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WEAKLY CLEAN AND EXCHANGE UNI RINGS*

СЛАБКО ЧИСТІ ТА ОБМІННІ УНІКІЛЬЦЯ

We find a criterion for a commutative ring to be weakly clean UNI, as well as a criterion for an exchange ring to have strongly invo-clean unit group. These two assertions somewhat improve earlier author's results (Ukr. Math. J., 2017) and (Internat. J. Algebra, 2017).

Знайдено критерій того, що комутативне кільце є слабко чистим унікільцем, а також критерій того, що обмінне кільце має сильну інво-чисту одиничну групу. Ці два твердження дещо покращують результати автора, що були опубліковані раніше в (Ukr. Math. J., 2017) та (Internat. J. Algebra, 2017).

1. Introduction and background. Everywhere in the text, all rings are assumed to be associative, containing the identity element 1 which differs from the zero element 0. As usual, for such a ring R, $U(R)$ stands for the group of all units in R, $\mathrm{Inv}(R)$ for its subset of all involutions, $\mathrm{Nil}(R)$ for the set of all nilpotents in R with subset $\mathrm{Nil}_2(R)$ consisting of all nilpotents of order less than or equal to 2, $\mathrm{Id}(R)$ for the set of all idempotents in R, $J(R)$ for the Jacobson radical of R, and $Z(R)$ for the center of R. Our further terminology and notations are in agreement with [11].

In [14] was introduced the following paramount notion.

Definition 1.1. *A ring R is called clean if, for every $r \in R$, there exist $u \in U(R)$ and $e \in \mathrm{Id}(R)$ with $r = u + e$.*

If the unit u is replaced by the unipotent $1 + q$ for some $q \in \mathrm{Nil}(R)$, we obtain the proper subclass of nil-clean rings.

This was naturally generalized in [1] to the following concept.

Definition 1.2. *A ring R is called weakly clean if, for each $r \in R$, there exist $u \in U(R)$ and $e \in \mathrm{Id}(R)$ with $r = u + e$ or $r = u - e$.*

If the unit u is again replaced by the unipotent $1 + q$ for some $q \in \mathrm{Nil}(R)$, we detect the proper subclass of weakly nil-clean rings.

It is straightforward to see that any clean ring is weakly clean, but the converse is irreversible. In fact, there are commutative rings which are weakly clean but *not* clean; e.g., such is the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ as well as the ring $\mathbb{Z}_{(15)}$, where $\mathbb{Z}_{(k)} = \left\{ \dfrac{r}{s} \in \mathbb{Q} : k \nmid s \right\}$, $k \in \mathbb{N}$, is a prime or $(k, s) = 1$ for any other natural k. However, if $2 \in J(R)$, then these two classes do coincide (see [3]). Indeed, one can represent $u - e = (u - 2e) + e$, where $u - 2e \in U(R) + J(R) = U(R)$.

Definition 1.3. *A ring R is called exchange if, for any $r \in R$, there exist $e \in \mathrm{Id}(rR)$ such that $1 - e \in (1 - r)R$.*

It is well known that all clean rings are exchange, while the converse fails in general. However, for Abelian rings (i.e., for rings R with $\mathrm{Id}(R) \subseteq Z(R)$), these two notions are equivalent.

Moreover, the weakly clean and exchange concepts are independent one to other.

On the other vein, following Călugăreanu, in [10] was given the following definition.

Definition 1.4. *A ring R is said to be UU, provided that $U(R) = 1 + \mathrm{Nil}(R)$.*

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It is clear that in these rings $2 \in \mathrm{Nil}(R)$ and so $2 \in J(R)$.

The above property of units was generalized in [4] like this.

Definition 1.5. *A ring R is said to be WUU, provided that $U(R) = 1 + \mathrm{Nil}(R)$ or $U(R) = -1 + \mathrm{Nil}(R)$, i.e., $U(R) = \pm 1 + \mathrm{Nil}(R)$.*

Apparently, UU rings are themselves WUU, but the converse is manifestly false by considering rings $\mathbb{Z}$ and $\mathbb{Z}_3 \cong \mathbb{Z}/(3)$. Nevertheless, the WUU notion was extended in [9] as follows.

