UDC 517.953

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FINITE SPEED OF PROPAGATION FOR THE THIN-FILM EQUATION IN THE SPHERICAL GEOMETRY

СКІНЧЕННА ШВИДКІСТЬ ПОШИРЕННЯ ЗБУРЕНЬ ДЛЯ РІВНЯННЯ ТЕЧІЇ ТОНКОЇ ПЛІВКИ ВЗДОВЖ КУЛІ

We show that a double degenerate thin-film equation obtained in modeling of a flow of viscous coating on the spherical surface has a finite speed of propagation for nonnegative strong solutions and, hence, there exists an interface or a free boundary separating the regions, where the solution $u > 0$ and $u = 0$. Using local entropy estimates, we also obtain the upper bound for the rate of the interface propagation.

Показано, що рівняння тонких плівок із подвійним виродженням, яке виникає з моделювання потоку в'язкого покриття на сферичній поверхні, має скінченну швидкість поширення носія невід'ємного сильного розв'язку, а отже, існує інтерфейс або вільна межа, що розділяє області, де розв'язок $u > 0$ і $u = 0$. Крім того, за допомогою локальної ентропійної оцінки отримано оцінку зверху для швидкості поширення інтерфейсу.

1. Introduction. In this paper, we study a particular case of the following doubly degenerate fourth-order parabolic equation:

$$u_t + \Big[u^n(1-x^2)\big(a - bx + c\big(2u + \big((1-x^2)u_x\big)_x\big)_x\big)\Big]_x = 0 \quad \text{in} \quad Q_T, \tag{1.1}$$

where $u(x,t)$ represents the thickness of the thin film, the dimensionless parameters a, b and c describe the effects of gravity, rotation and surface tension, $Q_T = \Omega \times (0,T)$, $n > 0$, $T > 0$, and $\Omega = (-1,1)$. For $n = 3$ (no-slip regime) this equation describes the dynamics of a thin viscous liquid film on the outer surface of a solid sphere. For $n = 2$ the classical Navier slip condition is recovered. On the other hand, parameter range $n \in (0,2)$ $(n \in (2,3))$ in the equation (1.1) corresponds to strong (weak) wetting slip regime. More general dynamics of the liquid film for the case when the draining of the film due to gravity was balanced by centrifugal forces arising from the rotation of the sphere about a vertical axis and by capillary forces due to surface tension was considered in [11]. In addition, Marangoni effects due to temperature gradients were taken into account in [12]. The spherical model without the surface tension and Marangoni effects was studied in [17, 18].

We are interested in time evolution of the support of nonnegative strong solutions to

$$u_t + \big((1-x^2)|u|^n((1-x^2)u_x)_{xx}\big)_x = 0. \tag{1.2}$$

Equation (1.2) is a particular case of (1.1) with $a = b = 0$ with an absence of the second-order diffusion term. Existence of weak solutions for (1.2) in a weighted Sobolev space was shown in [13] and existence of more regular nonnegative strong solutions of (1.2) was recently proved in [16]. Unlike the classical thin-film equation

$$u_t + \big(|u|^n u_{xxx}\big)_x = 0, \tag{1.3}$$

the qualitative behavior of solutions for double degenerate thin-film equation (1.2) is still not well understood. Note that the model equation (1.3) describes the coating flow of a thin viscous film on a flat surface under the surface tension effect. Depending on the value of the parameter n, nonnegative solutions of this equation posses some interesting properties. For example, in 1990, Bernis and Friedman [2] defined and constructed nonnegative weak solutions of the equation (1.3) when $n \geq 1$, and it was also shown that for $n \geq 4$, with a uniformly positive initial condition, there exists a unique positive classical solution. Later on, in 1994, Bertozzi et al. [6] generalised this positivity property for the case $n \geq \frac{7}{2}$. In 1995, Beretta et al. [1] proved the existence of nonnegative weak solutions for the equation (1.3) if $n > 0$, and the existence of strong ones for $0 < n < 3$. Also, they could show that this positivity-preserving property holds at almost every time t in the case $n \geq 2$. This positivity-preservation result was generalised for a cylindrical surface was obtained in [7]. Furthermore, for $n \geq \frac{3}{2}$ the solution's support to (1.3) is nondecreasing in time, and the support remains constant if $n \geq 4$. The existence (nonexistence) of compactly supported spreading source type solution to (1.3) was demonstrated for $0 < n < 3$ $(n \geq 3)$ in [5]. One of interesting qualitative properties of nonlinear parabolic thin-film equations is finite speed of support propagation that is not the case when the parabolic equation is a linear one. This property was first shown in [3] if $0 < n < 2$, and in [4, 10] if $2 \leq n < 3$ for nonnegative strong solutions of (1.3). A similar result on a cylindrical surface was obtained in [8].

