by $\eta_i^{p_i-1}(x)$ and $\xi_i^{p_i-1}(x)$, $i=1,2,\ldots,n$, respectively, we obtain

$$\eta_i^{p_i-1}(x)\,|u_i(x)|^{p_i}\le(\xi_i(x)\eta_i(x))^{p_i-1}\int_a^x r_i(s)\left|u_i'(s)\right|^{p_i}ds \tag{2.14}$$

and

$$\xi_i^{p_i-1}(x)\,|u_i(x)|^{p_i}\le(\xi_i(x)\eta_i(x))^{p_i-1}\int_x^b r_i(s)\left|u_i'(s)\right|^{p_i}ds \tag{2.15}$$

for $x \in [a,b]$ and $i = 1,2,\ldots,n$. Thus, adding the inequalities (2.14) and (2.15), we have

$$|u_i(x)|^{p_i}\left(\xi_i^{p_i-1}(x)+\eta_i^{p_i-1}(x)\right) \le (\xi_i(x)\eta_i(x))^{p_i-1}\int\limits_a^b r_i(s)\left|u_i'(s)\right|^{p_i} ds$$

for $x \in [a,b]$ and $i = 1,2,\ldots,n$. It is easy to see that the functions $\xi_i^{p_i-1}(x)+\eta_i^{p_i-1}(x)$ take the minimum values at $c_i \in (a,b)$ such that $\xi_i(c_i) = \eta_i(c_i)$ for $i = 1,2,\ldots,n$. Thus, we get

$$|u_i(x)|^{p_i}\left(\xi_i^{p_i-1}(c_i)+\eta_i^{p_i-1}(c_i)\right) \le (\xi_i(x)\eta_i(x))^{p_i-1}\int\limits_a^b r_i(s)\left|u_i'(s)\right|^{p_i} ds$$

for $i = 1,2,\ldots,n$. Since $\xi_i(c_i)+\eta_i(c_i) = \xi_i(x)+\eta_i(x)$ $\forall x, c_i \in (a,b)$, and $\xi_i(c_i) = \dfrac{\xi_i(x)+\eta_i(x)}{2} =$ $= \dfrac{1}{2}\displaystyle\int_a^b r_i^{1/(1-p_i)}(s)ds$, we obtain

$$|u_i(x)|^{p_i}\left[2^{2-p_i}\left(\xi_i(x)+\eta_i(x)\right)^{p_i-1}\right] = |u_i(x)|^{p_i}\left[2\xi_i^{p_i-1}(c_i)\right] \le$$

$$\le (\xi_i(x)\eta_i(x))^{p_i-1}\int\limits_a^b r_i(s)\left|u_i'(s)\right|^{p_i} ds$$

and hence

$$|u_i(x)|^{p_i} \le R_i E_i(x), \tag{2.16}$$

where $R_i = \displaystyle\int_a^b r_i(s)\left|u_i'(s)\right|^{p_i} ds$ for $i = 1,2,\ldots,n$. After that by using similar technique to the proof of Theorem 3.1 in Tang and He [24], it can be showed that the equality case in (2.16) does not hold. Thus, we can write

$$|u_i(x)|^{p_i} < R_i E_i(x)$$

for $x \in (a,b)$ and $i = 1,2,\ldots,n$. The rest of the proof is the same as in the proof of Theorem 2.1, and hence is omitted.

Theorem 2.2 is proved.

***Remark* 2.1.** Note that $p_i > 1$ for $i = 1,2,\ldots,n$ in both Theorems 2.1 and 2.2. However, it is easy to see from the inequality (1.10) that if we take $1 < p_i < 2$ for $i = 1,2,\ldots,n$, then the inequality (2.13) is better than (2.1) in the sense that (2.1) follows from (2.13), but not conversely. Similarly, from the inequality (1.11), if $p_i > 2$ for $i = 1,2,\ldots,n$, then the inequality (2.1) is better than (2.13) in the sense that (2.13) follows from (2.1), but not conversely. In addition, if $p_i = 2$ for $i = 1,2,\ldots,n$, then Theorem 2.1 coincides with Theorem 2.2.

By using the inequality (1.12) in Theorem 2.1 or (1.13) in Theorem 2.2, we obtain the following result.