Definition 1.6. *A ring R is said to be UNI, provided that $U(R) = [\mathrm{Inv}(R) \cap Z(R)] + \mathrm{Nil}(R)$.*

Evidently, WUU rings are themselves UNI, but the converse is untrue. However, if 2 is nilpotent, these two concepts are tantamount, thus enlarging an assertion from [9]. In fact, for any involution v, it must be that $(1-v)^2 = 2(1-v)$ whence by induction we have $(1-v)^{m+1} = 2^m(1-v)$ for every natural number m. Consequently, $1-v$ is a nilpotent and thus, for any central involution i, it follows that $1-i$ is a central nilpotent. That is why, $\mathrm{Inv}(R) \cap Z(R) \subseteq 1 + \mathrm{Nil}(R) \cap Z(R)$ implying that

$$U(R) = \mathrm{Inv}(R) \cap Z(R) + \mathrm{Nil}(R) \subseteq 1 + \mathrm{Nil}(R) \cap Z(R) + \mathrm{Nil}(R) = 1 + \mathrm{Nil}(R),$$

as required.

In [10] was found a criterion when clean ring is UU (compare with Theorem 2.1 below). Since in UU rings the element 2 is always nilpotent, this is tantamount to a criterion when weakly clean ring is UU. After that, in order to supersede this, in [4] and [7] was found a criterion when (weakly) clean ring is WUU (compare with Theorem 2.2 below). These two things were partially improved in [9] to commutative clean UNI rings and arbitrary weakly nil-clean UNI rings.

In the spirit of ring definitions from [5] and [6], we are now able to expand the UNI notions like these.

Definition 1.7. *Let R be a ring. We shall say that $U(R)$ is invo-fine if the equality $U(R) = \mathrm{Inv}(R) + \mathrm{Nil}(R)$ holds.*

A little stronger condition, however, is the next one definition.

Definition 1.8. *Let R be a ring. We shall say that $U(R)$ is strongly invo-fine if, for every $u \in U(R)$, there are $v \in \mathrm{Inv}(R)$ and $q \in \mathrm{Nil}(R)$ such that $u = v + q$ with $vq = qv$.*

A quick look shows that UNI rings and strongly invo-fine rings have strongly invo-fine unit groups. Also, in the commutative case UNI and invo-fine concepts are tantamount. Besides, invo-fine rings have invo-fine unit groups. As showed in [6], all (strongly, uniquely) invo-fine rings are just $\mathbb{Z}_2$ and $\mathbb{Z}_3$, but however rings with (strongly) invo-fine unit groups form a larger class which merits a proper investigation. Notice that in the general noncommutative case Definitions 1.7 and 1.8 are independent as these two sorts of rings form wide classes (certain relevant examples are given in [10]).

And so, the purpose of the present paper is to strengthen these achievements quoted above by finding a necessary and sufficient condition when a commutative ring is weakly clean UNI and an arbitrary ring is exchange with strongly invo-fine unit group. This is successfully done below in Theorems 2.3 and 2.4, respectively. The general problems for an arbitrary weakly clean UNI ring and for an arbitrary exchange ring with invo-fine unit group are still unsettled.