Our main result for the thin-film equation on the spherical surface is the finite speed of the interface propagation in the special case of the strong slip regime $n \in (1, 2)$. Proof of the finite speed of propagation property is based on local entropy estimate and Stampacchia's lemma. Moreover, we obtain an upper bound the time evolution of the support as: $\Gamma(t) \leq C_0 t^{\frac{1}{n+4}}$. This bound coincides with the asymptotic behaviour of self-similar type solutions to (1.3) (see [5]).

2. Main result. We study the thin-film equation

$$u_t + \big((1-x^2)|u|^n\big((1-x^2)u_x\big)_{xx}\big)_x = 0 \quad \text{in} \quad Q_T \tag{2.1}$$

with the no-flux boundary conditions

$$(1-x^2)u_x = (1-x^2)\left((1-x^2)u_x\right)_{xx} = 0 \quad \text{at} \quad x = \pm 1, \quad t > 0, \tag{2.2}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{2.3}$$

Here $n > 0$, $Q_T = \Omega \times (0,T)$, $\Omega := (-1,1)$, and $T > 0$. Integrating the equation (2.1) by using boundary conditions (2.2), we obtain the mass conservation property

$$\int\limits_{\Omega} u(x,t)dx = \int\limits_{\Omega} u_0(x)dx =: M > 0. \tag{2.4}$$

Consider initial data $u_0(x) \geq 0$ for all $x \in \bar{\Omega}$ satisfying

$$\int\limits_{\Omega} \{u_0^2(x) + (1-x^2)u_{0,x}^2(x)\}dx < \infty. \tag{2.5}$$

Definition 2.1 (weak solution)**.** *Let* $n > 0$. *A function* u *is a weak solution of the problem* (2.1)–(2.3) *with initial data* u_0 *satisfying* (2.5) *if* $u(x,t)$ *has the properties*

$$(1-x^2)^{\frac{\beta}{2}} u \in C^{\frac{\alpha}{2},\frac{\alpha}{8}}_{x,t}(\bar{Q}_T), \qquad 0 < \alpha < \beta \leq \frac{2}{n},$$

$$u_t \in L^2\big(0,T;(H^1(\Omega))^*\big), \qquad (1-x^2)^{\frac{1}{2}} u_x \in L^\infty\big(0,T;L^2(\Omega)\big),$$

$$(1-x^2)^{\frac{1}{2}} |u|^{\frac{n}{2}} \big((1-x^2)u_x\big)_{xx} \in L^2(P),$$

and u satisfies (2.1) *in weak sense*:

$$\int\limits_0^T \langle u_t, \phi\rangle_{(H^1)^*,H^1}\, dt - \iint\limits_P (1-x^2)|u|^n\big((1-x^2)u_x\big)_{xx}\phi_x dx dt = 0$$

for all $\phi \in L^2\big(0,T;H^1(\Omega)\big)$, where $P := \bar{Q}_T \setminus \big\{\{u=0\}\cup\{t=0\}\big\}$,

$$(1-x^2)^{\frac{1}{2}} u_x(.,t) \to (1-x^2)^{\frac{1}{2}} u_{0,x}(.) \quad \textit{strongly in} \quad L^2(\Omega) \quad \textit{as} \quad t \to 0,$$

and boundary conditions (2.2) *hold at all points of the lateral boundary, where $\{u \neq 0\}$.*

Let us denote by

$$0 \leq G_0(z) := \begin{cases} \dfrac{z^{2-n} - A^{2-n}}{(n-1)(n-2)} - \dfrac{A^{1-n}}{1-n}(z-A) & \text{if} \quad n \neq 1,2, \\ z\ln z - z(\ln A + 1) + A & \text{if} \quad n = 1, \\ \ln\left(\dfrac{A}{z}\right) + \dfrac{z}{A} - 1 & \text{if} \quad n = 2, \end{cases}$$

where $A = 0$ if $n \in (1,2)$ and $A > 0$ if else.

Theorem 2.1. *Assume that $n \geq 1$ and initial data u_0 satisfies $\int_\Omega G_0(u_0)\,dx < +\infty$, then the problem* (2.1) – (2.3) *has a nonnegative weak solution, u, in the sense of Definition* 2.1, *such that*

$$(1-x^2)u_x \in L^2\big(0,T;H^1(\Omega)\big), \qquad (1-x^2)^{\frac{\gamma}{2}} u_x \in L^2(Q_T), \quad \gamma \in (0,1],$$

$$u \in L^\infty\big(0,T;L^2(\Omega)\big), \qquad (1-x^2)^{\frac{\mu}{2}} u \in L^2(Q_T), \quad \mu \in (-1,\beta].$$

The solution in the sense of Theorem 2.1 is called a *strong solution*. The existence of these solutions was proved in [16]. Our aim is to establish the finite speed of propagation property for a strong solution u of (2.1).

