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R. Jedynak, $\boxed{\textbf{J. Gilewicz}}$ (K. Pulaski Univ. Technology and Humanities, Poland)

MAGIC EFFICIENCY OF APPROXIMATION OF SMOOTH FUNCTIONS BY WEIGHTED MEANS OF TWO N-POINT PADÉ APPROXIMANTS

ЧАРІВНА ЕФЕКТИВНІСТЬ НАБЛИЖЕННЯ ГЛАДКИХ ФУНКЦІЙ ЗВАЖЕНИМИ СЕРЕДНІМИ ДВОХ N-ТОЧКОВИХ ПАДЕ АПРОКСИМАЦІЙ

We consider the approximation of smooth functions by two weighted N-point Padé approximants. We present numerical examples and the inequalities between the Stietjes function and its N-point Padé approximant.

Статтю присвячено наближенню гладких функцій двома N-точковими наближеннями Паде з вагами. Наведено числові приклади, нерівності між функцією Стільтьєса та її N-точковим наближенням Паде.

We present the new developments of this intuitively discovered method of approximation of real functions f characterized by the two-sided estimate (TSE) property by weighted means of two N-point Padé approximants (NPA). This method was first presented in [13] and is briefly explained in the Introduction. In Section 2 we recall the inequalities between the Stieltjes functions and NPAs proved in [6, 7]. In the review article [6] these inequalities where extended outside the interval $[x_1, x_N]$ of definition of the NPA. Unfortunately, in general these extended inequalities do not hold for the entire interval $]x_N, \infty[$, which is illustrated by an example. In Section 3 we prove a surprising result: the weights used in the weighted approximation are convex. In Section 4 we discuss some questions related to the rescaling of the reference functions used in our method, the problem of the transformation of the function to be approximated, and the problem of the TSE property. Section 5 contains some illustrative examples.

1. Introduction. Let f be a function we wish to approximate on the interval $[x_1, x_N]$ knowing $p_1 > 1,\ p_2, \ldots, p_N$ coefficients of Taylor series expansion of f at the points $x_1, x_2, \ldots, x_N$. We start by computing two neighboring NPAs of f, namely $f_1 = [m/n]$ and $f_2 = [m-1/n]$ of f. The property TSE says that these two NPA f_1 and f_2 (the second being computed with a reduced amount of information by removing one coefficient from the expansion of f at one point x_i, in general at x_1) bound f in each interval $[x_i, x_{i+1}]$ on opposite sides. The TSE property holds for Stieltjes functions [7], but also for many other functions of practical interest (it holds for a great number of convex functions, but not all). In such a case, further steps become relatively simple. We select a known function s having the TSE property rescaling s such that its values $s(x_i)$ are as close as possible to the values $f(x_i)$. We then compute the approximants $s_1 = [m/n]$ and $s_2 = [m-1/n]$ using the values at the points x_i and determine for all x the weight function α from the equation

$$s(x) = \alpha(x)s_1(x) + (1-\alpha(x))s_2(x). \tag{1}$$

Applying this weight to calculate the weighted mean $\alpha f_1 + (1-\alpha)f_2$ we obtain a significantly improved approximation of f. This intuitive method of weighted means approximation was presented in [13] and gives more accurate results than all previously proposed methods: see the examples of the non-Stieltjes functions e^{-x}, e^{-x}/x and another one, playing an important role in tribology [2, 12, 15, 18]. The similar idea, the use of weight functions, which is a unit factorization to the

Taylor series, was proposed by O. M. Lytvyn and V. L. Rvachev for approximations by partial sums of the Taylor series [17], and then it was used for continuous fractions. The issue of continuous fractions is broadly discussed in our previous paper [8].

2. Inequalities between a Stieltjes function and its N-point Padé approximant. Let f be an analytic function at N different real points

$$-R < x_1 < x_2 < \ldots < x_N < \infty$$

having the Taylor expansion

$$\sum_{k=0}^{p_j-1} c_k(x_j)(x-x_j)^k + O\Big((x-x_j)^{p_j}\Big), \quad j = 1,\ldots,N. \tag{2}$$

Then the NPA to f, if it exists, is a rational function P_m/Q_n denoted as follows [9, 14]:

$$[m/n]^{p_1 p_2 \ldots p_N}_{x_1 x_2 \ldots x_N}(x) = \frac{a_0 + a_1 x + \ldots + a_m x^m}{1 + b_1 x + \ldots + b_n x^n}, \qquad m+n+1 = p = p_1 + p_2 + \ldots + p_N, \tag{3}$$

and satisfying the following relations:

$$f(x) - [m/n](x) = O\Big((x-x_j)^{p_j}\Big), \quad j = 1,2,\ldots,N,$$

where each p_j represents the number of coefficients $c_k(x_j)$ of the expansion (2) actually used for the computation of the NPA given by (3). In the following we deal only with subdiagonal $[n-1/n]$ and diagonal $[n/n]$ NPAs. Let us introduce a nondecreasing step-wise function L

$$L(x) = \sum_{j=1}^{N} p_j H(x - x_j),$$

where H is the Heaviside function. The value $L(x)$ denotes the total number of given coefficients of the power series expansions of f at all points $x_j \le x$:

$$L(x_k) = p_1 + p_2 + \ldots + p_k, \qquad L(x_N) = p = \sum_{j=1}^{N} p_j.$$

Theorem 1 (M. Barnsley). *Let s be a Stieltjes function defined as follows*:

$$s(z) = \int_0^{1/R} \frac{d\mu(t)}{1+tz}, \qquad z \in \mathbb{C}\backslash]-\infty, -R], \quad d\mu \ge 0,$$

then the subdiagonal $[k-1/k]$ and diagonal $[k/k]$ NPA to s obey the following inequality:

$$(-1)^{L(x)}[m/n](x) \le (-1)^{L(x)} s(x), \quad x \in]-R, \infty[.$$

Notice that the parity of L controls the position of the NPA with respect to s and so, playing with this parity, we can obtain two sided estimates of s. In particular, for $x < x_1$, all NPAs are less than s. Suppose, but it is without importance, that $p = L(x_N) = 2k+1$. Then our NPA is $[k/k]^{p_1 \dots p_N}_{x_1 \dots x_N}$. Removing one piece of information (one coefficient) from the development of s at x_1 we obtain $[k-1/k]^{p_1 - 1 \dots p_N}_{x_1 \dots x_N}$ which, starting from x_1, bounds s on the opposite side with respect to $[k/k]$. Immediately we can imagine a nice improvement of the approximation by computing the mean value of both these NPAs. However, the result is not so nice: $[k/k]$ is in fact better that this mean because $\big|s(x) - [k/k]\big| \ll \big|s(x) - [k-1/k]\big|$, which shows that the convergence of the Padé approximant to Stieltjes functions is very fast. This observation has suggested our weighted means approximation method.

The following theorem was proved in [6].