2. Main results. We first list the following technicality.

Lemma 2.1. *In a weakly clean UNI ring,* 6 *or* 30 *is nilpotent.*

Proof. In such a ring R, we write $3 = i + n + e$ or $3 = i + n - e$, where $i \in \mathrm{Inv}(R) \cap Z(R)$, $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Id}(R)$. Thus, $ne = en$ forcing that $(n+e)^2 - (n+e) = n^2 + 2ne - n \in \mathrm{Nil}(R)$ and $(n-e)^2 + (n-e) = n^2 - 2ne + n \in \mathrm{Nil}(R)$. Therefore, $(3-i)^2 - (3-i) \in \mathrm{Nil}(R)$ or $(3-i)^2 + (3-i) \in \mathrm{Nil}(R)$. In the first case, one has that $7 - 5i \in \mathrm{Nil}(R)$, while in the second one $13 - 7i \in \mathrm{Nil}(R)$. So, $5i \in 7 + \mathrm{Nil}(R)$ or $7i \in 13 + \mathrm{Nil}(R)$ giving by squaring that $24 \in \mathrm{Nil}(R)$ or $120 \in \mathrm{Nil}(R)$. Hence $6^3 = 9.24 \in \mathrm{Nil}(R)$, i.e., $6 \in \mathrm{Nil}(R)$ or $(30)^3 = 225.120 \in \mathrm{Nil}(R)$, i.e., $30 \in \mathrm{Nil}(R)$, as stated.

Lemma 2.1 is proved.

Recall the following result from [10].

Theorem 2.1. *Suppose that R is a ring. Then the next three points are equivalent*:

(1) *R is weakly clean UU*;

(2) *R is clean UU*;

(3) *$R/J(R)$ is Boolean and $J(R)$ is nil.*

Recall the following result from [4, 7].

Theorem 2.2. *Suppose R is a ring. Then the next three issues are equivalent*:

(1) *R is weakly clean WUU*;

(2) *R is clean WUU*;

(3) *$J(R)$ is nil and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a boolean ring.*

We now come to our chief statement.

Theorem 2.3. *Let R be a commutative ring. Then the next three items are equivalent*:

(1) *R is weakly clean UNI*;

(2) *R is clean UNI*;

(3) *$J(R)$ is nil with $R \cong L \times P$, where $L/J(L) \subseteq \prod_\lambda \mathbb{Z}_2$ and $P/J(P) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.*

Proof. The equivalence $(2) \iff (3)$ follows directly from [9]. Since the implication $(2) \Rightarrow (1)$ is self-evident, we will show the reverse one $(1) \Rightarrow (2)$. To that goal, we first observe that a homomorphic image of a weakly clean ring is again weakly clean as well as a direct factor of a UNI ring is also UNI. Furthermore, since $(2, 3, 5) = 1$, in view of Lemma 2.1, one can decompose as done in [4, 9, 7], respectively, $R \cong P \times L \times T$, where P, L, T are weakly UNI rings with $2 \in J(P)$, $3 \in J(L)$ and $5 \in J(T)$, provided that they are non-zero. Thus, if $T \neq \{0\}$, then $6 \in 1 + J(T) \subseteq U(T)$ yielding that both $2 \in U(T)$ and $3 \in U(T)$. However, in accordance with [9], this is impossible to be fulfilled in a UNI ring, whence it must be that $T = \{0\}$.

On the other hand, exploiting [3], we derive that P is clean, because it is weakly clean for which $2 \in J(P)$. Besides, owing to [9], $J(L)$ is nil and $L/J(L)$ has to be weakly clean UNI of characteristic 3. Since $L/J(L)$ is commutative, it does not contain non-zero nilpotent elements, that is, $\mathrm{Nil}(L/J(L)) = \{0\}$. So, $L/J(L) = \mathrm{Inv}(L/J(L)) \pm \mathrm{Id}(L/J(L))$ writing $x' = v' + e'$ or $x' = v' - e'$ for any $x' \in L/J(L)$ and some $v' \in \mathrm{Inv}(L/J(L))$, $e' \in \mathrm{Id}(L/J(L))$. Since $x'^3 = (v' + e')^3 = v'^3 + e'^3 = v' + e' = x'$ and $x'^3 = (v' - e')^3 = v'^3 - e'^3 = v' - e' = x'$, it follows that $L/J(L)$ is unit-regular. This means with the aid of [2] that $L/J(L)$ is clean and thus it allows us to infer with [14] at hand that L remains clean. That is why, $R \cong P \times L$ is clean as being the direct product of two clean rings, as wanted.

Theorem 2.3 is proved.

A question which immediately arises is whether or not R is (weakly) clean UNI if, and only if, $J(R)$ is nil and $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ, μ? We next continue with a paralleling technical claim to that in Lemma 2.1.

