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APPLICATION OF DZYADYK'S POLYNOMIAL KERNELS IN THE CONSTRUCTIVE FUNCTION THEORY

ЗАСТОСУВАННЯ ПОЛІНОМІАЛЬНИХ ЯДЕР ДЗЯДИКА В КОНСТРУКТИВНІЙ ТЕОРІЇ ФУНКЦІЙ

This is a survey of recent results in the constructive theory of functions of complex variable obtained by the author through the application of the theory of Dzjadyk's kernels combined with the methods and results from modern geometric function theory and the theory of quasiconformal mappings.

Наведено огляд нових результатів у конструктивній теорії функцій, що отримані автором із застосуванням теорії ядер Дзядика в поєднанні з методами та результатами сучасної геометричної теорії функцій і теорії квазіконформних відображень.

1. Introduction. In [9] (see also [10], Chapter IX, § 7) Dzjadyk introduced and thoroughly researched his polynomial kernels $K_{r,m,k,n}(\zeta, z)$. We believe that these kernels possess the strongest approximation properties out of all the kernels approximating the Cauchy kernel $1/(\zeta - z)$. This is a survey of some recent results in the constructive theory of functions of complex variable obtained by the author through the application of the theory of Dzjadyk's kernels $K_{r,m,k,n}(\zeta, z)$ combined with the methods and results from modern geometric function theory and the theory of quasiconformal mappings.

The paper is organized as follows. In Section 2, we discuss a conjecture on the rate of polynomial approximation on the compact set of the plane to a complex extension of the absolute value function. The conjecture was stated by Grothmann and Saff in 1988 (see [12]). We also mention Gaier's conjecture on the polynomial approximation of piecewise analytic functions on a compact set consisting of two touching discs.

Section 3 is devoted to a Jackson – Mergelyan type theorem on approximation of a function by reciprocals of complex polynomials. The function is continuous on a quasismooth (in the sense of Lavrentiev) arc in $\mathbb{C}$.

2. Polynomial approximation on touching domains. In connection with the distribution of the zeros of some "near best" approximating polynomials, Grothmann and Saff stated the following conjecture.

Let $\mathbb{P}_n$ be the set of complex algebraic polynomials of degree at most $n \in \mathbb{N} \cup \{0\}$, where $\mathbb{N} := \{1, 2, \ldots\}$. For a compact set $K \subset \mathbb{C}, n \in \mathbb{N}$, and a function $f : K \to \mathbb{C}$, set

$$E_n(f, K) := \inf_{p \in \mathbb{P}_n} \|f - p\|_K,$$

where we use the notation $\|f\|_K := \sup_{z \in K} |f(z)|$ for the uniform norm.

For $\alpha > 0$, consider the piecewise analytic function

$$f_\alpha(z) = \begin{cases} z^\alpha & \text{if} \quad \Re(z) > 0, \\ (-z)^\alpha & \text{if} \quad \Re(z) < 0, \\ 0 & \text{if} \quad z = 0, \end{cases} \tag{2.1}$$

which is the "analytic continuation" of $|x|^\alpha$, $x \in \mathbb{R}$, so that $f_1(x) = |x|$ for the real $x \in \mathbb{R}$.

Grothmann and Saff conjectured that if

$$K = \{z \in \mathbb{C} \colon |\Re(z)| \le 2, |\Im(z)| \le |\Re(z)|^2\}$$

is a closed "parabolic region", then

$$E_n(f_1, K) = O(n^{-1}) \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Since $E_n(f_1, K) \ge E_n\big(|x|, [-1, 1]\big)$, the rate of approximation in (2.2) cannot be improved.

Saff and Totik [23] confirmed the above conjecture by constructing polynomials that "overconverge" to f on certain compact subsets of $\mathbb{C} \setminus i\mathbb{R}$, where $i\mathbb{R} := \{z \in \mathbb{C} \colon \Re(z) = 0\}$. By the way, another proof of (2.2) can be derived from the results of Dzjadyk [10, p. 440] (Theorem 1).

Motivated by the Grothmann and Saff conjecture, Anderson and Fuchs [1], in the same 1988, inquired about the structure of a compact set E with $[0, 1] \subset E \subset \{z \in \mathbb{C} \colon \Re(z) > 0\} \cup \{0\}$ such that f_1 satisfies (2.2) with $K = E \cup (-E)$. Here $-E := \{z \in \mathbb{C} \colon -z \in E\}$. In Theorem 3 below, we use purely geometric terms to give full description to the type of "touching domains" with the above property. Following [7, p. 322], we call such sets continua with the de la Vallée Poussin property, or, for short, VP-property.