Theorem 2.2 (finite speed of propagation)**.** *Assume that $1 < n < 2$, the initial data satisfies the hypotheses of Theorem* 2.1 *and the support of the initial data satisfies $\operatorname{supp}(u_0) \subset \Omega \setminus (-r_0, r_0)$, where $\Omega = (-1,1)$ and $r_0 \in (0,1)$. Let u be the strong solution from Theorem* 2.1. *Then there exists a time $T^* > 0$ and a nondecreasing function $\Gamma(t) \in C\big([0,T^*]\big)$, $\Gamma(0) = 0$ such that u has finite speed propagation, i. e.,*

$$\operatorname{supp}(u(\cdot,t)) \subseteq \big[-r_0 + \Gamma(t), r_0 - \Gamma(t)\big] \subset \Omega$$

for all $t \in [0,T^*]$. Moreover, $\Gamma_{\text{opt}}(t) = C_0 t^{\frac{1}{n+4}}$ for all $t \in [0,T^*]$.

3. Proof of Theorem 2.2. ***3.1. Local entropy estimate.***

Lemma 3.1. *Assume that $1 < n < 2$ and $\nu > 1$. Let $\zeta \in C^{1,2}_{t,x}(\bar{Q}_T)$ such that its support satisfies $\operatorname{supp}(\zeta) \subseteq \Omega$ and $(\zeta^4)_x = 0$ on $\partial\Omega$. Then there exist positive constants C_1, C_2 are independent of Ω, such that for all $T > 0$ the strong solution u of Theorem 2.1 satisfies*

$$\int\limits_{\Omega} (1-x^2)^{\nu}\zeta^4(x,T)G_0(u)\,dx - \iint\limits_{Q_T} (1-x^2)^{\nu}(\zeta^4)_t G_0(u)\,dxdt+$$

$$+\frac{1}{4}\iint\limits_{Q_T} (1-x^2)^{\nu+2}u_{xx}^2\zeta^4\,dxdt \le \int\limits_{\Omega} (1-x^2)^{\nu}\zeta^4(x,0)\,G_0(u_0)\,dx+$$

$$+C_1\iint\limits_{Q_T} (1-x^2)^{\nu}u_x^2\big[\zeta^4+\zeta^2\zeta_x^2+\zeta^3|\zeta_{xx}|\big]\,dxdt+$$

$$+C_2\iint\limits_{Q_T} (1-x^2)^{\nu-2}u^2\big[\zeta^4+\zeta_x^4+\zeta^2\zeta_{xx}^2\big]dxdt. \qquad (3.1)$$

Proof. Equation (2.1) is doubly degenerate when $u = 0$ and $x = \pm 1$. Therefore, for any $\varepsilon > 0$ and $\delta > 0$ we consider two-parametric regularised equations

$$u_{\varepsilon\delta,t} + \Big[(1-x^2+\delta)\big(|u_{\varepsilon\delta}|^n+\varepsilon\big)\Big((1-x^2+\delta)u_{\varepsilon\delta,x}\Big)_{xx}\Big]_x = 0 \quad \text{in} \quad Q_T \qquad (3.2)$$

with boundary conditions

$$u_{\varepsilon\delta,x} = \big((1-x^2+\delta)u_{\varepsilon\delta,x}\big)_{xx} = 0 \quad \text{at} \quad x = \pm 1,$$

and initial data

$$u_{\varepsilon\delta}(x,0) = u_{0,\varepsilon\delta}(x) \in C^{4+\gamma}(\bar{\Omega}), \qquad \gamma > 0,$$

where

$$u_{0,\varepsilon\delta}(x) \ge u_{0\delta}(x) + \varepsilon^{\theta}, \qquad \theta \in \left(0, \frac{1}{2(n-1)}\right),$$

$$u_{0,\varepsilon\delta} \to u_{0\delta} \quad \text{strongly in} \quad H^1(\Omega) \quad \text{as} \quad \varepsilon \to 0,$$

$$(1-x^2+\delta)^{\frac{1}{2}}u_{0x,\delta} \to (1-x^2)^{\frac{1}{2}}u_{0,x} \quad \text{strongly in} \quad L^2(\Omega) \quad \text{as} \quad \delta \to 0.$$

The parameters $\varepsilon > 0$ and $\delta > 0$ in (3.2) make the problem regular up to the boundary (i.e., uniformly parabolic). The existence of a local in time solution of (3.2) is guaranteed by the classical Schauder estimates (see [9]). Now suppose that $u_{\varepsilon\delta}$ is a solution of equation (3.2) and that it is continuously differentiable with respect to the time variable and fourth order continuously differentiable with respect to the spatial variable. For the full detailed proof of existence of strong solutions please refer to [16].