Lemma 2.2. *If R is a clean ring with strongly invo-clean $U(R)$, then $6 \in \mathrm{Nil}(R)$. In particular, $R \cong R_1 \times R_2$, where both R_1 and R_2 are clean rings with strongly invo-clean unit groups $U(R_1)$ and $U(R_2)$ such that $2 \in J(R_1)$ and $3 \in J(R_2)$.*

Proof. Write $3 = u + e = v + q + e$, where $u \in U(R)$; $v \in \mathrm{Inv}(R)$ and $q \in \mathrm{Nil}(R)$ with $vq = qv$; $e \in \mathrm{Id}(R)$. Therefore, it is readily seen that $qe = eq$ and, resultantly, $(3-v)^2 - (3-v) = = (q+e)^2 - (q+e) = q^2 + 2qe - q \in \mathrm{Nil}(R)$, i.e., $10 - 6v - 3 + v = 7 - 5v \in \mathrm{Nil}(R)$. This gives $5v \in 7 + \mathrm{Nil}(R)$ and squaring the containment we derive that $24 \in \mathrm{Nil}(R)$. This ensures that $6^3 = 216 = 9.24 \in \mathrm{Nil}(R)$ forcing that $6 \in \mathrm{Nil}(R)$, as asserted.

The second part-half is straightforward and its proof is leaved to the interested reader (for more details, see [4, 8, 9], respectively).

Lemma 2.2 is proved.

Lemma 2.3. *For every ring $R \neq \{0\}$ the group $U(\mathbb{M}_2(R))$ is not strongly invo-fine.*

Proof. Consider the invertible matrix $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ with the inverse $\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$. If we assume in a way of contradiction that it is presentable as a sum of an involution matrix and a nilpotent matrix which commute each to other, then one must have that $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ should be a nilpotent. But $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ whence $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ is a nilpotent. But as we already have just seen, this matrix is invertible, getting the wanted contradiction.

Lemma 2.3 is proved.

The following construction is well-known and will be used in the subsequent lemma. And so, if R is a ring with $r \in R$ such that $r^2 - r \in J(R)$, which is a nil ideal, it follows that there is $e \in \mathrm{Id}(R)$ with $er = re$ and $e - r \in J(R)$. In fact, $(r^2 - r)^m = 0$ for some $m \in \mathbb{N}$, and expanding this by the Newton's binomial formula, we obtain that $r^m - r^{m+1}\sum_{i=1}^m (-1)^{i-1} C_i^m r^{i-1} = 0$. Putting $a = = \sum_{i=1}^m (-1)^{i-1} C_i^m r^{i-1}$, we conclude that $r^m = r^{m+1}a$ and $ra = ar$. Taking $e = (ra)^m = r^m a^m$, it follows easily that $er = re$ and $e^2 = e$, as promised.

We are now ready to prove the following lemma.

Lemma 2.4. *Let R be a ring.*

(i) *If $U(R)$ is strongly invo-fine, then $J(R)$ is nil and $U(R/J(R))$ is strongly invo-fine.*

(ii) *If $U(R/J(R)) = \mathrm{Inv}(R/J(R))$ and $J(R)$ is nil with $3 \in J(R)$, then $U(R)$ is strongly invo-fine.*

Proof. (i) Given $z \in J(R)$, we write that $1+z = v+q$, where $v \in \mathrm{Inv}(R)$ and $q \in \mathrm{Nil}(R)$ with $vq = qv$, because $1+J(R) \leq U(R)$. Squaring this equality, we find that $z^2+2z = 2vq+q^2 \in \mathrm{Nil}(R)$. Replacing z by $-z$, we have that $z^2-2z \in \mathrm{Nil}(R)$. So, the sum $(z^2+2z)+(z^2-2z) = 2z^2 \in \mathrm{Nil}(R)$ since its members commute. Furthermore, since $z^2 \in J(R)$, adapting the same trick for this element, we infer that $z^4 + 2z^2 \in \mathrm{Nil}(R)$. Thus the difference $(z^4 + 2z^2) - 2z^2 = z^4 \in \mathrm{Nil}(R)$ because its members commute. Finally, $z \in \mathrm{Nil}(R)$, as required.