Theorem 2. *In the two first inequalities, the second NPA is computed with one piece of information removed at x_1. In the two last inequalities, the second NPA is computed with one piece of information removed at an arbitrary point x_r with $r \le i$:*

$$k \ge 1, \quad p = 2k+1, \quad x \in \,]-R, x_1[\,:$$

$$0 < s(x) - [k/k]^{p_1 \dots p_N}_{x_1 \dots x_N} < \frac{x_1 - x}{x_1 + R}\Big\{s(x) - [k-1/k]^{p_1 - 1 \dots p_N}_{x_1 \dots x_N}\Big\},$$

$$k \ge 0, \quad p = 2k+2, \quad x \in \,]-R, x_1[\,:$$

$$0 < s(x) - [k/k+1]^{p_1 \dots p_N}_{x_1 \dots x_N} < \frac{x_1 - x}{x_1 + R}\Big\{s(x) - [k/k]^{p_1 - 1 \dots p_N}_{x_1 \dots x_N}\Big\},$$

$$k \ge 1, \quad p = 2k+1, \quad 1 \le i \le N-1, \quad x \in \,]x_i, x_{i+1}[\,:$$

$$0 < (-1)^{L(x)}\Big\{s(x) - [k/k]^{p_1 \dots p_N}_{x_1 \dots x_N}\Big\} < \frac{x_{i+1} - x}{x_{i+1} - x_i}(-1)^{L(x)-1}\Big\{s(x) - [k-1/k]^{p_1 \dots p_r - 1 \dots p_N}_{x_1 \dots x_r \dots \dots x_N}\Big\},$$

$$k \ge 0, \quad p = 2k+2, \quad 1 \le i \le N-1, \quad x \in \,]x_i, x_{i+1}[\,:$$

$$0 < (-1)^{L(x)}\Big\{s(x) - [k/k+1]^{p_1 \dots p_N}_{x_1 \dots x_N}\Big\} < \frac{x_{i+1} - x}{x_{i+1} - x_i}(-1)^{L(x)-1}\Big\{f(x) - [k/k]^{p_1 \dots p_r - 1 \dots p_N}_{x_1 \dots x_r \dots \dots x_N}\Big\}.$$

The two last inequalities presented in the original theorem [6] (Theorem 4.3, inequalities (74) and (75)) for the interval $]x_N, \infty[$ are not valid in general. They depend on each particular case: on the Stieltjes function s and on the chosen NPAs. The classical inequalities between the ordinary PAs, i.e., the 1-point Padé approximant to the Stieltjes functions [5, p. 264–265] in $]x_N, \infty[\, \equiv \,]0, \infty[$ do not imply general inequalities for errors in this interval.

Example. We consider three classical Padé Padé approximants to the Stieltjes function

$$s(x) = \frac{\ln(1+x)}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \dots :$$

$$[0/0](x) = 1, \qquad [0/1](x) = \frac{2}{2+x} \qquad \text{and} \qquad [1/1](x) = \frac{6+x}{6+4x}.$$

In this case $N = 1$, $x_1 = x_N = 0$ and $s(x) - [0/1](x) < [0/0](x) - s(x)$, that is,

$$\frac{\ln(1+x)}{x} - \frac{2}{2+x} < 1 - \frac{\ln(1+x)}{x} \tag{4}$$

holds for all $x > 0$, as shown in Fig. 1.

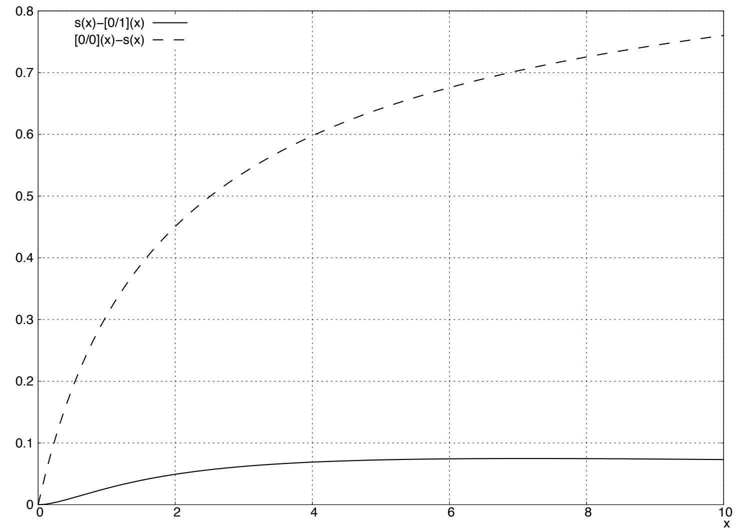


Fig. 1. In this case inequality (4) holds on all interval $]x_N, \infty[\, = \,]0, \infty[$.

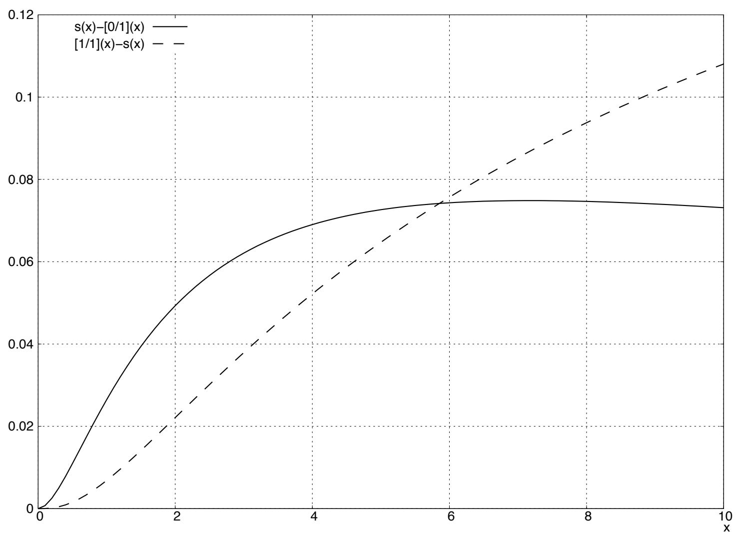


Fig. 2. In this case inequality (5) holds only on the interval $]x_1, 5+\varepsilon[\, = \,]0, 5+\varepsilon[\, \neq \,]x_1, \infty[$.

On the contrary, the inequality $[1/1](x) - s(x) < s(x) - [0/1](x)$, that is,

$$\frac{6+x}{6+4x} - \frac{\ln(1+x)}{x} < \frac{\ln(1+x)}{x} - \frac{2}{2+x} \tag{5}$$

is satisfied only for $x \in \,]0, 5+\varepsilon[$ and is false for $x = 6$, as shown in Fig. 2.

3. Convexity of the weight function. We prove a new surprising result: the weight $\alpha(x) = \dfrac{s(x) - s_2(x)}{s_1(x) - s_2(x)}$ defined by s and two oscillating NPAs is a smooth convex function. Then, we can use α in the formulae (1) continuously. We prove this property in the particular case of the Stieltjes function $s(x) = \dfrac{\ln(1+x)}{x}$ and two NPAs, $s_1(x) = [2/1]_{012}^{211}(x)$ and $s_2(x) = [1/1]_{012}^{111}(x)$.

The power series expansions of s at $x = 0, 1, 2$ are

$$s(x) = \frac{\ln(1+x)}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 + \ldots =$$

$$= \ln 2 + \left(\frac{1}{2} - \ln 2\right)(x-1) + \left(\ln 2 - \frac{5}{8}(x-1)^2\right) + \ldots =$$

$$= \frac{\ln 3}{2} + \left(\frac{1}{6} - \frac{\ln 3}{4}\right)(x-2) + \left(\frac{\ln 3}{8} - \frac{1}{9}\right)(x-2)^2 + \ldots,$$

$$s_1(x) = [2/1]_{012}^{211}(x) =$$

$$= \frac{(4 - 8\ln 2 + 2\ln 3) + (-6 + 12\ln 2 - 2\ln 3)x + (2 - 4\ln 2 + \ln 2 \times \ln 3)x^2}{(4 - 8\ln 2 + 2\ln 3) + (-4 + 8\ln 2 - \ln 3)x} =$$

$$= \frac{1. + .184866x - .017006x^2}{1. + .684866x},$$

$$s_2(x) = [1/1]_{012}^{111}(x) = \frac{2(2\ln 2 - \ln 3) + (-2\ln 2 + 2\ln 3 - \ln 2 \times \ln 3)x}{2(2\ln 2 - \ln 3) + (2 - 4\ln 2 + \ln 3)x} = \frac{1. + .085911x}{1. + .566639x},$$