Also motivated by the Grothmann and Saff conjecture, Gaier [11] considered a more general problem where he suggested to approximate the function f_α for arbitrary $\alpha > 0$ by polynomials. One of his major results can be stated as follows. Let E be a closed domain in $\{z \in \mathbb{C} \colon \Re(z) > 0\} \cup \{0\}$ which is symmetric with respect to $\mathbb{R}$ and bounded by a Jordan curve passing through $z = 0$ which is smooth except at the origin. Let the upper half of ∂E be a Jordan arc J represented by

$$J \colon y = g(x), \quad 0 \le x \le A, \tag{2.3}$$

with $g(0) = g(A) = 0$. The behavior of J is only of consequence near $z = 0$.

Theorem 1 [11]. *Let $0 < \alpha \le 1$ and assume that g in* (2.3) *satisfies the following conditions: for some $0 < a < A$, the function $g(x)/x$ is continuous and increasing on $(0, a)$ and*

$$\int_0^a \frac{g(x)}{x^2}\, dx < \infty.$$

Then

$$E_n\big(f_\alpha, E \cup (-E)\big) = O(n^{-\alpha}) \quad \text{as} \quad n \to \infty. \tag{2.4}$$

Hence, $E \cup (-E)$ from Theorem 1 possesses the VP-property. In Theorem 4 below, we generalize (2.4) to the case of an arbitrary continuum with the VP-property and $\alpha > 0$.

In the same paper [11], Gaier also studied other types of continua. The case of two touching discs seems to be the most difficult for his analysis. Let

$$D(z, r) := \{\zeta \in \mathbb{C} \colon |\zeta - z| < r\}, \quad z \in \mathbb{C}, \quad r > 0.$$

Theorem 2 [11]. *Let $\alpha > 1$ and assume that K satisfies*

$$D(0, a) \cap K = D(0, a) \cap \left(\overline{D(-A, A)} \cup \overline{D(A, A)} \right)$$

for some $0 < a < A$. Then

$$E_n(f_\alpha, K) = O\big((\log n)^{-\alpha+1}\big) \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Gaier conjectured that (2.5) holds with the exponent $-\alpha$ instead of $-\alpha+1$ and for all $\alpha>0$. This conjecture is surprising since the best known estimate of $E_n(f_\alpha, K)$, $0<\alpha\le 1$, in this case, is $\mathrm{O}((\log n)^{-\alpha/2})$. The result follows from the classical Mergelyan theorem (see [25], Chapter 1, § 7, Theorem 8).

Next, we formulate the main results of this section. Let G_1 and G_2 be open, bounded Jordan domains such that

$$[-1,0)\subset G_1,\quad \overline{G_1}\subset\big\{z\in\mathbb{C}\colon \Re(z)<0\big\}\cup\{0\},$$
$$(0,1]\subset G_2,\quad \overline{G_2}\subset\big\{z\in\mathbb{C}\colon \Re z>0\big\}\cup\{0\}.$$

Following [7, p. 322], we say that a continuum

$$K:=\overline{G_1}\cup\overline{G_2} \tag{2.6}$$

consisting of two touching domains has the VP-property, if for f_1, defined by (2.1), (2.2) holds.

From now on till the end of this section, we always assume that the continuum K is defined by (2.6), usually without further mention. Let $\phi:[0,1]\to\overline{\mathbb{R}}_+:=[0,\infty)$ be a continuous function such that $\phi(x)/x$ is increasing on $(0,1)$ and $\lim_{x\to 0+}(\phi(x)/x)=0$. For $0<\varepsilon\le 1$, let

$$S_\varepsilon(\phi):=\big\{z\in\mathbb{C}\colon \big|\Re(z)\big|\le\varepsilon,\ |\Im(z)|\le\phi\big(\big|\Re(z)\big|\big)\big\}.$$

According to [7, p. 331] (Theorem 1.4), if K has the VP-property, then there exist a sufficiently small constant ε and a function ϕ as above such that

$$K_{2\varepsilon}:=K\cap\overline{D(0,2\varepsilon)}\subset S_{2\varepsilon}(\phi). \tag{2.7}$$

Without loss of generality, we can assume that

$$\left|\frac{\Im(z)}{\Re(z)}\right|<0.01,\quad z\in K_{2\varepsilon}. \tag{2.8}$$

For $z\in\mathbb{C}\setminus\mathbb{R}$, denote by Δ_z the closed equilateral triangle with one vertex at z and one side on $\mathbb{R}$. Consider the "envelope" of $K_\delta, 0<\delta\le 2\varepsilon$, i.e., the set

$$E_\delta=E_\delta(K):=\bigcup_{z\in\partial K\cap\overline{D(0,\delta)}}\Delta_z\supset K_\delta,$$

where $\Delta_0:=\{0\}$.

