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NOTES ON THE LIGHTLIKE HYPERSURFACES ALONG SPACELIKE SUBMANIFOLDS

ПРО СВІТЛОПОДІБНІ ГІПЕРПОВЕРХНІ ВЗДОВЖ ПРОСТОРОВОПОДІБНИХ ПІДМНОГОВИДІВ

In the light of the method of construction of lightlike hypersurfaces along spacelike submanifolds, we give a relation between the second fundamental form of a spacelike submanifold and the screen second fundamental form of the corresponding lightlike hypersurface. In addition, we investigate the conditions for a lightlike hypersurface of this kind to be screen conformal.

У світлі методу побудови світлоподібних гіперповерхонь уздовж просторовоподібних підмноговидів отримано співвідношення між другою фундаментальною формою просторовоподібного підмноговиду та екранною другою фундаментальною формою відповідної світлоподібної гіперповерхні. Крім того, вивчено умови, за яких така світлоподібна гіперповерхня є екранно конформною.

1. Introduction. The theory of lightlike hypersurfaces has a special place in differential geometry and theoretical physics. In the general relativity, lightlike hypersurfaces play an important role since they are considered as the models for different horizon types of black holes. A black hole is a region of spacetime which contains a huge amount of mass compacted into an extremely small volume. The gravity inside a black hole is so strong that, even light with a remarkable speed can not escape, see [1] . Since Einstein's theory of gravitation was first published in 1915, so many research papers on the mathematical and physical theory of black holes, have been published. For further information about black holes and applications of lightlike hypersurfaces, see [2, 4, 6].

The lightlike hypersurfaces along spacelike submanifolds was introduced by Izumiya and Sato in [7]. They constructed the lightlike hypersurfaces as ruled hypersurfaces based on spacelike submanifolds with lightlike rulings. There are several types of ruled surfaces in the Lorentz – Minkowski space (see, for example, [1, 8]). In this paper, we investigate the geometric properties of the lightlike hypersurfaces along spacelike submanifolds defined by

$$LH_M(p,\xi,t) = X(u) + tLG(n^T)(u,\xi).$$

Since we have degenerate metric on the tangent space, considering a lightlike hypersurface together with its screen distribution provides simplicity. Thus, we define the screen second fundamental form and give the lightcone Weingarten equations for the screen distribution of the lightlike hypersurface mentioned above. Then we find a relation between the second fundamental form of the spacelike submanifold and the screen second fundamental form of corresponding lightlike hypersurface. Also, we put forward the conditions to be screen conformal of the lightlike hypersurface. As an example, we show that the event horizon in Schwarzschild spacetime is actually the lightlike hypersurface along a spacelike submanifold. Then we support our theory with some other examples.

2. Preliminaries. Let $x = (x_0, x_1, \ldots, x_n),\ y = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$, then the pseudoscalar product of x and y is defined by

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

$\big(\mathbb{R}^{n+1}, \langle , \rangle\big)$ is called Lorentz – Minkowski $(n+1)$-space and represented by $\mathbb{R}^{n+1}_1$. A non-zero vector $x \in \mathbb{R}^{n+1}_1$ is spacelike, timelike or lightlike if $\langle x, x \rangle > 0$, $\langle x, x \rangle < 0$ or $\langle x, x \rangle = 0$, respectively. The norm of a non-null vector is defined by $\|x\| = \sqrt{|\langle x, x \rangle|}$. The canonical projection $\pi \colon \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^n$, where $\pi(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. The lightcone with vertex a is defined as follows:

$$LC_a = \big\{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}_1 \mid \langle x - a, x - a \rangle = 0\big\}$$

and we denote $LC^* = LC_0 \backslash \{0\}$.