$$\alpha(x) = \frac{\left(x\left(\ln\left(\frac{256}{3}\right) - 4\right) + 4 - 8\ln(2) + \ln(9)\right)}{x^2(x^2 - 3x + 2)\left(4 - (\ln(81) - 16)\ln^2(2) + \ln^2(3)\ln(2) - \ln(3)\ln(4) - \ln\left(\frac{65536}{9}\right)\right)} \times$$

$$\times \frac{\left(\left(\ln\left(\frac{16}{9}\right) - x\left(\ln\left(\frac{16}{3}\right) - 2\right)\right)\ln(x+1) - x\left(x\left(\ln\left(\frac{9}{4}\right) - \ln(2)\ln(3)\right) + \ln\left(\frac{16}{9}\right)\right)\right)}{x^2(x^2 - 3x + 2)\left(4 - (\ln(81) - 16)\ln^2(2) + \ln^2(3)\ln(2) - \ln(3)\ln(4) - \ln\left(\frac{65536}{9}\right)\right)}$$

We can observe that the weight $\alpha(x)$ has the indeterminate form $0/0$ at the points $x_i = 0, 1, 2$. They can be quickly evaluated with the help of L'Hospital's rule. Fig. 3 shows plots of derivative of numerator $s'(x) - s_2'(x)$ and derivative of denominator $s_1'(x) - s_2'(x)$ of $\alpha(x)$. It is clear that for those mentioned before points exist limits because denominator is not equal to 0. For example

$$\alpha(0) = \lim_{x \to 0} \frac{s(x) - s_2(x)}{s_1(x) - s_2(x)} = \frac{0}{0} = \lim_{x \to 0} \frac{s'(x) - s_2'(x)}{s_1'(x) - s_2'(x)} = 1.$$

The weight α is convex: the second derivative is positive, as shown in Fig. 4. We remark also that if the numerical values are rounded to four digits, the resulting α is incorrect. This is due to the very fast convergence of NPAs to Stieltjes functions, i.e., $\left|s(x) - s_1(x)\right| \ll \left|s(x) - s_2(x)\right|$ and then a high numerical precision is necessary to obtain a correct result.

4. The TSE property, rescaling, modification of approximated function. In the next section we show the practical application of the proposed method of approximation on a few examples. They cover the inverse Langevin function with its transformed function and a statistical integral that is popular in tribology. Before we apply this method, we have to consider some additional requirements. They concern the problem of selecting the appropriate reference function s, its rescaling, and choice of the NPA.

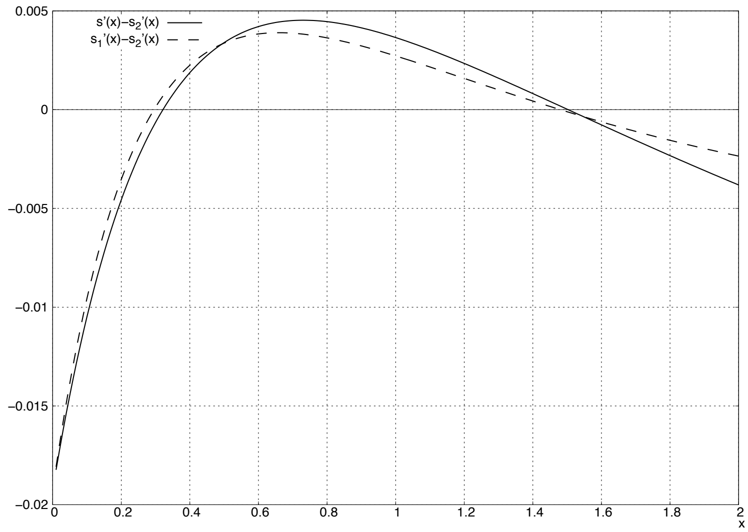


Fig. 3. Plots of derivative of numerator $s'(x) - s_2'(x)$ and derivative of denominator $s_1'(x) - s_2'(x)$ of $\alpha(x)$.

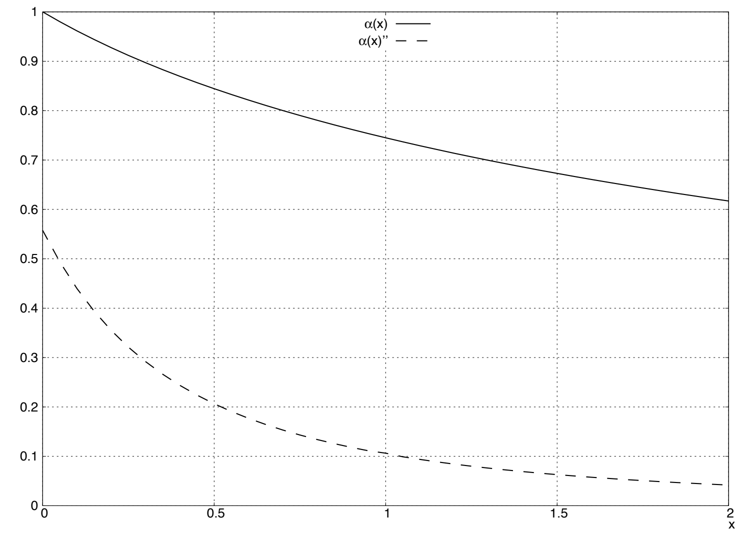


Fig. 4. Convexity of the weight α.

As discussed in our previous paper, we choose, for a reference function, functions that possess the TSE property, i.e., the existence of the two sided estimates and the same kind of smoothness as the approximated function f. Many real functions have these properties, such as Stieltjes functions, and any function $(a + bx)h(x)$ where h is a Stieltjes function. Other examples used in our research include the function $(a + bx)\operatorname{ctg} h(x)$, which is not Stieltjes.

To simplify the problem of rescaling the reference function s to be as close as possible to f, we use a simple condition at one point x^* of the interval $[x_1, x_N]$:

$$s(x^*) = f(x^*).$$

In our examples we use the inverse Langevin function and its transformation. Because the well-known approximation formulas for this function are based on $[3/1]$ and $[3/2]$ PA: we choose the NPAs $f_1[3/1]$ and $f_2[2/1]$ for our computation of the inverse Langevin function. For computation of transformed function we use the NPAs $f_1[2/1]$ and $f_2[1/1]$.

In our further studies we use α_s, which means the average value of the function $\alpha(x)$ on $[x_1, x_N]$. It is computed from the following formula:

$$\alpha_s = \frac{1}{x_N - x_1} \int\limits_{x_1}^{x_N} \alpha(x)\, dx.$$

The mean value simplifies the computation of new approximation formulas and gives quite reasonable results in comparison to using the continuous function $\alpha(x)$ in certain cases. In the case of using the continuous weight $\alpha(x)$ we obtain the approximation formula

$$m = \alpha(x) f_1 + (1 - \alpha(x)) f_2,$$

and in the case of using the mean value α_s we get

$$m_s = \alpha_s f_1 + (1 - \alpha_s) f_2. \tag{6}$$

Our examples show that these approximations can work equally well.