***Corollary* 2.1.** *Let the hypotheses* (H_1) *and* (H_2) *hold. If system* (1.1) *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1,2,\ldots,n$, *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1,2,\ldots,n$ *are not identically zero on* $[a,b]$, *then the inequality*

$$1<\prod_{i=1}^{n}\left(F_i^{\frac{1}{p_i}}F_{i+1}^{\frac{\alpha_i-1}{p_{i+1}}}\int\limits_a^b|f_i(s)|\,ds\right)^{\frac{1}{\alpha_i-1}}$$

holds, where $F_{n+1}:=F_1$.

If we get the conditions $\alpha_i=p_i=p_1$ and $r_i(x)\geq k_i>0$ for $i=1,2,\ldots,n$ in Theorems 2.1 and 2.2, then we obtain the following results, respectively.

***Corollary* 2.2.** *Let the hypotheses* (H_1) *and* (H_2) *hold. If system* (1.1) *with* $\alpha_i=p_i=p_1$ *and* $r_i(x)\geq k_i>0$ *for* $i=1,2,\ldots,n$ *has a real nontrivial solution* $(u_1(x),u_2(x),\ldots,\,u_n(x))$ *such that* $u_i(a)=0=u_i(b)$ *for* $i=1,2,\ldots,n$, *where* $a,\ b\in\mathbb{R}$ *with* $a<b$ *are consecutive zeros and* u_i *for* $i=1,2,\ldots,n$ *are not identically zero on* $[a,b]$, *then the inequality*

$$1<\left(\prod_{i=1}^{n}k_i^{\frac{1}{p_1}}k_{i+1}^{1-\frac{1}{p_1}}\right)^{-1}\prod_{i=1}^{n}\int\limits_a^b|f_i(s)|\left[\frac{1}{(s-a)^{1-p_1}+(b-s)^{1-p_1}}\right]ds$$

holds, where $k_{n+1}:=k_1$.

***Corollary* 2.3.** *Let the hypotheses* (H_1) *and* (H_2) *hold. If system* (1.1) *with* $\alpha_i=p_i=p_1$ *and* $r_i(x)\geq k_i>0$ *for* $i=1,2,\ldots,n$ *has a real nontrivial solution* $(u_1(x),u_2(x),\ldots,\,u_n(x))$ *such that* $u_i(a)=0=u_i(b)$ *for* $i=1,2,\ldots,n$, *where* $a,b\in\mathbb{R}$ *with* $a<b$ *are consecutive zeros and* u_i *for* $i=1,2,\ldots,n$ *are not identically zero on* $[a,b]$, *then the inequality*

$$1<\left(\prod_{i=1}^{n}k_i^{\frac{1}{p_1}}k_{i+1}^{1-\frac{1}{p_1}}\right)^{-1}\prod_{i=1}^{n}\int\limits_a^b|f_i(s)|\left[2^{p_1-2}\left(\frac{b-a}{(s-a)(b-s)}\right)^{1-p_1}\right]ds \tag{2.17}$$

holds, where $k_{n+1}:=k_1$.

***Remark* 2.2.** Note that since $k_{n+1}:=k_1$, we obtain $\prod_{i=1}^{n}k_i^{\frac{1}{p_1}}k_{i+1}^{1-\frac{1}{p_1}}=\prod_{i=1}^{n}k_i$. Let $f_i(x)\geq$ ≥ 0 for $i=1,2,\ldots,n$ hold. It is easy to see that if we take $0<\prod_{i=1}^{n}k_i<1$, then the inequality (2.17) is better than (1.7) in the sense that (1.7) follows from (2.17), but not conversely. Similarly, if $\prod_{i=1}^{n}k_i>1$, then the inequality (1.7) is better than (2.17) in the sense that (2.17) follows from (1.7), but not conversely. In addition, if $\prod_{i=1}^{n}k_i=1$, then Corollary 2.2 coincides with Theorem C given by Rodrigues [22].

***Remark* 2.3.** Let $\alpha_i=p_i=p_1$, $r_i(x)=1$, and $f_i(x)\geq 0$ for $i=1,2,\ldots,n$ hold. It is easy to see from the inequality (1.10) that if we take $1<p_1<2$, then Theorem A given by Sim and Lee [23] is better than Theorem 2.1 in the sense that (2.1) follows from (1.5), but not conversely. Similarly, from the inequality (1.11), if $p_1>2$, then Theorem 2.1 is better than Theorem A in the sense that (1.5) follows from (2.1), but not conversely. In addition, if $p_1=2$, then Theorems A, 2.1, and 2.2 are equivalent to each other.