Let $X \colon U \to \mathbb{R}^{n+1}_1$ be a spacelike embedding of codimension k, where $U \subset \mathbb{R}^s$ is an open subset. Take N_pM as the pseudonormal space of M at p in $\mathbb{R}^{n+1}_1$, which is a k-dimensional Lorentzian subspace of $T_p\mathbb{R}^{n+1}_1$. On the pseudonormal space N_pM, there are following pseudospheres:

$$N_p(M; -1) = \{v \in N_pM \mid \langle v, v \rangle = -1\},$$

$$N_p(M; 1) = \{v \in N_pM \mid \langle v, v \rangle = 1\},$$

so that it can be written the following unit spherical normal bundles over M:

$$N(M; -1) = \bigcup\nolimits_{p \in M} N_p(M; -1) \text{ and } N(M; 1) = \bigcup\nolimits_{p \in M} N_p(M; 1).$$

There is always a future directed unit timelike normal vector field $n^T(u) \in N_p(M; -1)$. One can also choose a pseudonormal section $n^S(u) \in \big(\mathrm{Sp}\,\{n^T(u)\}\big)^{\perp} \cap N(M; 1)$ at least locally. Then $\langle n^S, n^S \rangle = 1$ and $\langle n^S, n^T \rangle = 0$. A $(k-1)$-dimensional spacelike unit sphere is defined by

$$N_1(M)_p[n^T] = \big\{\xi \in N_p(M; 1) \mid \langle \xi, n^T \rangle = 0\big\}$$

and a spacelike unit $k-2$ spherical bundle over M is defined by

$$N_1(M)[n^T] = \bigcup_{p \in M} N_1(M)_p[n^T].$$

The vector field $n^T + n^S$ is taken as a lightlike normal vector field along M, see [7].

Definition 1. *The mapping* $LG(n^T) \colon N_1(M)[n^T] \to LC^*$, *defined by*

$$LG(n^T)(u, \xi) = n^T(u) + \xi,$$

is called the lightcone Gauss image of $N_1(M)[n^T]$ [7].

Definition 2. *A hypersurface*

$$LH_M(n^T) \colon N_1(M)[n^T] \times \mathbb{R} \to \mathbb{R}^{n+1}_1$$

given by $LH_M(p, \xi, t) = X(u) + t(n^T + \xi)(u) = X(u) + tLG(n^T)(u, \xi)$, *where* $p = X(u)$, *is called the lightlike hypersurface along* M *relative to* n^T [7].

3. Lightlike hypersurfaces. In the light of the information given in [5], we consider the lightlike hypersurface along a spacelike submanifold $M = X(U)$ in $\mathbb{R}_1^{n+1}$ mentioned in the previous section. Take $LH_M\,(u,\xi,t) = Y\,(u,\xi,t)$ to ease the calculations. If we take $\tilde{U} = N_1(M)[n^T] \times \mathbb{R}$, we can represent the hypersurface by $Y(\tilde{U}) = \tilde{M}$. It is easy to see that the dimension of $T_P\tilde{M} = n$ and M is a submanifold of $\tilde{M}$. We can find the basis of $T_P\tilde{M}$ by taking the partial derivatives as

$$\begin{gathered} Y_{u_i}(u,\xi,t) = X_{u_i}(u) + t\left(n^T_{u_i} + \xi_{u_i}\right)(u), \quad i = 1,\ldots,s, \\ Y_{\xi_m}(u,\xi,t) = t\xi_{\xi_m}(u), \quad m = 1,\ldots,k-2, \\ Y_t(u,\xi,t) = (n^T + \xi)(u), \end{gathered} \tag{1}$$

where $u \in U$ and $\xi \in N_1(M)[n^T]$. Here we note that the sections Y_{u_i} and Y_{ξ_m} are spacelike and Y_t is lightlike.

From now, we assume $a_1 = u_1,\ldots,a_s = u_s,\ a_{s+1} = \xi_1,\ldots,a_{n-1} = \xi_{k-2},\ a_n = t$ for the easement of the calculations. We have a pseudo-Riemannian metric on $\tilde{M} = Y(\tilde{U})$ which is the lightcone first fundamental form defined by $ds^2 = \sum_{i=1}^{n} \tilde{g}_{ij} da_i da_j$, where $\tilde{g}_{ij} = \langle Y_{a_i}, Y_{a_j}\rangle$. Now let $V \in \operatorname{Rad}(T\tilde{M})$ and we choose a screen distribution of $\tilde{M}$ as $S(T\tilde{M}) = \operatorname{Sp}\{Y_{u_i}, Y_{\xi_m}\}$. Then the lightcone second fundamental form with respect to the pair $\left(n^T,\xi\right)$ of $S(T\tilde{M})$ is

$$\tilde{h}_{ij}(n^T) = \langle -V_{a_i}, Y_{a_j}\rangle, \quad \text{where} \quad i = 1,\ldots,n, \quad j = 1,\ldots,n-1. \tag{2}$$

We know that the radical distribution of the lightlike hypersurface $\tilde{M}$ is $\operatorname{Rad}\left(T\tilde{M}\right) = \operatorname{Sp}\{Y_t\}$. If we use the global null splitting theorem [1], to figure out the lightcone Weingarten operator of the screen distribution $S(T\tilde{M})$ of $\tilde{M}$, we can give the following theorem.