5. Examples. In this section we analyse different attempts to approximate the inverse Langevin function and the Gaussian distribution of asperity heights in a tribology problem. The Langevin function is defined by

$$\mathcal{L}(x) = \coth x - 1/x = 1 + \frac{2}{e^{2x} - 1} - \frac{1}{x} = \frac{(x-1)e^{2x} + 1 + x}{x(e^{2x} - 1)}$$

but its inverse $\mathcal{L}^{-1}$ has no analytical form. However, their power series expansions can be calculated arbitrarily far [11]. For instance, its development at $x = 0$ is

$$\mathcal{L}^{-1}(x) = 3x + \frac{9x^3}{5} + \frac{297x^5}{175} + \frac{1539x^7}{875} + \frac{126117x^9}{67375} + \frac{43733439x^{11}}{21896875} + \frac{231321177x^{13}}{109484375} +$$
$$+ \frac{20495009043x^{15}}{9306171875} + \frac{1073585186448381x^{17}}{476522530859375} + \frac{4387445039583x^{19}}{1944989921875} + O\left(x^{21}\right).$$

This series can be divided by x and multiplied by $(1 - x)$. In fact it is not difficult to observe that $\mathcal{L}^{-1}$ has an asymptote at $x = 1$. $\mathcal{L}^{-1}$ is defined on $[0, 1]$ and plays an important role in the theoretical physics analysis of polymers [1, 20, 21]. Their values in a neighbourhood of $x = 1$ are of a little practical interest. For instance, the case $x = 1$ corresponds to the unreal case of a polymer with all molecules aligned. A simple approximation of $\mathcal{L}^{-1}$ is needed in many theoretical calculations. The following attempts at approximations will be presented in the following:

1) simple PA,
2) NPA,
3) different optimization methods,
4) our general method of weighted means approximation of two neighbouring NPAs.

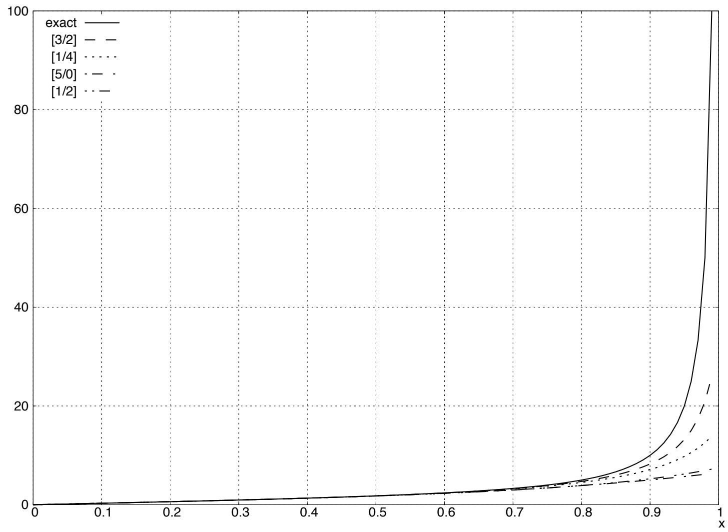


Fig. 5. Comparison of different PAs of the inverse Langevin function for $x \in [0, 1]$.

The well known and the most popular PA $[3/2]$ is given by Cohen [3] but it is usually used in the rounded form

$$\mathcal{L}^{-1}(x) = x\frac{3 - x^2}{1 - x^2}, \quad \epsilon_{\max} \approx 4.9\%.$$

Thanks to the process of rounding off, the formula has the required singularity at $x = 1$. This approximant has been widely accepted and developed by numerous researchers.

The proposals made by Darabi and Itskov [4]

$$\mathcal{L}^{-1}(x) = x\frac{3 - 3x + x^2}{1 - x}, \quad \epsilon_{\max} \approx 2.6\%$$

and Jedynak [10]

$$\mathcal{L}^{-1}(x) = x\frac{3.0 - 2.6x + 0.7x^2}{(1 - x)(1 + 0.1x)}, \quad \epsilon_{\max} \approx 1.5\%$$

are more exact than the PA. They are derived by the use of NPA and are presented in rounded forms. $\epsilon_{\max}$ shows the maximal relative error for the appropriate formula.

Fig. 5 compares the different PAs of the inverse Langevin function. Fig. 6 presents the relative errors of these approximants.

Quite different approaches from those mentioned above can be found in the third attempt. These methods use mathematical software to minimize the maximal error of the approximation formula by modifying the coefficients of previous approximations in each particular case to obtain a better result. A representative example of this method is the solution obtained by Marchi and Arruda.

In our study we would like to concentrate on the Padé approach therefore we do not discuss further optimization methods in this paper.

In our further research we want to find the answer to the question whether the inverse Langevin function has the TSE property. We also wish to find a function simply related to the original having the TSE property. We initially considered 3 candidates: $R_1(x) = \dfrac{\mathcal{L}^{-1}(x)}{x}$, $R_2(x) = (1 - x)\mathcal{L}^{-1}(x)$

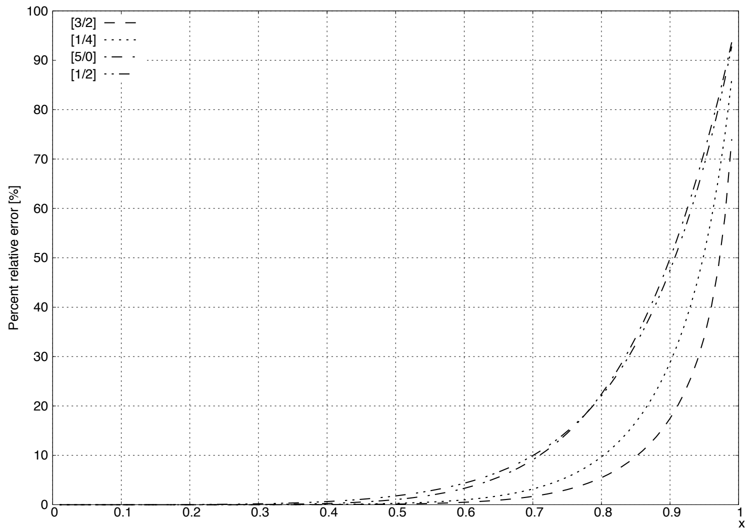


Fig. 6. Graphs of relative error for $x \in [0, 1]$.

and $R_3(x) = (1-x)\dfrac{\mathcal{L}^{-1}(x)}{x}$. Our detailed inspection showed that only the first transformed function has the partial TSE property.

In the next subsection we show in detail our intensive research on R_1 function and next on the inverse Langevin function. This section also documents extended study on the Gaussian integral, which was introduced in our previous paper [13].

5.1. The case of $f(x) = R_1(x)$. In this example we consider the following information at four points $x = 0,\ 0.3,\ 0.6,\ 0.99$: $f(x_1),\ f'(x_1),\ f(x_2),\ f(x_3),\ f(x_4)$ to compute the NPAs $f_1 = [2/2]^{2111}_{x_1x_2x_3x_4}$ and $f_2 = [1/2]^{1111}_{x_1x_2x_3x_4}$. Our reference function after rescaling process is

$$s(x) = 3x\coth x = 3x\frac{e^{2x}+1}{e^{2x}-1}.$$

Fig. 7 presents R_1 and s and their first derivatives.