Multiplying the equation (3.2) by $\phi(x,t)G'_\varepsilon(u_{\varepsilon\delta})$, integrating over Ω, and then integrating by parts yield

$$\frac{d}{dt}\int\limits_{\Omega}\phi\, G_{\varepsilon}(u_{\varepsilon\delta})\,dx-$$

$$-\int\limits_{\Omega}\phi_t G_{\varepsilon}(u_{\varepsilon\delta})\,dx=\int\limits_{\Omega}(1-x^2+\delta)u_{\varepsilon\delta,x}\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_{xx}\phi\,dx+$$

$$+\int\limits_{\Omega}(1-x^2+\delta)\big(|u_{\varepsilon\delta}|^n+\varepsilon\big)G'_{\varepsilon}(u_{\varepsilon\delta})\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_{xx}\phi_x\,dx=$$

$$=-\int\limits_{\Omega}\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_x^2\phi\,dx-\int\limits_{\Omega}(1-x^2+\delta)u_{\varepsilon\delta,x}\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_x\phi_x\,dx-$$

$$-\int\limits_{\Omega}\big[(1-x^2+\delta)(|u_{\varepsilon\delta}|^n+\varepsilon)G'_{\varepsilon}(u_{\varepsilon\delta})\phi_x\big]_x\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_x\,dx=$$

$$=-\int\limits_{\Omega}\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_x^2\phi\,dx+\frac{1}{2}\int\limits_{\Omega}\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]^2\phi_{xx}\,dx-$$

$$-\int\limits_{\Omega}\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_x\big(|u_{\varepsilon\delta}|^n+\varepsilon\big)G'_{\varepsilon}(u_{\varepsilon\delta})\big((1-x^2+\delta)\phi_x\big)_x\,dx-$$

$$-\int\limits_{\Omega}\big[(1-x^2+\delta)u_{\varepsilon\delta,x}\big]_x(1-x^2+\delta)\Big[\big(|u_{\varepsilon\delta}|^n+\varepsilon\big)G'_{\varepsilon}(u_{\varepsilon\delta})\Big]'_u u_{\varepsilon\delta,x}\phi_x dx. \tag{3.3}$$

Integrating (3.3) in time and taking the regularising parameter $\varepsilon\to 0$, by applying the Young inequality and $z^nG'_0(z)=\dfrac{1}{1-n}z$, we finally get

$$\int\limits_{\Omega}\phi\, G_0(u_\delta)\,dx-\iint\limits_{Q_T}\phi_t G_0(u_\delta)\,dxdt+\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]_x^2\phi\,dxdt\le$$

$$\le\int\limits_{\Omega}\phi\, G_0(u_{0,\delta})\,dx+\frac{1}{2}\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]^2\phi_{xx}\,dxdt-$$

$$-\frac{1}{1-n}\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]_x u_\delta((1-x^2+\delta)\phi_x)_x\,dxdt-$$

$$-\frac{1}{1-n}\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]_x(1-x^2+\delta)u_{\delta,x}\phi_x\,dxdt\le$$

$$\le\int\limits_{\Omega}\phi\, G_0(u_{0,\delta})\,dx+\mu\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]_x^2\phi\,dxdt+$$

$$+\frac{2-n}{2(1-n)}\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]^2\phi_{xx}\,dxdt+\frac{1}{4\mu(1-n)^2}\iint\limits_{Q_T}u_\delta^2\frac{\big((1-x^2+\delta)\phi_x\big)_x^2}{\phi}\,dxdt, \quad (3.4)$$

where $\mu>0$. Choosing $\mu=\dfrac{1}{2}$ in (3.4), we arrive at

$$\int\limits_\Omega \phi\, G_0(u_\delta)\,dx-$$

$$-\iint\limits_{Q_T}\phi_t G_0(u_\delta)\,dxdt+\frac{1}{2}\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]_x^2\phi\,dxdt\le$$

$$\le\int\limits_\Omega \phi\, G_0(u_{0,\delta})\,dx+\frac{2-n}{2(1-n)}\iint\limits_{Q_T}\big[(1-x^2+\delta)u_{\delta,x}\big]^2|\phi_{xx}|\,dxdt+$$

$$+\frac{1}{2(1-n)^2}\iint\limits_{Q_T}u_\delta^2\frac{\big((1-x^2+\delta)\phi_x\big)_x^2}{\phi}\,dxdt. \quad (3.5)$$

Letting $\delta\to0$ in (3.5), we deduce that

$$\int\limits_\Omega\phi(T)\,G_0(u)\,dx-\iint\limits_{Q_T}\phi_t G_0(u)\,dxdt+$$

$$+\frac{1}{2}\iint\limits_{Q_T}\big[(1-x^2)u_x\big]_x^2\phi\,dxdt\le\int\limits_\Omega\phi(0)\,G_0(u_0)\,dx+$$

$$+\frac{2-n}{2(1-n)}\iint\limits_{Q_T}\big[(1-x^2)u_x\big]^2|\phi_{xx}|\,dxdt+\frac{1}{2(1-n)^2}\iint\limits_{Q_T}u^2\frac{\big((1-x^2)\phi_x\big)_x^2}{\phi}\,dxdt. \quad (3.6)$$