Theorem 1. *The lightcone Weingarten equations of the screen distribution $S(T\tilde{M})$ of $\tilde{M}$ are*

$$\Pi\left(V_{a_i}\right) = -\sum_{j=1}^{n-1} \tilde{h}_i^j(n^T) Y_{a_j}, \quad i = 1,\ldots,n,$$

where $\tilde{h}_i^j(n^T) = \left(\tilde{h}_{ik}(n^T)\right)\left(\tilde{g}^{kj}\right)$, $\tilde{g}^{kj} = (\tilde{g}_{kj})^{-1}$ and Π is the canonical projection of $\chi\big(T\tilde{M}\big)$ on $\chi\big(S(T\tilde{M})\big)$.

***Corollary* 1.** *Let $\tilde{M}$ be the lightlike hypersurface along a spacelike submanifold M. Then the following equation gives the relation between the lightcone second fundamental forms of $S(T\tilde{M})$ and M:*

$$\tilde{h}_{i\alpha} = \sum_{\beta=1}^{s} \tilde{h}_i^\beta \left(g_{\alpha\beta} - 2th_{\alpha\beta} + \frac{t^2}{\lambda^2} \sum_{l,q=1}^{n-1} \tilde{h}_\beta^l \tilde{h}_\alpha^q \tilde{g}_{lq} \right),$$

where $i = 1,\ldots,n, \alpha = 1,\ldots,s$ and $\lambda \in \mathbb{R}$.

4. Screen conformality. Since the screen distribution is non degenerate, it is very important when we investigate a lightlike hypersurface. Screen conformality provides getting information about the structure of the lightlike hypersurface with the help of its screen distribution. In the following theorem, we use the components of the lightlike vector field in the radical distribution to show the screen conformality of a lightlike hypersurface. Hence, this method makes remarkable simplicity.

Theorem 2. *A lightlike hypersurface $\tilde{M}$ along a spacelike submanifold M is screen conformal, if $(n^T - \xi)^i \neq 0$ and $\dfrac{\partial\left(\xi \pm n^T\right)^i}{\partial a_j} = 0$, where $i \neq j,\ i, j = 1, \ldots, n$.*

Proof. From now, we will denote the metric and connection of $\mathbb{R}_1^{n+1}$ by $\langle , \rangle$ and $\bar{\nabla}$, respectively. There is a section $N \in tr\,(TM)$ which is given by $N = \dfrac{1}{2}(\xi - n^T)$ and it ensures $\langle N, N\rangle = 0$, $\langle N, Y_{u_i}\rangle = \langle N, Y_{\xi_m}\rangle = 0$, $\langle N, Y_t\rangle = 1$. Since $B(X, Y_t) = 0$, see [1], where $X \in \chi(\tilde{M})$, we have $\bar{\nabla}_X Y_t = \nabla_X Y_t$. We can write $Y_t = \sum_{A=1}^{n} (n^T + \xi)^A \dfrac{\partial}{\partial a_A}$ and $X = \sum_{j=1}^{n} x_j \dfrac{\partial}{\partial a_j}$, then we get

$$\bar{\nabla}_X Y_t = \sum_{A=1}^{n}\sum_{j=1}^{n} x_j \frac{\partial (n^T + \xi)^A}{a_j} \frac{\partial}{\partial a_A}.$$