We obtain the following NPAs:

$$f_1(x) = \frac{-1.35932x^2 + 0.166147x + 3}{-1.05775x^2 + 0.0553938x + 1},$$

$$f_2(x) = \frac{2.99999 - 1.6867x}{-0.405847x^2 - 0.595011x + 1}$$

and the continuous weight function $\alpha(x)$ shown in Fig. 8:

$$\alpha(x) = \frac{0.157155\bigl(x^2 + 0.0531582x + 16.0798\bigr)}{x^4 - 1.89x^3 + 1.071x^2 - 0.1782x}\times$$

$$\times\frac{\bigl((x^2 - 2.23917x - 1.7268)x\coth(x) + 2.16865x + 1.7268\bigr)}{x^4 - 1.89x^3 + 1.071x^2 - 0.1782x}$$

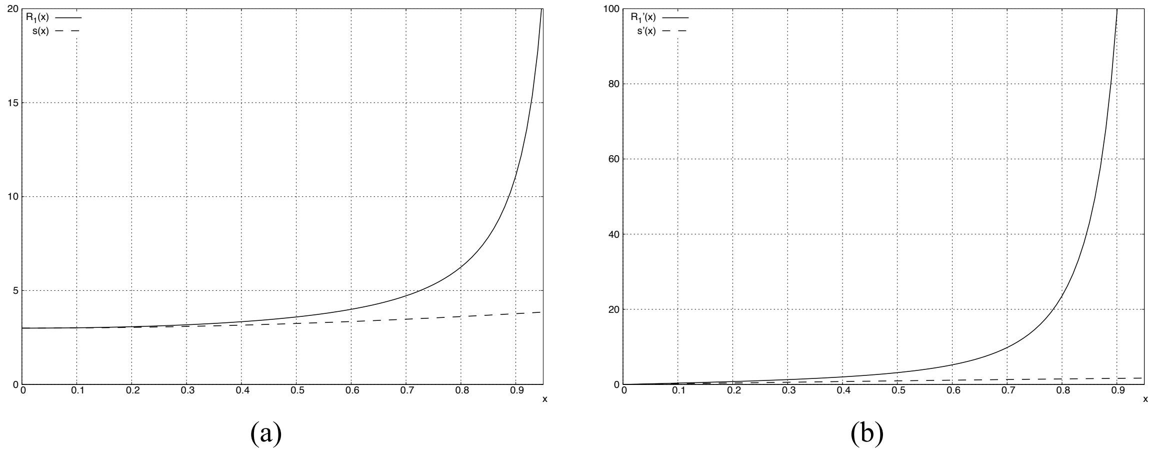


Fig. 7. The functions R_1 and s (a) and their first derivatives for $x \in [0, 1]$ (b).

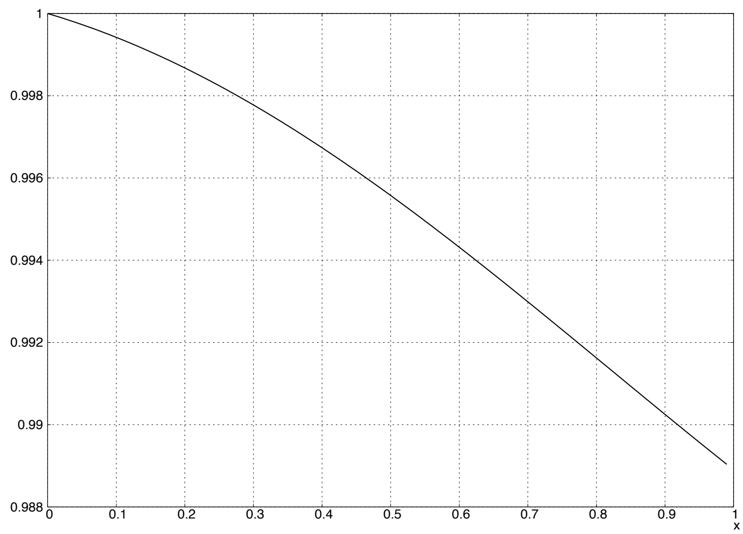
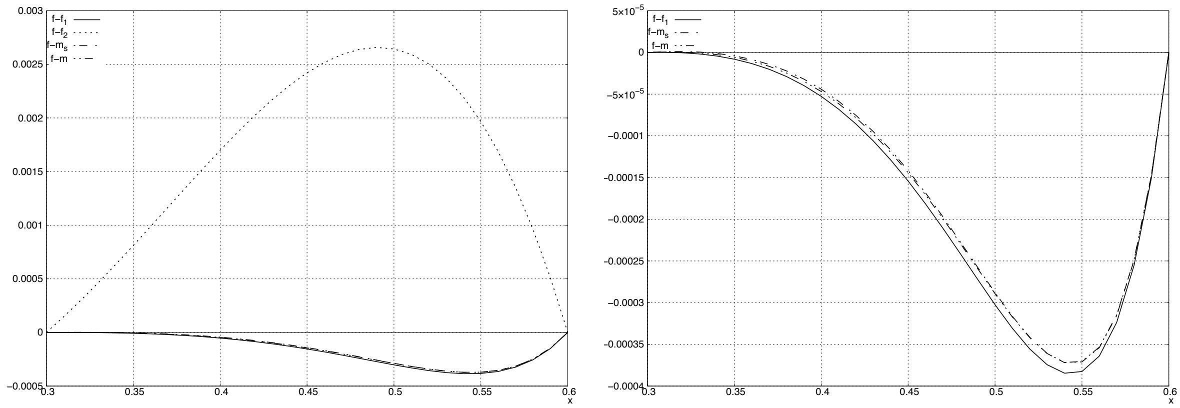


Fig. 8. Plot of α for $x \in [0, 1]$.

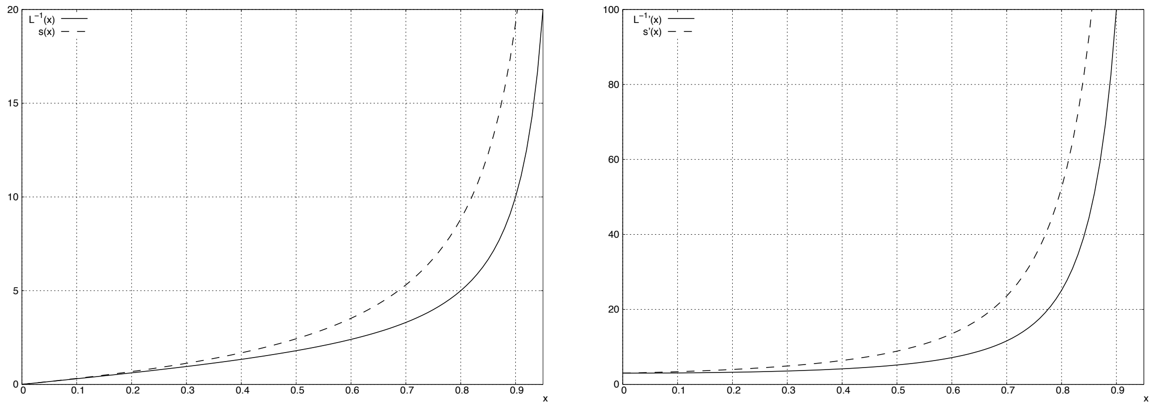
The error functions of the NPAs $f_1,\ f_2,\ m$ and m_s are depicted in Fig. 9 (a). From this figure we can conclude that R_1 has partial TSE property in the interval $[0.3, 0.6]$. Fig. 9 (b) shows the plots from the previous picture without f_2. It documents a slight superiority of the continuous α and simplified approximation with the weighted mean α_s over f_1 in this interval. We observe that in the interval $[0.3,\ 0.95]$, the new method gives the best results (Table 1).

Table 1 presents the numerical results. Recall that m in the third column corresponds to the continuous approximation used on the interval $[0, 1]$ and the fourth column corresponds to the approximation m_s (6) on the interval $[0, 1]$ computed with the mean weight $\alpha_s = 0.99509$. We can observe that only at the beginning of the interval $[0, 0.3]$ are our new proposals a bit worse than the 'good' NPA f_1. It is surprising that the approximation with the mean weight α_s gives the best results on the interval $[0.3, 0.5]$.



(a) (b)

Fig. 9. Plot of errors of f_1, f_2, m, and m_s (*a*) and without f_2 (*b*).