3. Lyapunov-type inequalities for system (1.2). For system (1.2), one of the main results of this section is the following theorem.

Theorem 3.1. *Let the hypothesis* (H_1) *with* $j=1$ *hold. If system* (1.2) *has a real nontrivial solution* $(u_1(x),u_2(x),\ldots,\,u_n(x))$ *such that* $u_i(a)=0=u_i(b)$ *for* $i=1,2,\ldots,n$, *where* a, $b\in\mathbb{R}$ *with* $a<b$ *are consecutive zeros and* u_i *for* $i=1,2,\ldots,n$ *are not identically zero on* $[a,b]$, *then the inequality*

$$1 < \sum_{i=1}^{n} \int_a^b |f_i(s)|\, D_1(s)\, ds \tag{3.1}$$

holds.

Proof. Let $u_i(a) = 0 = u_i(b)$ where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, and u_i for $i = 1, 2, \ldots, n$ are not identically zero on $[a, b]$. It follows from the similar technique to the proof of Theorem 2.1, we obtain the inequalities

$$|u_i(x)|^{p_1} < M_i D_1(x), \tag{3.2}$$

$$|u_i(x)| < M_i^{1/p_1} D_1^{1/p_1}(x), \tag{3.3}$$

and

$$|u_i(x)|^{p_1-1} < M_i^{(p_1-1)/p_1} D_1^{(p_1-1)/p_1}(x), \tag{3.4}$$

where $M_i = \int_a^b r_1(s) \left|u_i'(s)\right|^{p_1} ds$ for $i = 1, 2, \ldots, n$. On the other hand, multiplying the ith equation of system (1.2) by u_i, integrating from a to b and taking into account that $u_i(a) = 0 = u_i(b)$, we have from the inequalities (3.2), (3.3), and (3.4) that

$$M_i \le \int_a^b |f_i(s)|\, |u_i(s)|^{p_1} ds + \int_a^b |f_i(s)|\, |u_i(s)| \sum_{\substack{j=1 \\ j \neq i}}^{n} |u_j(s)|^{p_1-1} ds <$$

$$< M_i \int_a^b |f_i(s)|\, D_1(s)\, ds + M_i^{1/p_1} \sum_{\substack{j=1 \\ j \neq i}}^{n} M_j^{(p_1-1)/p_1} \int_a^b |f_i(s)|\, D_1(s)\, ds$$

and hence

$$M_i^{(p_1-1)/p_1} < \sum_{j=1}^{n} M_j^{(p_1-1)/p_1} \int_a^b |f_i(s)|\, D_1(s)\, ds \tag{3.5}$$

for $i = 1, 2, \ldots, n$. It is easy to see that by using similar technique to the proof of Theorem 2.1, we obtain $M_i > 0$ for $i = 1, 2, \ldots, n$. By summing the inequalities (3.5), we have

$$\sum_{i=1}^{n} M_i^{(p_1-1)/p_1} < \sum_{j=1}^{n} M_j^{(p_1-1)/p_1} \sum_{i=1}^{n} \int_a^b |f_i(s)|\, D_1(s)\, ds$$

and hence

$$1 < \sum_{i=1}^{n} \int_a^b |f_i(s)|\, D_1(s)\, ds,$$

which completes the proof.

Theorem 3.2. *Let the hypothesis* (H_1) *with* $j = 1$ *hold. If system* (1.2) *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n$, *where* a, $b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 < \sum_{i=1}^{n} \int_{a}^{b} |f_i(s)| E_1(s)\, ds \tag{3.6}$$

holds.