Taking $\phi(x,t)=(1-x^2)^\nu\zeta^4(x,t)$ in (3.6) for $\nu>1$, we have

$$\int\limits_\Omega(1-x^2)^\nu\zeta^4(T)G_0(u)\,dx-\iint\limits_{Q_T}(1-x^2)^\nu(\zeta^4)_tG_0(u)\,dxdt+$$

$$+\frac{1}{2}\iint\limits_{Q_T}(1-x^2)^\nu\big[(1-x^2)u_x\big]_x^2\zeta^4\,dxdt\le\int\limits_\Omega(1-x^2)^\nu\zeta^4(0)\,G_0(u_0)\,dx+$$

$$+\tilde C_1\iint\limits_{Q_T}\big[(1-x^2)u_x\big]^2\big[(1-x^2)^{\nu-2}\zeta^4+(1-x^2)^\nu\big(\zeta^2\zeta_x^2+\zeta^3|\zeta_{xx}|\big)\big]\,dxdt+$$

$$+C_2\iint\limits_{Q_T}u^2\big[(1-x^2)^{\nu-2}\zeta^4+(1-x^2)^{\nu+2}\zeta_x^4+(1-x^2)^{\nu+2}\zeta^2\zeta_{xx}^2\big]\,dxdt\le$$

$$\leq \int\limits_{\Omega} (1-x^2)^{\nu}\zeta^4(0)\, G_0(u_0)\, dx + \tilde{C}_1 \iint\limits_{Q_T} (1-x^2)^{\nu} u_x^2 \big[\zeta^4 + \zeta^2\zeta_x^2 + \zeta^3|\zeta_{xx}|\big]\, dxdt +$$

$$+C_2 \iint\limits_{Q_T} (1-x^2)^{\nu-2} u^2 \big[\zeta^4 + \zeta_x^4 + \zeta^2\zeta_{xx}^2\big]\, dxdt,$$

where $\tilde{C}_1 = \dfrac{2-n}{1-n}\max\big\{5\nu, 2\nu(\nu+1), 2(3+2\nu)\big\}$, $C_2 = \dfrac{16(\nu+1)^4}{(1-n)^2}$. From here, due to

$$\frac{1}{2}\iint\limits_{Q_T} (1-x^2)^{\nu}\big[(1-x^2)u_x\big]_x^2 \zeta^4\, dxdt = \frac{1}{2}\iint\limits_{Q_T} (1-x^2)^{\nu+2} u_{xx}^2 \zeta^4\, dxdt -$$

$$-2\iint\limits_{Q_T} x(1-x^2)^{\nu+1} u_x u_{xx}\zeta^4\, dxdt + 2\iint\limits_{Q_T} x^2(1-x^2)^{\nu} u_x^2\zeta^4\, dxdt \geq$$

$$\geq \frac{1}{4}\iint\limits_{Q_T} (1-x^2)^{\nu+2} u_{xx}^2\zeta^4\, dxdt - 2\iint\limits_{Q_T} (1-x^2)^{\nu} u_x^2\zeta^4\, dxdt,$$

we deduce (3.1) with $C_1 = \tilde{C}_1 + 2$.

Lemma 3.1 is proved.

3.2. Finite speed of propagation. For an arbitrary $s > 0$ and $0 < \delta \leq s$ we consider the families of sets

$$\Omega(s) := \{x \in \bar{\Omega}\colon |x| \leq s\}, \qquad Q_T(s) = (0,T)\times\Omega(s), \qquad K_T(s,\delta) = Q_T(s)\setminus Q_T(s-\delta).$$

We introduce a nonnegative cutoff function $\eta(\tau)$ from the space $C^2(\mathbb{R}^1)$ with the following properties:

$$\eta(\tau) = \begin{cases} 1 & \text{if} \quad \tau \leq 0, \\ -\tau^3(6\tau^2 - 15\tau + 10) + 1 & \text{if} \quad 0 < \tau < 1, \\ 0 & \text{if} \quad \tau \geq 1. \end{cases}$$

Next, we introduce our main cut-off functions $\eta_{s,\delta}(x) \in C^2(\bar{\Omega})$ such that $0 \leq \eta_{s,\delta}(x) \leq 1$ for all $x \in \bar{\Omega}$ and possess the following properties:

$$\eta_{s,\delta}(x) = \eta\left(\frac{|x| - (s-\delta)}{\delta}\right) = \begin{cases} 1, & x \in \Omega(s-\delta), \\ 0, & x \in \Omega\setminus\Omega(s), \end{cases}$$

$$|(\eta_{s,\delta})_x| \leq \frac{15}{8\delta}, \qquad |(\eta_{s,\delta})_{xx}| \leq \frac{5(\sqrt{3}-1)}{\delta^2}$$

for all $s > 0$ and $0 < \delta \leq s$. Choosing $\zeta^4(x,t) = \eta_{s,\delta}(x)e^{-\frac{t}{T}}$ in (3.1), we arrive at

$$\int\limits_{\Omega(s-\delta)} (1-x^2)^{\nu} u^{2-n}(T)\, dx + \frac{1}{T}\iint\limits_{Q_T(s-\delta)} (1-x^2)^{\nu} u^{2-n}\, dxdt +$$