(a) (b)

Fig. 10. Plot of $\mathcal{L}^{-1}$ and s functions (*a*) and their first derivatives for $x \in [0, 1]$ (*b*).

5.2. The case of $-f(x) = \mathcal{L}^{-1}(x)$. In this case we use the following information at four points $x = 0,\ 0.2,\ 0.6,\ 0.9:$ $f(x_1),\ f'(x_1),\ f(x_2),\ f(x_3),\ f(x_4)$ to compute the NPAs $f_1 = [3/1]^{2111}_{x_1x_2x_3x_4}$ and $f_2 = [2/1]^{1111}_{x_1x_2x_3x_4}$. The same points were used by Darabi and Itskov [4] to compute their $[3/1]$ approximation formula for the inverse Langevin function.

We choose the following reference function for this case:

$$s(x) = 3\frac{\ln(x+1)}{1-x}.$$

Fig. 10 presents the functions $\mathcal{L}^{-1}$ and s and their first derivatives.

We obtain the following NPAs:

$$f_1(x) = \frac{x(3 - 2.86755x + 0.934859x^2)}{1 - 0.993467x},$$

$$f_2(x) = \frac{x(2.89653 - 2.25965x)}{1 - 1.02483x}$$

Table 1. Errors of approximations for $f(x) = R_1(x)$

x	$f - f_1$	$f - m$	$f - m_s$
0.	0	0	0
0.05	–0.0000219185	–0.0000228992	–0.0000399383
0.1	–0.000056229	–0.000059273	–0.0000820745
0.15	–0.0000706737	–0.0000755992	–0.0000966069
0.2	–0.0000562392	–0.0000617667	–0.0000766606
0.25	–0.0000236783	–0.000027706	–0.000034875
0.3	0	0	0
0.35	–0.0000231763	–0.0000167292	–0.0000116906
0.4	–0.000131737	–0.000117268	–0.000110236
0.45	–0.000342822	–0.000320593	–0.000314725
0.5	–0.000604877	–0.000578399	–0.000575948
0.55	–0.000695581	–0.000673674	–0.000674612
0.6	0	0	0
0.65	0.00297636	0.00292337	0.00293611
0.7	0.0116858	0.0115257	0.011576
0.75	0.0339223	0.0335597	0.0336958
0.8	0.0870261	0.0862857	0.0866021
0.85	0.209815	0.20834	0.209035
0.9	0.501434	0.498365	0.499928
0.95	1.34894	1.34133	1.34545

and the continuous weight function $\alpha(x)$ which is shown in Fig. 11:

$$\alpha(x) = \frac{1.57378(x - 1.00402)(x(x^2 - 3.92326x + 2.92326) + (3.00879x - 2.99161)\ln(x + 1))}{(x - 1)x(x^3 - 1.7x^2 + 0.84x - 0.108)} \tag{7}$$

The error functions of NPAs f_1, f_2, m and m_s are depicted in Fig. 12. They clearly show that our function has the TSE property in the whole interval. We inspected also percent relative error of our continuous weighted mean approximation and compared to two well-known formulas: Jedynak $[3/2]$ and Darabi & Itskov $[3/1]$. Fig. 13 shows plots of errors for those mentioned before formulas. The maximum relative error of our new proposition in the interval $[0, 0.9]$ is equal to 0.35%. Our previous approximation formula $[3/2]$ gives 1.5 % in this interval, while Darabi & Itskov $[3/1]$ 2.6%. We are very surprised this excellent accuracy. We remind the fact that our 'good' NPA is the same as Darabi & Itskov $[3/1]$. The final approximation formula of our previous NPA $[3/2]$ approximation formula and Darabi & Itskov $[3/1]$ one slightly differ from NPAs $[3/2]$ and $[3/1]$ because they were a little rounded to capture the asymptotic property of approximation formula at the point $x = 1$. In the literature we can find two other proposals of rational approximation $R_{3,1}$ of the inverse Langevin

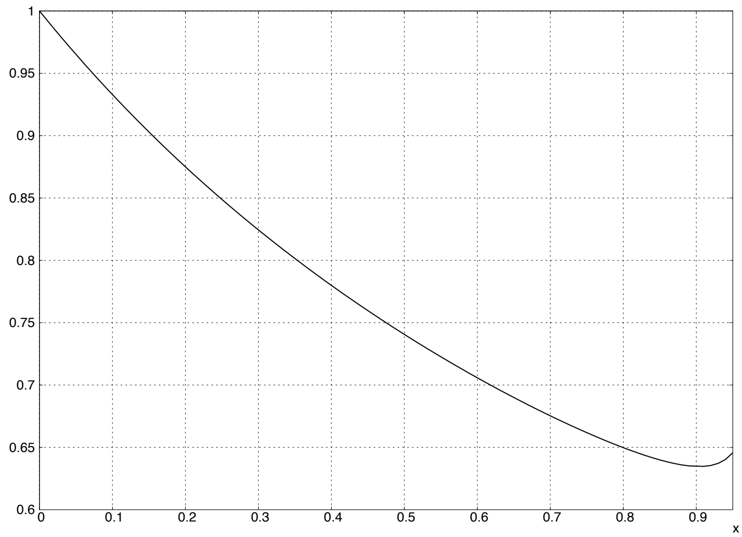


Fig. 11. Plot of α for $x \in [0, 1]$.

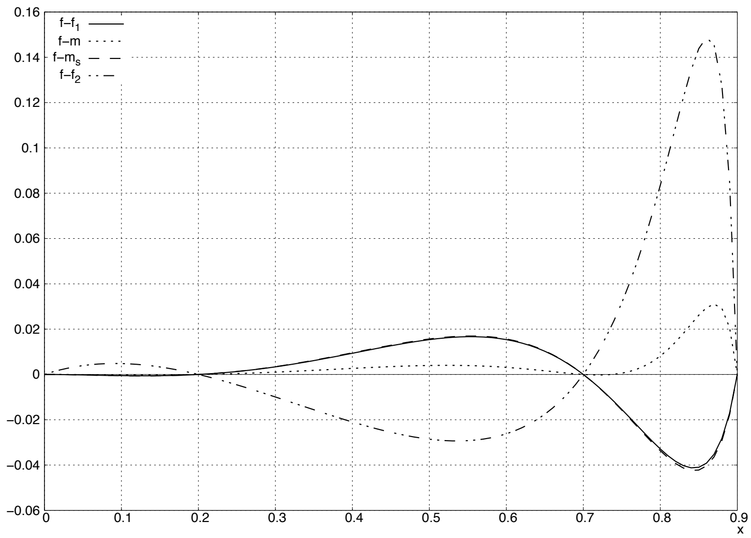


Fig. 12. Plot of errors of f_1, f_2, m and m_s. Information at the following points $x = 0,\ 0.2,\ 0.6,\ 0.9$ is used for those approximations.

function by Marchi and Arruda [19] and Kröger [16]. In both cases, the maximum relative error of approximation formula is about 0.95% in that mentioned before interval. We cite these facts to emphasize high precision of our new numerical method of approximation.