About the second part, utilizing the folklore fact that $U(R) \to U(R/J(R))$ is always a surjective homomorphism, induced by the natural ring map $R \to R/J(R)$, with kernel $1 + J(R)$, and using the fact that homomorphic images of commuting involutions and nilpotents are again such elements, we are set.

(ii) Choosing an arbitrary $u \in U(R)$ and seeing that $u + J(R) \in U(R/J(R)) = \mathrm{Inv}(R/J(R))$, it follows that $u^2 - 1 \in J(R)$. This can be written as $(-u-1)^2 - (-u-1) = u^2 + 2u + 1 + u + 1 = u^2 - 1 + 3u + 3 \in J(R)$. Since $J(R)$ is nil, with the aid of the classical construction given above, we conclude that there is an idempotent f such that f commutes with $-u-1$, and so with u, and such that $f - (-u-1) = f + u + 1 \in J(R)$. Hence, $u \in -1 - f + J(R)$ and thus

$$u \in (2f-1) - 3f + J(R) = 2f - 1 + J(R) \in \mathrm{Inv}(R) + \mathrm{Nil}(R).$$

Since u commutes with $2f-1$, it follows that $2f-1$ will commute with the existing element from $J(R)$ as well, and we are done.

Lemma 2.4 is proved.

We now have accumulated all the information necessary to prove our second basic assertion, which settles Problem 1 from [9] and corresponds to Theorem 2.3 (2).

Theorem 2.4. *A ring R is exchange with strongly invo-fine $U(R)$ if and only if $J(R)$ is nil and $R \cong R_1 \times R_2$, where $R_1/J(R_1) \subseteq \prod_\lambda \mathbb{Z}_2$ and $R_2/J(R_2) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.*

Proof. $\Rightarrow$. First, assume $J(R) = \{0\}$ and assume $\mathrm{Nil}_2(R) \neq \{0\}$. Since R is an I-ring in the sense of [13], [12] (Theorem 2.1) applies to find an idempotent f in R and a non-zero ring T such that the isomorphism $fRf \cong \mathbb{M}_2(T)$ is fulfilled. Therefore, one deduces that $U(fRf) \cong U(\mathbb{M}_2(T))$.

We claim now that the square of any element of the group $U(fRf)$ is unipotent, that is, the sum of 1 and nilpotent. Indeed, letting $u \in U(fRf)$ with the inverse $w \in fRf$, we derive that $u + (1-f) \in U(R)$ with the inverse $w + (1-f)$. Writing $u + (1-f) = v + q$ for some $v \in \mathrm{Inv}(R)$ and $q \in \mathrm{Nil}(R)$ with $vq = qv$, and taking into account that $u(1-f) = (1-f)u = 0$, by squaring the above equality about u, we infer that $u^2 + 1 - f = 1 + d$ for some $d \in \mathrm{Nil}(R)$. Hence $u^2 = f + d$ and since $u^2 - f \in fRf$, we conclude that $d \in fRf \cap \mathrm{Nil}(R) = \mathrm{Nil}(fRf)$, as required.

However, the proof of Lemma 2.3 demonstrates that this is not case for the group $U(\mathbb{M}_2(T))$. Thus the above group isomorphism is impossible and, finally, this contradiction argues that $\mathrm{Nil}_2(R) = \{0\}$ and hence $\mathrm{Nil}(R) = \{0\}$. So, R being reduced yields that R is Abelian and thus, in virtue of [14], it must be clean. Also, $U(R) = \mathrm{Inv}(R)$. Consequently, $R = U(R) + \mathrm{Id}(R) = \mathrm{Inv}(R) + \mathrm{Id}(R)$. Now, in terms of [8], the ring R is strongly invo-clean. Thus we can now apply [8] (Corollary 2.17) accomplished with [10] (Theorem B) to get that $R \cong R_1 \times R_2$, where $R_1 \cong B$, where B is a Boolean ring, and $R_2 \subseteq \prod_\mu \mathbb{Z}_3$. That is why, we may identify $R \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