Proof. Let $u_i(a) = 0 = u_i(b)$, where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u_i for $i = 1, 2, \ldots, n$ are not identically zero on $[a, b]$. It follows from the similar technique to the proof of Theorem 2.2, we have the inequalities

$$|u_i(x)|^{p_1} < M_i E_1(x),$$

$$|u_i(x)| < M_i^{1/p_1} E_1^{1/p_1}(x),$$

and

$$|u_i(x)|^{p_1 - 1} < M_i^{(p_1-1)/p_1} E_1^{(p_1-1)/p_1}(x),$$

where $M_i = \displaystyle\int_a^b r_1(s) \left|u_i'(s)\right|^{p_1} ds$ for $i = 1, 2, \ldots, n$. The rest of the proof is the same as in the proof of Theorem 3.1 and hence is omitted.

Remark **3.1.** Note that $p_1 > 1$ in both Theorems 3.1 and 3.2. However, it is easy to see from the inequality (1.10) that if we take $1 < p_1 < 2$, then the inequality (3.6) is better than (3.1) in the sense that (3.1) follows from (3.6), but not conversely. Similarly, from the inequality (1.11), if $p_1 > 2$, then the inequality (3.1) is better than (3.6) in the sense that (3.6) follows from (3.1), but not conversely. In addition, if $p_1 = 2$, then Theorem 3.1 coincides with Theorem 3.2.

By using the inequality (1.12) in Theorem 3.1 or (1.13) in Theorem 3.2, we have the following result.

Corollary **3.1.** *Let the hypothesis* (H_1) *with* $j = 1$ *hold. If system* (1.2) *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n$, *where* a, $b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 < F_1 \sum_{i=1}^{n} \int_{a}^{b} |f_i(s)|\, ds$$

holds.

If we get the condition $r_1(x) \geq k_1 > 0$ in Theorems 3.1 and 3.2, we obtain the following results, respectively.

Corollary **3.2.** *Let the hypothesis* (H_1) *with* $j = 1$ *hold. If system* (1.2) *with* $r_1(x) \geq k_1 > 0$ *has a real nontrivial solution* $(u_1(x), u_2(x), \ldots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \ldots, n$, *where* a, $b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \ldots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$1 < \frac{1}{k_1} \sum_{i=1}^{n} \int_a^b |f_i(s)| \left[\frac{1}{(s-a)^{1-p_1} + (b-s)^{1-p_1}} \right] ds \tag{3.7}$$

holds.

***Corollary* 3.3.** *Let the hypothesis (H_1) with $j = 1$ hold. If system (1.2) with $r_1(x) \geq k_1 > 0$ has a real nontrivial solution $(u_1(x), u_2(x), \ldots, u_n(x))$ such that $u_i(a) = 0 = u_i(b)$ for $i = 1, 2, \ldots, n$, where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u_i for $i = 1, 2, \ldots, n$ are not identically zero on $[a, b]$, then the inequality*

$$1 < \frac{1}{k_1} \sum_{i=1}^{n} \int_a^b |f_i(s)| \left[2^{p_1-2} \left(\frac{(s-a)(b-s)}{b-a} \right)^{p_1-1} \right] ds \tag{3.8}$$

holds.

***Remark* 3.2.** Let $r_1(x) \geq k_1 > 0$ in system (1.2), $r_i(x) = r_1(x) \geq k_1 > 0$ for $i = 1, 2, \ldots, n$ in system (1.8), and $f_i(x) \geq 0$ for $i = 1, 2, \ldots, n$ hold. It is easy to see from the inequality (1.11) that if we take $p_1 > 2$, then the inequality (3.7) is better than (1.9) in the sense that (1.9) follows from (3.7), but not conversely. Moreover, it is easy to see that the inequality (3.8) is better than (1.9) in the sense that (1.9) follows from (3.8), but not conversely. In addition, if $n = 1$, then Corollary 3.2 coincides with Theorem D given by Rodrigues [22].

***Remark* 3.3.** Let $f_i(x) \geq 0$ for $i = 1, 2, \ldots, n$ and $r_1(x) = 1$. If we take $n \neq 1$, the inequality (3.6) is better than (1.6) in the sense that (1.6) follows from (3.6), but not conversely. Thus, Theorem 3.2 improves and generalizes Theorem B given by Sim and Lee [23]. If $n = 1$, then Theorem 3.2 coincides with Theorem B. Moreover, from the inequality (1.11), if $p_1 > 2$, then Theorem 3.1 gives a better result than Theorem B. If $n = 1$ and $p_1 = 2$, then Theorems 3.1, 3.2, and B are equivalent to each other.

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