It is seen that our solution approximates very accurately the inverse Langevin function in the interval $[0, 0.9]$. Outside of this interval we observe a decrease in accuracy. We suppose that this decrease results from the location of the last node $x = 0.9$. We decided to do an experiment with different values of this point to confirm this assumption. The obtained results for $x = 0.95$ are illustrated in Fig. 14 and for $x = 0.99$ in Fig. 15. They clearly show that our solution is very

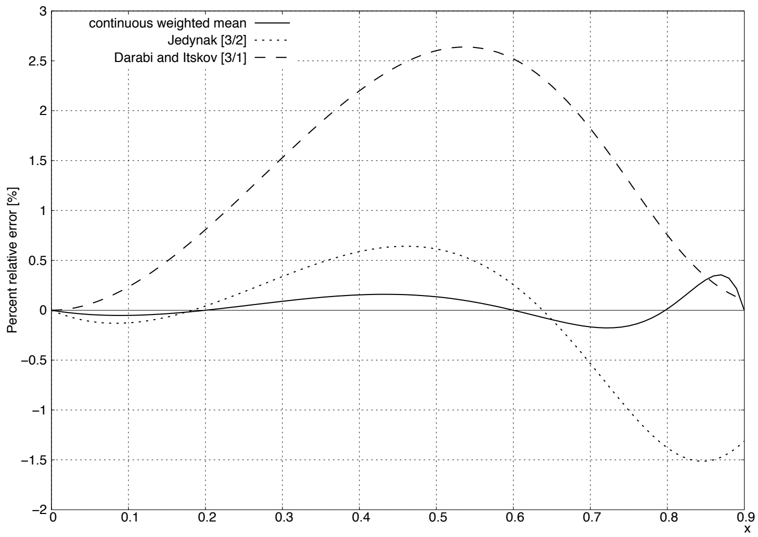


Fig. 13. Plot of percent relative error of new proposition (weighted mean), Jedynak $[3/2]$ and Darabi & Itskov $[3/1]$.

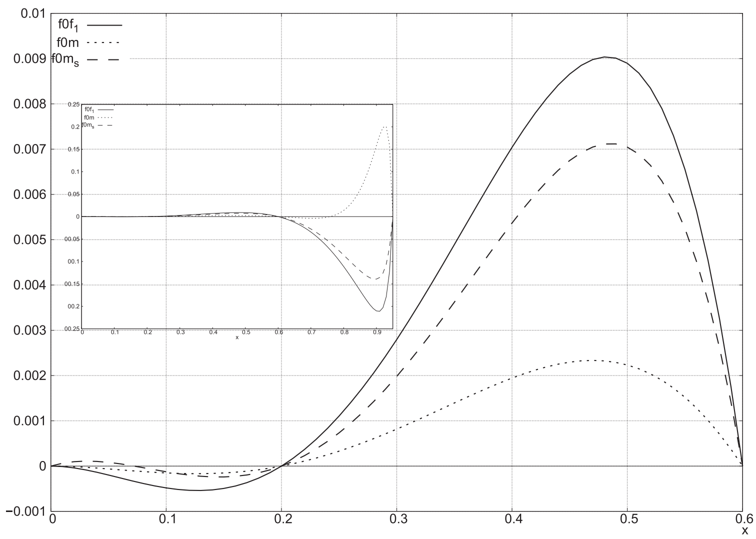


Fig. 14. Plot of errors of $f_1,\ f_2,\ m$ and m_s. Information at the points $x = 0,\ 0.2,\ 0.6,\ 0.95$ is used for those approximations.

sensitive to the location of the mentioned node. Both figures document that m and m_s give a more exact approximation than 'good' NPA (f_1) in a bigger interval than in the first case with $x = 0.9$.

Using the mean weights $m_s = 1.10592 f_1 - 0.10592 f_2$ we get the results which are presented in Table 2.

We notice that in the interval $[0, 0.9]$ the best results are given by the approximation with continuous α. Our simplified approximation m_s with the mean weight α_s gives slightly worse results than f_1. This situation changes radically when we consider higher values of the last node (it is $x = 0.95$ or $x = 0.99$). This effect is documented by Figs. 14 and 15.

Table 2. Errors of approximations for function $f(x) = \mathcal{L}^{-1}(x)$

x	$f - f_1$	$f - m$	$f - m_s$
0.	0	0	0
0.05	–0.000194303	–0.0000646495	–0.000588925
0.1	–0.000473798	–0.000155885	–0.00097594
0.15	–0.000484451	–0.000157643	–0.000841124
0.2	0	0	0
0.25	0.0010798	0.000343309	0.00159598
0.3	0.00272264	0.000854084	0.00384982
0.35	0.00474827	0.00146613	0.00649942
0.4	0.00680342	0.00205991	0.00908647
0.45	0.00832601	0.00245746	0.0109124
0.5	0.00849974	0.00242318	0.010981
0.55	0.00620616	0.00168379	0.00793335
0.6	0	0	0
0.65	–0.0118168	–0.00263578	–0.0149539
0.7	–0.030816	–0.00551996	–0.0390682
0.75	–0.0572276	–0.00624079	–0.0731967
0.8	–0.086495	0.000631018	–0.112846
0.85	–0.098469	0.0203343	–0.133422
0.9	0	0	0
0.95	1.07753	–1.77336	1.92977

5.3. Approximation of the Gaussian distribution from a tribology problem. We demonstrated in [12] that the simple PA gives more accurate results than all previously proposed formulas. In a subsequent paper [13], we showed that the accuracy can be further improved by using the method of weighted means. We used discrete values of α and their means. In contrast to that method, we now apply continuous values of the function α and their mean in the interval $[x_1, x_N]$. The reference function $s(x)$ is drawn from our previous paper [13]. Below we give the explicit formulas for the NPAs. They are calculated by the use of the information for four points: 0.5; 1; 1.5 and 2. The NPAs are denoted by $g_1 = [2/3]$ and $g_2 = [1/3]$

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \int\limits_x^\infty (t-x)^{\frac{5}{2}} e^{-\frac{t^2}{2}} \, dt,$$

$$s(x) = \frac{\ln\left(\dfrac{x}{2}+1\right)}{\dfrac{x}{2}} (-.279408x + .616634).$$

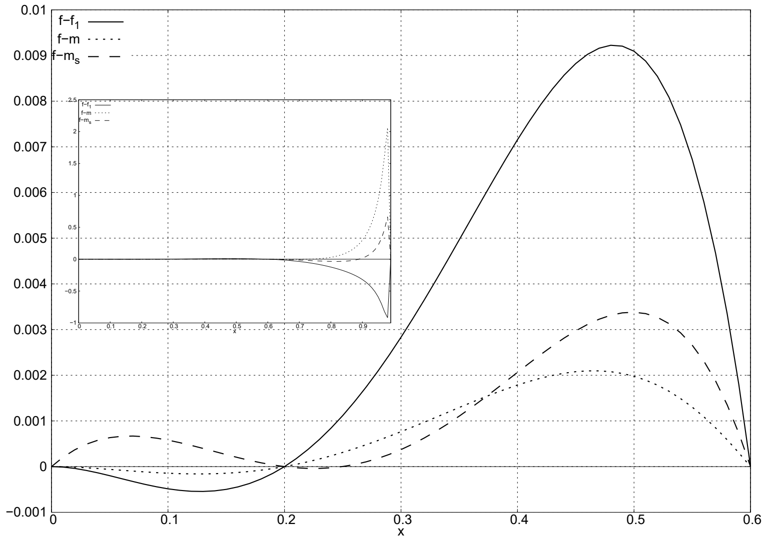
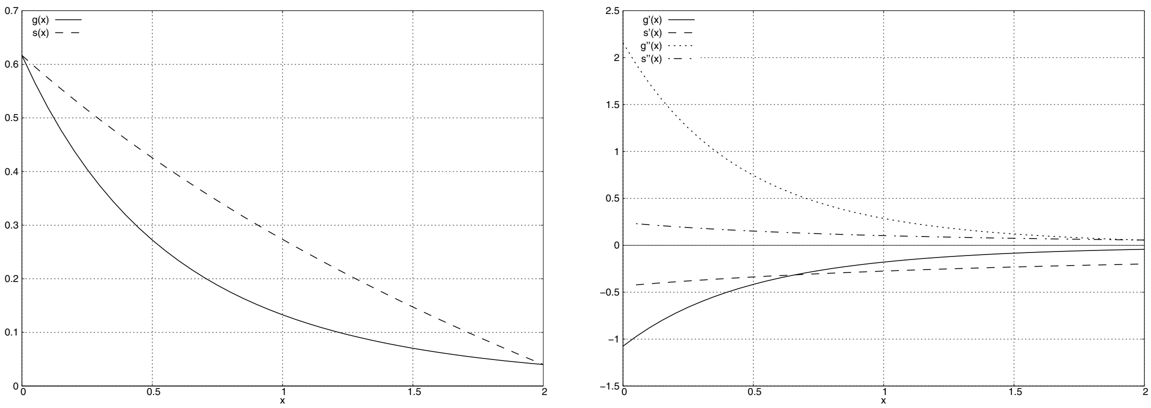


Fig. 15. Plot of errors of $f_1,\ f_2,\ m$ and m_s. Information at the following points $x = 0,\ 0.2,\ 0.6,\ 0.99$ is used for those approximations.