Furthermore, to treat the general case when $J(R) \neq \{0\}$, we detect with the aid of [14] that the quotient $R/J(R)$ is also exchange. Likewise, with Lemma 2.4 at hand, the ideal $J(R)$ is nil, and the group $U(R/J(R)) \cong U(R)/(1 + J(R))$ remains strongly invo-fine. We may now employ what we have shown above in the first part to deduce that $R/J(R)$ is clean. Again [14] works to get that R is clean. Now Lemma 2.2 is applicable to decompose $R \cong R_1 \times R_2$, where R_1 is clean with $2 \in J(R_1)$ and R_2 is clean with $3 \in J(R_2)$. Since $U(R) \cong U(R_1) \times U(R_2)$, and it is not hardly checked with the help of standard coordinate-wise arguments that a direct factor of a strongly invo-fine group is again strongly invo-fine, we infer that both $U(R_1)$ and $U(R_2)$ are strongly invo-fine. Thus $R_1/J(R_1)$ and $R_2/J(R_2)$ are both clean with strongly invo-fine unit groups. We next apply the first part above to conclude the pursued relations.

$\Leftarrow$. That R is exchange follows immediately by application of [14], because both R_1 and R_2 are so. In fact, so are $R_1/J(R_1)$ and $R_2/J(R_2)$ being unit-regular rings, since their elements satisfy the equation $x^3 = x$, as well as $J(R_1)$ and $J(R_2)$ are nil.

On the other side, $U(R_1)/(1 + J(R_1)) \cong U(R_1/J(R_1)) = \{1\}$ and so $U(R_1) = 1 + J(R_1) = = 1 + \mathrm{Nil}(R_1)$ implying that $U(R_1)$ is a strongly invo-fine group. Moreover, $U(R_2/J(R_2)) = = \mathrm{Inv}(R_2/J(R_2))$ and thus Lemma 2.4 (ii) assures that $U(R_2)$ is a strongly invo-fine group, as $J(R_2)$ is nil. Since the direct product of two strongly invo-fine groups is plainly verified to be again a strongly invo-fine group, so will be $U(R)$ and we are finished.

A question which immediately arises is whether or not R is exchange with strongly invo-fine $U(R)$ if, and only if, $J(R)$ is nil and $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ, μ?

A brief information concerning the inheritance of some properties by the corner subring of a given ring is like this: If a ring R is respectively a UU ring or a WUU ring, then so is its corner subring eRe for any $e \in \mathrm{Id}(R)$; unfortunately this is generally not the case for clean and weakly clean rings. In this direction, the affirmation below resolves [9] (Problem 2) as well as it expands the corresponding claims from [10, 4], respectively.

Proposition 2.1. *If R is a UNI ring, then the corner ring eRe is also UNI for any $e \in \mathrm{Id}(R)$.*

Proof. Given $u \in U(eRe)$ with inverse $w \in U(eRe)$, it plainly follows that $u+(1-e) \in U(R)$ with inverse $w + (1 - e) \in U(R)$. Thus one may write that $u + (1 - e) = v + q$, where $v \in \in \mathrm{Inv}(R) \cap Z(R)$ and $q \in \mathrm{Nil}(R)$. Since $ue = eu = u$, by multiplying both sides of $u+1-e = v+q$ with e on the left and right, we deduce that $u = ve + qe = ev + eq$. Since $ve = ev$, it must be that $qe = eq \in \mathrm{Nil}(R)$. But $ve = ve^2 = vee = eve \in eRe$, so that $u - ve = qe \in \mathrm{Nil}(eRe)$. Moreover, $(ve)^2 = v^2e^2 = e$ and hence $ve \in \mathrm{Inv}(eRe)$. Finally, because $v \in Z(R)$, for any $r \in R$ we have that $ve.ere = vere = evre = erve = ervee = ere.ve$, and so it follows that $u = ve + qe \in \mathrm{Inv}(eRe) \cap Z(eRe) + \mathrm{Nil}(eRe)$, as expected.