$$+C \iint\limits_{Q_T(s-\delta)} (1-x^2)^{\nu+2} u_{xx}^2 \, dxdt \le e \int\limits_{\Omega(s)} (1-x^2)^{\nu} u_0^{2-n}(x)\, dx +$$

$$\le \frac{C}{\delta^2} \iint\limits_{K_T(s,\delta)} (1-x^2)^{\nu} u_x^2 \, dxdt + \frac{C}{\delta^4} \iint\limits_{K_T(s,\delta)} (1-x^2)^{\nu-2} u^2 \, dxdt \tag{3.7}$$

for all $0 < \delta \le s$. Here and throughout this proof, C denotes a positive constant independent of Ω. By (3.7) we deduce

$$(1-(s-\delta)^2)^{\nu} \int\limits_{\Omega(s-\delta)} u^{2-n}(T)\, dx + \frac{(1-(s-\delta)^2)^{\nu}}{T} \iint\limits_{Q_T(s-\delta)} u^{2-n} \, dxdt +$$

$$+C(1-(s-\delta)^2)^{\nu} \iint\limits_{Q_T(s-\delta)} (1-x^2)^2 u_{xx}^2 \, dxdt \le \frac{C(1-(s-\delta)^2)^{\nu}}{\delta^2} \iint\limits_{K_T(s,\delta)} u_x^2 \, dxdt +$$

$$+\frac{C(1-(s-\delta)^2)^{\nu}}{\delta^4} \iint\limits_{K_T(s,\delta)} (1-x^2)^{-2} u^2 \, dxdt,$$

whence

$$\int\limits_{\Omega(s-\delta)} u^{2-n}(T)\, dx + \frac{1}{T} \iint\limits_{Q_T(s-\delta)} u^{2-n} \, dxdt +$$

$$+\,C(1-r_0^2)^2 \iint\limits_{Q_T(s-\delta)} u_{xx}^2 \, dxdt \le \frac{C}{\delta^2} \iint\limits_{K_T(s,\delta)} u_x^2 \, dxdt + \frac{C(1-r_0^2)^{-2}}{\delta^4} \iint\limits_{K_T(s,\delta)} u^2 \, dxdt =: R(s) \tag{3.8}$$

for all $0 < \delta \le s \le r_0$. We apply Lemma A.1 in the region $\Omega(s-\delta)$ to a function $v := u$ with $a = d = j = 2$, $b = 2-n$, $k = 0$ (or $k = 1$), $N = 1$, and $\theta_1 = \dfrac{n}{8-3n}$ $\left(\text{or } \theta_2 = \dfrac{4-n}{8-3n}\right)$. Integrating the resulted inequalities with respect to time and taking into account (3.8), we arrive at the following relations:

$$A(s-\delta) \le C(1-r_0^2)^{-\alpha_1} T^{\beta_1} \big(R(s)\big)^{1+\kappa_1} + C\, T \big(R(s)\big)^{1+\kappa_3}, \tag{3.9}$$

$$B(s-\delta) \le C(1-r_0^2)^{-\alpha_2} T^{\beta_2} \big(R(s)\big)^{1+\kappa_2} + C\, T \big(R(s)\big)^{1+\kappa_3}, \tag{3.10}$$

where

$$A(s) := \iint\limits_{Q_T(s)} u^2 dxdt, \qquad B(s) := \iint\limits_{Q_T(s-\delta)} u_x^2 dxdt,$$

$$\alpha_1 = \frac{4(n+4)}{8-3n}, \qquad \alpha_2 = \frac{4(6-n)}{8-3n}, \qquad \beta_1 = \frac{4(2-n)}{8-3n}, \qquad \beta_2 = \frac{2(2-n)}{8-3n},$$

$$\kappa_1 = \frac{4n}{8-3n}, \qquad \kappa_2 = \frac{2n}{8-3n}, \qquad \kappa_3 = \frac{n}{2-n}.$$

Since all integrals on the right-hand sides of (3.9), (3.10) vanish as $T \to 0$ and $u \in L^2(0,T;$ $H^1\big(-r_0, r_0)\big)$, then for sufficiently small T we get

$$A(s-\delta) \le C_3(1-r_0^2)^{-\alpha_1} T^{\beta_1} \big(\delta^{-4} A(s) + \delta^{-2} B(s)\big)^{1+\kappa_1}, \tag{3.11}$$

$$B(s-\delta) \le C_4(1-r_0^2)^{-\alpha_2} T^{\beta_2} \big(\delta^{-4} A(s) + \delta^{-2} B(s)\big)^{1+\kappa_2}, \tag{3.12}$$

where $C_3,\ C_4$ are a positive constant depending on all known parameters and independent of Ω. Let us denote by

$$D(s) := A^{1+\kappa_2}(s) + B^{1+\kappa_1}(s), \qquad \kappa = (1+\kappa_1)(1+\kappa_2),$$

$$C_5(T) := 2^{\kappa-1} \max\Big\{ \big[C_3(1-r_0^2)^{-\alpha_1} T^{\beta_1}\big]^{1+\kappa_2}, \big[C_4(1-r_0^2)^{-\alpha_2} T^{\beta_2}\big]^{1+\kappa_1} \Big\}.$$