Since $\dfrac{\partial(\xi + n^T)^i}{\partial a_j} = 0$, where $i \neq j,\ i, j = 1, \ldots, n$, we denote

$$\bar{\nabla}_X Y_t = Z = \left(x_1 \frac{\partial (n^T + \xi)^1}{a_1}, \ldots, x_n \frac{\partial (n^T + \xi)^n}{a_n} \right),$$

where $Z \in \chi\big(T\tilde{M}\big)$. Therefore, we obtain

$$A^*_{Y_t} X + \tau(X) Y_t + \nabla_X Y_t = 0.$$

It can be seen that $\tau(X) = 0$ by using the equations $\langle \bar{\nabla}_X N, Y_t\rangle = \tau(X)$, see [3] and (1). Then $A^*_{Y_t} X = -PZ$, where P is the projection on $S(T\tilde{M})$. For every $X \in \chi\big(S(T\tilde{M})\big)$ we can write $X = \sum_{A=1}^{n} X^A \dfrac{\partial}{\partial a_A}$. Then we get

$$\sum_{A=1}^{n} X^A (n^T + \xi)^A = 0,$$
$$\nabla_{Y_t} X = \bar{\nabla}_{Y_t} X = \sum_{A=1}^{n}\sum_{i=1}^{n} (n^T + \xi)^i \frac{\partial X^A}{\partial a_i} \frac{\partial}{\partial a_A}. \tag{3}$$

If we take the partial derivatives of (1), since $\dfrac{\partial(\xi + n^T)^i}{\partial a_j} = 0$, where $i \neq j,\ i, j = 1, \ldots, n$, we obtain $\langle \nabla_{Y_t} X, Y_t\rangle = 0$. Therefore, $\nabla_{Y_t} X \in \chi\big(S(T\tilde{M})\big)$ and $A_N Y_t = 0$.

For $X, W \in \chi\big(S(T\tilde{M})\big)$,

$$g\left(A_N X, W\right) = C(X, W) = g(\nabla_X W, N) = \left\langle \bar{\nabla}_X W, N\right\rangle.$$

If we write

$$\bar{\nabla}_X N = \frac{1}{2}\sum_{A=1}^{n}\sum_{i=1}^{n} X^A \frac{\partial (n^T - \xi)^i}{\partial a_A} \frac{\partial}{\partial a_i},$$

since $\dfrac{\partial(\xi - n^T)^i}{\partial a_j} = 0,\ i \neq j,\ i, j = 1, \ldots, n$, we have

$$g\left(A_N X, W\right) = -\langle \bar{\nabla}_X N, W\rangle.$$

Hence, we write

$$A_N X = -\frac{1}{2}\begin{bmatrix} X^1 \dfrac{\partial (n^T - \xi)^1}{\partial a_1} \\ \vdots \\ X^n \dfrac{\partial (n^T - \xi)^n}{\partial a_n} \end{bmatrix}_{n\times 1} \qquad A^*_{Y_t} X = -PZ = -\begin{bmatrix} X^1 \dfrac{\partial (n^T + \xi)^1}{\partial a_1} \\ \vdots \\ X^n \dfrac{\partial (n^T + \xi)^n}{\partial a_n} \end{bmatrix}_{n\times 1}.$$

Finding the map

$$\Phi = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{bmatrix}_{n\times n}$$

which satisfies the equation $A_N X = \Phi A^*_{Y_t} X$, completes the proof. For this, from the equation

$$\alpha_{11} X^1 \frac{\partial (n^T + \xi)^1}{\partial a_1} + \ldots + \alpha_{1n} X^n \frac{\partial (n^T + \xi)^n}{\partial a_n} = X^1 \frac{\partial (n^T - \xi)^1}{\partial a_1},$$

we see that all the coefficients are zero except the ones on diagonal line. We calculate α_{ii} as

$$\alpha_{11} = -\frac{(n^T + \xi)^1}{(n^T - \xi)^1}, \quad \alpha_{22} = \frac{(n^T + \xi)^2}{(n^T - \xi)^2}, \quad \ldots \quad , \quad \alpha_{nn} = \frac{(n^T + \xi)^n}{(n^T - \xi)^n}.$$

Theorem 2 is proved.

Theorem 3. *Let $\tilde{M}$ be a lightlike hypersurface along a spacelike submanifold M. The Gauss image $LG(n^T)$ of M is a geodesic line in $\tilde{M}$.*

5. Examples.

***Example* 1.** The Schwarzschild spacetime in Eddington – Finkelstein coordinates $(u, r, \vartheta, \varphi)$ is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 + 2dudr + r^2 d\Omega^2,$$

where $M > 0$ denotes the mass and $d\Omega^2 = d\vartheta^2 + \sin^2\theta d\varphi^2$ denotes the volume element of the standard sphere. The event horizon is the surface given by

$$r = r_0 = 2M.$$

This is a lightlike hypersurface foliated by metric spheres of constant radius $r = r_0$ and generated by the lightlike vector field $L = 2\dfrac{\partial}{\partial u}$, see [9].