(a) (b)

Fig. 16. Functions $g(x)$ and s (*a*) and their first and second derivatives for $x \in [0, 2]$ (*b*).

Fig. 16 presents the functions $g(x)$ and s and their first and second derivatives.

We obtain the following NPAs:

$$g_1(x) = \frac{0.0131008x^2 - 0.133611x + 0.617026}{0.287364x^3 + 0.917832x^2 + 1.53293x + 1},$$

$$g_2(x) = \frac{0.621943 - 0.0611792x}{0.500488x^3 + 1.00966x^2 + 1.71168x + 1}$$

and the continuous weight function $\alpha(x)$ which is shown in Fig. 17.

Using the mean weights $m_s = 0.989638g_1 + 0.010362g_2$ we get the results which are presented in Table 3. We adopted some values from our previous paper [13] to compare with those computed

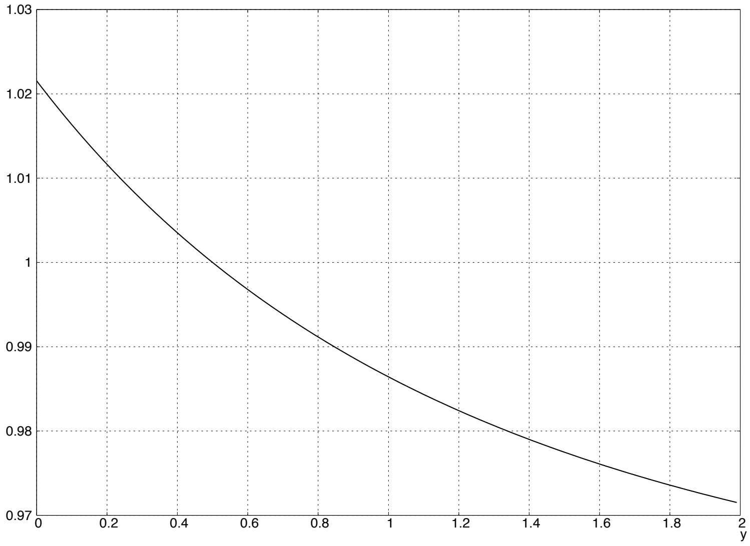


Fig. 17. Plot of α for $x \in [0, 2]$.

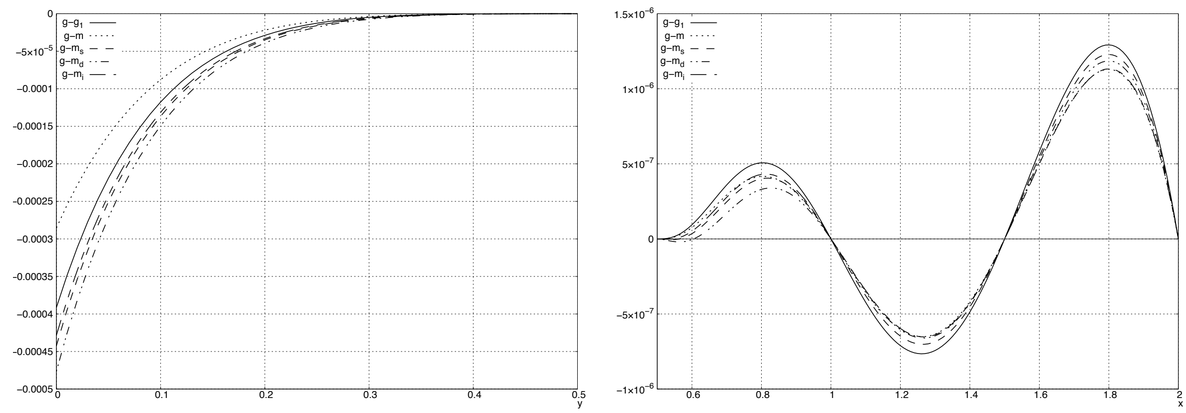
Table 3. Errors of approximation for $g(x)$

x	$g - g_1$	$g - m_i$	$g - m_d$	$g - m$	$g - m_s$
0.	–0.00039	–0.00043	–0.00048	–0.00029	–0.00044
0.25	–0.0000126	–0.000015	–0.000018	–0.00000969	–0.0000158
0.5	0	0	0	0	0
0.75	.000000464	.00000038	.00000027	.00000038	.000000348
1.	0	0	0	0	0
1.25	.000000763	.00000065	.00000066	.000000649	.000000699
1.5	0	0	0	0	0
1.75	.00000124	.0000011	.0000011	.00000108	.00000117
2.	0	0	0	0	0

by the use of the new method. The third and fourth columns are computed from discrete values but the last two columns from the new continuous method.

Note that only in the interval $[0, 0.5[$ is the NPA g_1 a little better than the results coming from the ‘old method’ m_1 and m_d. The new method with the continuous α gives a more exact approximation at this interval. This fact is clearly demonstrated in Fig. 18 (a). Fig. 18 (b) shows advantage of all our approximation methods compared to the ’good’ NPA g_1 in the interval $[0.5, 2]$. Our new method with continuous α gives a more exact approximation at interval $[1, 2]$.

6. Conclusions. The presented method, introduced in a previous paper, develops our cutting-edge idea of constructing the weighted mean approximation of a given function. We show that the weight $\alpha(x)$ defined by s and two oscillating NPAs is a smooth convex function. Then, we can use it in formula (1) continuously. This approach improves our ‘old’ method which relies on calculating



(a) (b)

Fig. 18. Errors of approximations with reference to [12, 13]. A slight superiority of the continuous α (presented by curve $g - m$) over f_1 (presented by curve $g - f_1$) is detectable in these graphs.

the discrete values of the weights. We observe the fact that the accuracy of the approximation is sensitive to the proximity of both the reference and the approximated functions.

The examples presented in the paper clearly demonstrate that the idea is very interesting and promises good results in the numerical approximation of functions.

The weighted means approximation for the inverse Langevin function compared with the approximations in Jedynak [10] and Darabi and Itskov [4] gives us satisfaction. Also the approximation by continuous weighted means of two NPAs of the integral of the Gaussian distribution used in tribology [18] is better than the discrete one presented in our previous research.

Because our method is substantially based on numerical investigations, many questions concerning this method remain open:

1) characterize the functions having the TSE property,

2) improve the choice and the rescaling of the auxiliary functions defining the weight,

3) analyse the cases where the second NPA is calculated by removing one coefficient from the expansion of f at the points x_i with $i \neq 1$,

4) what are the limits of the use of this method for functions without the TSE property?

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