Without loss of generality, we can define the function

$$\tilde{D}(s) = D(s) \quad \text{if} \quad s \in (0, r_0] \quad \text{and} \quad \tilde{D}(s) = 0 \quad \text{if} \quad s > r_0.$$

Then by (3.11), (3.12) we arrive at

$$\tilde{D}(s-\delta) \le C_5(T)\big(\delta^{-4\kappa} \tilde{D}^{1+\kappa_1}(s) + \delta^{-2\kappa} \tilde{D}^{1+\kappa_2}(s)\big) \tag{3.13}$$

for all $s \in \mathbb{R}^+$ and $\delta \in (0, r_0]$. Choosing

$$\delta(s) = \max\Big\{ \big[4C_5(T)\tilde{D}^{\kappa_1}(s)\big]^{\frac{1}{4\kappa}}, \big[4C_5(T)\tilde{D}^{\kappa_2}(s)\big]^{\frac{1}{2\kappa}} \Big\}$$

in (3.13), we find that

$$\tilde{D}\big(s-\delta(s)\big) \le \frac{1}{2}\tilde{D}(s),$$

whence it follows

$$\delta\big(s-\delta(s)\big) \le \gamma \delta(s) \quad \forall\, s \in \mathbb{R}^+, \tag{3.14}$$

where $\gamma = \max\big\{2^{-\frac{\kappa_1}{4\kappa}}, 2^{-\frac{\kappa_2}{2\kappa}}\big\} < 1$. Applying Stampacchia's lemma (see Lemma A.2) to (3.14), we obtain

$$\delta(s) = 0 \quad \text{for all} \quad s \le r_0 - \frac{\delta(r_0)}{1-\gamma}.$$

Next, we will find the upper bound for $\delta(r_0)$. In view of Theorem 2.1, $(1-x^2)^{\frac{\nu-2}{2}} u \in L^2(Q_T)$ and $(1-x^2)^{\frac{\nu}{2}} u_x \in L^2(Q_T)$ for any $\nu > 1$, then the right-hand side of (3.7) is bounded for all $T > 0$. So, taking $s = 2r_0$ and $\delta = r_0$ in (3.9) and (3.10), we obtain $\tilde{D}(r_0) \le C_6\, C_5(T)$, whence

$$\delta(r_0) \le C_7 (1-r_0^2)^{-\frac{2(6-n)}{8-3n}} T^{\frac{2-n}{8-3n}}.$$

This implies the upper bound for speed of propagation to solution support, i.e.,

$$\Gamma(T) \le r_0 - C_8 T^{\frac{2-n}{8-3n}} \quad \text{for all} \quad T \le T^* := \left(\frac{r_0}{C_8}\right)^{\frac{8-3n}{2-n}} \tag{3.15}$$

for any $r_0 \in (0,1)$, where $C_8 = \dfrac{C_7}{1-\gamma}(1-r_0^2)^{-\frac{2(6-n)}{8-3n}}$.

3.3. Exact upper bound for speed of propagation. In this subsection, we refine the estimate (3.15). Throughout this subsection, C denotes a positive constant independent of Ω. Applying Lemma A.1 in the region $\Omega(s)\setminus\Omega(s-\delta)$ to a function $v:=u$ with $a=d=j=2,\ b=1,\ k=0$ (or $k=1$), $N=1$, and $\theta_1=\dfrac{1}{5}$ $\Big($or $\theta_2=\dfrac{3}{5}\Big)$, and integrating the resulted inequalities with respect to time, taking into account the mass conservation (2.4), we arrive at the following estimates:

$$\iint\limits_{K_T(s,\delta)} u^2\,dxdt \le C\,T^{1-\theta_1}M^{2(1-\theta_1)}\Bigg(\iint\limits_{K_T(s,\delta)} u_{xx}^2\,dxdt\Bigg)^{\theta_1}+C\,\delta^{-1}TM^2, \tag{3.16}$$

$$\iint\limits_{K_T(s,\delta)} u_x^2\,dxdt \le C\,T^{1-\theta_2}M^{2(1-\theta_2)}\Bigg(\iint\limits_{K_T(s,\delta)} u_{xx}^2\,dxdt\Bigg)^{\theta_2}+C\,\delta^{-3}TM^2. \tag{3.17}$$