Let the nonnegative functions $\psi^\pm:[-\varepsilon,\varepsilon]\to\mathbb{R}$ be defined by the condition

$$\big\{z\in E_{2\varepsilon}\colon |\Re(z)|\le\varepsilon\big\}=\big\{z\in\mathbb{C}\colon |\Re(z)|\le\varepsilon,\ -\psi^-(\Re(z))\le\Im(z)\le\psi^+(\Re(z))\big\}. \tag{2.9}$$

Theorem 3 [5]. *The continuum K has the VP-property if and only if both conditions below hold*:

(i) *there exist ε and ϕ satisfying* (2.7) *and* (2.8);

(ii) *for $\psi^\pm$ defined by* (2.9),

$$\int\limits_{-\varepsilon}^{\varepsilon}\frac{\psi^\pm(x)}{x^2}\,dx<\infty.$$

Since conditions (i) and (ii) have purely geometric nature, Theorem 3 provides a natural and intrinsic characterization of continua with the VP-property. In particular, it provides the justification of the Grothmann and Saff conjecture: take $\phi(x) = \psi^{\pm}(x) = x^2$.

Next, for $\alpha > 0$, denote by $PA_\alpha(K)$ the class of "piecewise analytic" functions

$$f(z) = \begin{cases} g_1(z) & \text{if} \quad z \in \overline{G_1}, \\ g_2(z) & \text{if} \quad z \in \overline{G_2}, \\ 0 & \text{if} \quad z = 0, \end{cases}$$

with the following two properties:

(i) $g_j, j = 1, 2$, has analytic continuation to $\mathbb{C} \setminus \{x \in \mathbb{R} : (-1)^{j-1}x \geq 0\}$;

(ii) for $z \in \mathbb{C} \setminus \mathbb{R}$ and $j = 1, 2$, we have $|g_j(z)| \leq |z|^\alpha$.

Theorem 4 [5]. *Let K have the VP-property. Then, for $\alpha > 0$ and $f \in PA_\alpha(K)$,*

$$E_n(f, K) = O(n^{-\alpha}) \quad \textit{as} \quad n \to \infty.$$

Since by [7, p. 331] (Theorem 1.5), the continuum from Theorem 1 has the VP-property, Theorem 4 implies Theorem 1 and extends it to the most general family of continua for which such statements can hold (at least for $\alpha = 1$).

Theorem 5 [5]. *Let K be as in Theorem* 2. *Then, for $\alpha > 0$ and $f \in PA_\alpha(K)$,*

$$E_n(f, K) = O\big((\log n)^{-\alpha}\big) \quad \textit{as} \quad n \to \infty. \tag{2.10}$$

This statement proves the Gaier conjecture.

Theorem 6 [5]. *Let $K = \overline{D(-1,1)} \cup \overline{D(1,1)}$, $\alpha > 0$, and*

$$f_\alpha^*(z) = \begin{cases} z^\alpha & \textit{if} \quad z \in \overline{D(1,1)}, \\ 0 & \textit{if} \quad z \in \overline{D(-1,1)}. \end{cases}$$

Then

$$E_n(f_\alpha^*, K) \geq c(\log n)^{-\alpha}, \quad n \in \mathbb{N} \setminus \{1\},$$

holds with a constant $c = c(\alpha) > 0$.

Hence, the estimate (2.10), conjectured by Gaier, cannot, in general, be improved.

3. Approximation of functions by reciprocals of polynomials. The starting point of our investigation is the following result by Levin and Saff [16, 17]. Let, as above, $I := [-1, 1]$ and for a continuous function $f : I \to \mathbb{C}$, let

$$E_{0,n}(f, I) := \inf_{p \in \mathbb{P}_n} \left\| f - \frac{1}{p} \right\|_I. \tag{3.1}$$

Theorem 7 [17]. *We have*

$$E_{0,n}(f, I) = O\big(\omega_{f,I}(n^{-1})\big) \quad \textit{as} \quad n \to \infty, \tag{3.2}$$

where $\omega_{f,I}$ denotes the modulus of continuity of f on I.

The approximation by reciprocals of polynomials is a well-known research direction in constructive function theory. The results related to Theorem 7 and further references can be found in [8, 15, 18, 21, 26].

It was observed in [16] that the use of complex polynomials in Theorem 7 is essential even in the case of real functions (which have a sign change in I). Since f and p in (3.1) and (3.2) are complex-valued, it seems quite natural to extend Theorem 7 to the case where I is changed to an arc L in the complex plane $\mathbb{C}$. Such an extension is the main subject of this section.

Note that Theorem 7 is an analogue of the classical Jackson theorem on polynomial approximation. Our results can be related in the same way to the Mergelyan's extension of the Jackson theorem for the complex plane (see [20] or [25], Chapter I, § 7) and to results in [2, 10, 13, 14, 19, 24] concerning polynomial approximation on arcs. A complete bibliography may be found in [10] (Chapter IX) and [6] (Chapter 5).