Proposition 2.1 is proved.

The next statement is paralleled and motivated by Lemma 2.3 alluded to above.

Proposition 2.2. *Suppose R is a commutative non-zero ring. Then, for each integer $n \geq 2$, the full matrix $n \times n$ ring $\mathbb{M}_n(R)$ is not UNI. Even more, it has unit group which is not invo-fine when $n = 2$.*

Proof. Since $\mathbb{M}_2(R)$ is isomorphic to a corner ring of $\mathbb{M}_n(R)$, in virtue of Proposition 2.1 it suffices to show that the ring $\mathbb{M}_2(R)$ is not UNI. We will prove even that its unit group is not invo-fine. To that aim, consider the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \in GL_2(R)$. Assume in a way of contradiction that it can be presented as the sum of an involution matrix and a nilpotent matrix. Therefore, $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -1-b \\ -1-c & 1-d \end{pmatrix}$ is an involution, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nilpotent. So, $\begin{pmatrix} -a & -1-b \\ -1-c & 1-d \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and, as it is well-known, $a + d$ is a nilpotent in R.

We next obtain the system of equations

$$a^2 + (1 + b)(1 + c) = 1,$$

$$a(1 + b) - (1 + b)(1 - d) = 0,$$

$$a(1 + c) - (1 + c)(1 - d) = 0,$$

$$(1 + c)(1 + b) + (1 - d)^2 = 1.$$

From the second equality we detect that $(1+b)(a+d-1) = 0$ and from the third that $(1+c)(a+ + d - 1) = 0$. As noted above, $a + d$ is nilpotent, so that $a + d - 1$ has to be a unit and hence $b = c = -1$. Consequently, $a^2 = (1 - d)^2 = 1$ and $d^2 = 2d$. Furthermore, $a^2 - (1 - d)^2 = = (a - 1 + d)(a + 1 - d) = 0$. Taking into account that $a - 1 + d$ is a unit, one obtains that $a + 1 - d = 0$, i.e., that $a - d = -1$. Once again, since $a + d = q$ is nilpotent, it follows immediately that $(a + d) - (a - d) = q + 1$ is a unit, whence $2d$ is a unit which insures d^2 and thus d is a unit. Finally, $d^2 = 2d$ guarantees that $d = 2$ and $a = 1$. That is why, the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ should be by assumption a nilpotent, but it is *not*. In fact, it may easily be verified that it is a unit with the inverse $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This contradicts our initial choice, thus giving the pursued claim after all.

***Remark* 2.1.** Contrasting with the commutative version, it is not the case that the trace of a nilpotent matrix over an arbitrary ring is again nilpotent. For example, let R be the quotient of the free (noncommutative) ring on two variables x, y by the ideal generated by x^2 and y^2, that is, $R = \mathbb{Z}\langle x, y\rangle/(x^2, y^2)$, and let A be the following 2×2 matrix over R:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix},$$

where $X = x + (x^2, y^2)$ and $Y = y + (x^2, y^2)$ denote the images of x and y, respectively, in R. Then, one checks that $A^2 = 0$, and so A is nilpotent of index 2. But the trace $X + Y$ of A is not a nilpotent, since any power of $X + Y$ is of the form $(XY)^m + (YX)^m$ or $Y(XY)^m + X(YX)^m$, once all terms containing X^2 or Y^2 are identified with 0. Expressions of these forms are not 0 in R, indeed, because expressions of the forms $(xy)^m + (yx)^m$ and $y(xy)^m + x(yx)^m$ are not elements of the ideal (x^2, y^2) in the ring $\mathbb{Z}\langle x, y\rangle$.

We finish off with a facilitating problem of interest (compare with Proposition 2.1 as well).

Problem. If R is a ring and $e \in \mathrm{Id}(R)$ such that $U(R)$ is (strongly) invo-fine, is then $U(eRe)$ also (strongly) invo-fine?

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