Using (3.16), (3.17) and Young inequality, from (3.8) we find

$$\int\limits_{\Omega(s-\delta)} u^{2-n}(T)\,dx+\frac{1}{T}\iint\limits_{Q_T(s-\delta)} u^{2-n}\,dxdt+C(1-r_0^2)^2\iint\limits_{Q_T(s-\delta)} u_{xx}^2\,dxdt \le$$

$$\le \varepsilon(1-r_0^2)^2\iint\limits_{K_T(s,\delta)} u_{xx}^2\,dxdt+C_\varepsilon\delta^{-5}(1-r_0^2)^{-3}TM^2,$$

where $\varepsilon>0$. Selecting $\varepsilon\in(0,2^{-5})$ enough small and making standard iteration process, we get

$$\int\limits_{\Omega(s-\delta)} u^{2-n}(T)\,dx+\frac{1}{T}\iint\limits_{Q_T(s-\delta)} u^{2-n}\,dxdt+$$

$$+C(1-r_0^2)^2\iint\limits_{Q_T(s-\delta)} u_{xx}^2\,dxdt\le C\,\delta^{-5}(1-r_0^2)^{-3}TM^2. \tag{3.18}$$

Taking $s=2\Gamma(T)$ and $\delta=\Gamma(T)$ in (3.18), we obtain

$$\iint\limits_{Q_T(\Gamma(T))} u_{xx}^2\,dxdt\le C\,\Gamma^{-5}(T)(1-r_0^2)^{-5}TM^2,$$

whence, similar to (3.16) and (3.17), we have

$$A(\Gamma(T))\le C\,\Gamma^{-1}(T)(1-r_0^2)^{-1}TM^2,$$

$$B(\Gamma(T))\le C\,\Gamma^{-3}(T)(1-r_0^2)^{-3}TM^2.$$

Hence,

$$\delta(\Gamma(T))\le C\max\Big\{[\Gamma^{-\kappa_1}(T)(1-r_0^2)^{-(\kappa_1+\alpha_1)}T^{\kappa_1+\beta_1}M^{2\kappa_1}]^{\frac{1}{4(1+\kappa_1)}},$$

$$[\Gamma^{-3\kappa_2}(T)(1-r_0^2)^{-(3\kappa_2+\alpha_2)}T^{\kappa_2+\beta_2}M^{2\kappa_2}]^{\frac{1}{2(1+\kappa_2)}}\Big\} =$$

$$= C_9 \max\Big\{\Gamma^{-\frac{n}{n+8}}(T)T^{\frac{2}{n+8}}, \Gamma^{-\frac{3n}{8-n}}(T)T^{\frac{2}{8-n}}\Big\}.$$

Thus, we get

$$\Gamma(T) + C_{10}\max\Big\{\Gamma^{-\frac{n}{n+8}}(T)T^{\frac{2}{n+8}}, \Gamma^{-\frac{3n}{8-n}}(T)T^{\frac{2}{8-n}}\Big\} \le r_0, \tag{3.19}$$

where $C_{10} = \dfrac{C_9}{1-\gamma}$. Now we use the following calculus result: let $a > 0$ and $b > 0$, then the function $f(x) = x + a\,x^{-b}$ for all $x \ge 0$ has minimum at $x_{\min} = (ab)^{\frac{1}{1+b}}$ and $f(x_{\min}) = \dfrac{1+b}{b}x_{\min}$. Hence, minimizing the right-hand side, we obtain

$$\Gamma_{\text{opt}}(T) = C_0 T^{\frac{1}{n+4}} \quad \text{for all} \quad T \le T^*.$$

Theorem 2.2 is proved.

Appendix A.

Lemma A.1 [14]. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise-smooth boundary, $a > 1$, $b \in (0, a)$, $d > 1$, and $0 \le k < j$, $k, j \in \mathbb{N}$, then there exist positive constants d_1 and d_2 ($d_2 = 0$ if Ω is unbounded) depending only on Ω, d, j, b, and N such that the following inequality is valid for every $v(x) \in W^{j,d}(\Omega) \cap L^b(\Omega)$:*

$$\left\| D^k v \right\|_{L^a(\Omega)} \le d_1 \left\| D^j v \right\|^{\theta}_{L^d(\Omega)} \|v\|^{1-\theta}_{L^b(\Omega)} + d_2 \|v\|_{L^b(\Omega)},$$

where $\theta = \dfrac{\frac{1}{b} + \frac{k}{N} - \frac{1}{a}}{\frac{1}{b} + \frac{j}{N} - \frac{1}{d}} \in \left[\dfrac{k}{j}, 1\right)$. *Note that if $\Omega = B(0,R) \setminus B(0,r)$, where $B(0,x)$ is ball with the radius x and the origin at 0, then $d_2 = c(R-r)^{-\frac{(a-b)N}{ab}-k}$.*

Lemma A.2 [15]. *Assume that $f(s)$ is nonnegative nondecreasing function satisfying the following inequality*:

$$f(s - f(s)) \le \varepsilon f(s) \quad \forall\, s \le s_0,$$

where $\varepsilon \in (0,1)$. Then $f(s) = 0$ for all $s \le s_0 - \dfrac{f(s_0)}{1-\varepsilon}$.

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Received 14.11.17