Let $L \subset \mathbb{C}$ be a bounded Jordan arc, i.e., its complement $\Omega := \overline{\mathbb{C}} \setminus L$ is a simply connected domain. Denote by $L(z_1, z_2)$ a subarc of L between points $z_1 \in L$ and $z_2 \in L$.

We assume that L is quasismooth (in the sense of Lavrentiev), which means that

$$\big|L(z_1, z_2)\big| \le c|z_1 - z_2|, \quad z_1, z_2 \in L,$$

where $c = c(L) \ge 1$ and $\big|L(z_1, z_2)\big|$ is the length of $L(z_1, z_2)$.

Let function Φ map Ω conformally and univalently onto $\mathbb{D}^*$ with standard normalization at infinity and let for $z \in L$ and $\tau > 0$,

$$L_\tau := \big\{\zeta \in \Omega\colon |\Phi(\zeta)| = 1 + \tau\big\}, \qquad \operatorname{int} L_\tau := \mathbb{C} \setminus \big\{z\colon |\Phi(z)| \ge 1 + \tau\big\},$$

$$\delta_{1/\tau} = \delta_{1/\tau}(L) := \sup_{\zeta \in \operatorname{int} L_\tau} d(\zeta, L_\tau).$$

Let $C(L)$ be the set of continuous functions $f: L \to \mathbb{C}$.

For $f \in C(L)$ and $\tau > 0$ set

$$\omega_{f,L}(\tau) := \sup_{\substack{z_1, z_2 \in L \\ |z_1 - z_2| \le \tau}} \big|f(z_1) - f(z_2)\big|.$$

It is known (see, for example, [3]) that, for $f \in C(L)$,

$$E_n(f, L) = O\big(\omega_{f,L}(\delta_n)\big) \quad \text{as} \quad n \to \infty. \tag{3.3}$$

We are interested in the extension (3.3) to the case of functions approximated by reciprocals of polynomials on a quasismooth arc in which the quantity n^{-1} in (3.2) is replaced by δ_n.

The main result of this section is the following statement. For $f \in C(L)$ and $n \in \mathbb{N}$, let

$$E_{0,n}(f, L) := \inf_{p \in \mathbb{P}_n} \left\| f - \frac{1}{p} \right\|_L .$$

Theorem 8 [4]. *For $f \in C(L)$,*

$$E_{0,n}(f, L) = O\big(\omega_{f,L}(\delta_n)\big) \quad \textit{as} \quad n \to \infty. \tag{3.4}$$

From the reasoning in [16, 17] one can make an intuitive conclusion that the functions vanishing on I are the worst ones for the approximation by reciprocals of polynomials. Surprisingly, for arcs the situation seems to be different. We demonstrate this idea by showing the sharpness of Theorem 8 for the linear functions $l_\zeta(z) := z - \zeta$.

For positive functions a and b we use the order inequality $a \preceq b$ if $a \leq cb$ with a constant $c > 0$. The expression $a \asymp b$ means that $a \preceq b$ and $b \preceq a$ simultaneously. Note that since $\omega_{l_\zeta,L}(\tau) = \tau, \tau > 0$, the estimate (3.4) implies

$$E_{0,n}(l_\zeta, L) \preceq \delta_n, \quad n \in \mathbb{N}, \quad \zeta \in \mathbb{C}.$$

Theorem 9 [4]. *For* $n \in \mathbb{N}$,

$$\sup_{\zeta \in \operatorname{int} L_{1/n}} E_{0,n}(l_\zeta, L) \succeq \delta_n, \tag{3.5}$$

$$\sup_{\zeta \in L} E_{0,n}(l_\zeta, L) \preceq d_n, \tag{3.6}$$

where $d_n = d_n(L) := \sup_{z \in L} d(z, L_{1/n})$.

If L is Dini-smooth, then according to the well-known distortion properties of Φ, which can be found in [22, p. 52] (Theorem 3.9) or [7, p. 32–36], we have $d_n \asymp \delta_n \asymp n^{-1}$. Therefore, in the case of sufficiently smooth L (3.4) looks exactly like (3.2).

It also follows from the properties of Φ (see [22, p. 52] (Theorem 3.9) or [7, p. 32–36]) that if L consists of a finite number of Dini-smooth arcs which meet under angles $\beta_k\pi$, $0 < \beta_k \leq 1$, $k = 1, \ldots, m$, then

$$d_n \asymp n^{-1}, \qquad \delta_n \asymp n^{-\beta}, \qquad \beta := \min_{1 \leq k \leq m} \beta_k.$$

Therefore, for the piecewise smooth arcs (3.4) is "worse" than (3.2). However, by (3.5) the estimate (3.4) is sharp. Moreover, according to (3.6) the "slowly approximable" linear functions have zeros outside of L.

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