Now, let we consider the spacelike submanifold defined by

$$X(\vartheta, \varphi) = (0, 2M, \vartheta, \varphi).$$

Then we can find a basis of the tangent space as

$$\left\{ X_\vartheta = \frac{\partial}{\partial \vartheta}, X_\varphi = \frac{\partial}{\partial \varphi} \right\}.$$

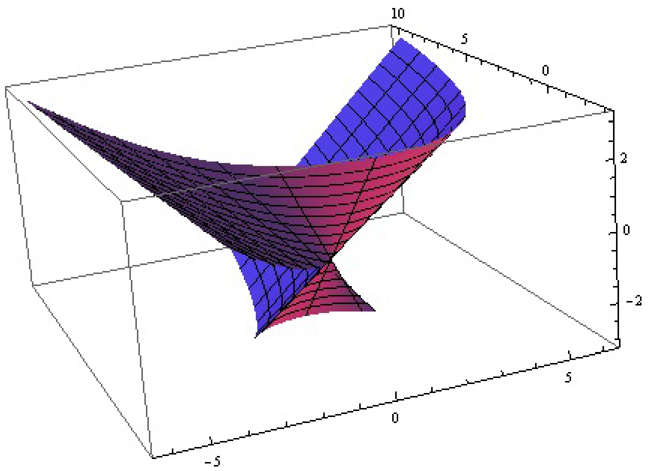

Fig. 1

One can see that the vector field L is in the normal space of X. Then we find an orthogonal basis of the normal space as $\left\{\eta = \frac{1}{2}\frac{\partial}{\partial u} + \frac{\partial}{\partial r},\ \zeta = \frac{1}{2}\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right\}$. Here we choose $\zeta = n^T$ and $\eta = \xi$ since ζ is timelike and η is spacelike. Hence, we have the lightcone Gauss image of the spacelike submanifold X defined by $(n^T + \xi) = \frac{\partial}{\partial u}$. Finally, we can rewrite the event horizon as

$$Y(u, \vartheta, \varphi) = (0, 2M, \vartheta, \varphi) + u(n^T + \xi).$$

***Example* 2.** In $\mathbb{R}^3_1$ with the metric $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2$ we take the spacelike curve $\alpha(u) = \big(0, u, u^2\big)$. Then we have the Frenet frame

$$T = \frac{1}{\sqrt{1+4u^2}}(0, 1, 2u),$$

$$N = \frac{1}{\sqrt{1+4u^2}}(0, -2u, 1),$$

$$B = (-1, 0, 0).$$

We can choose $B = n^T$ and $N = \xi$, then $n^T + \xi = \left(-1, \frac{-2u}{\sqrt{1+4u^2}}, \frac{1}{\sqrt{1+4u^2}}\right)$. Hence, the lightlike hypersurface along α is

$$Y(u, t) = \left(-t, u - \frac{2ut}{\sqrt{1+4u^2}}, u^2 + \frac{t}{\sqrt{1+4u^2}}\right),$$

where $t \in \mathbb{R}$. It can be seen in the Fig. 1.

If we use the notations in [3], we have

$$x_0 = -t, \qquad x_1 = u - \frac{2ut}{\sqrt{1+4u^2}}, \qquad x_2 = u^2 + \frac{t}{\sqrt{1+4u^2}},$$

$$\frac{\partial}{\partial u} = \left(1 - \frac{2t}{(1+4u^2)^{3/2}}\right)\frac{\partial}{\partial x_1} + \left(2u - \frac{4ut}{(1+4u^2)^{3/2}}\right)\frac{\partial}{\partial x_2},$$

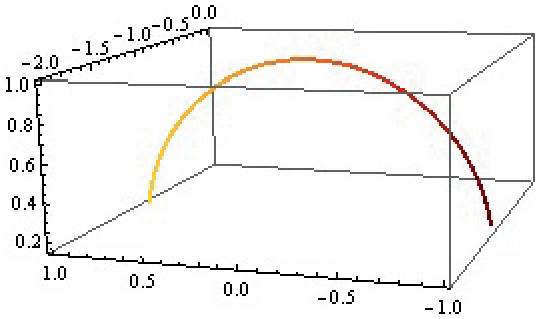

Fig. 2

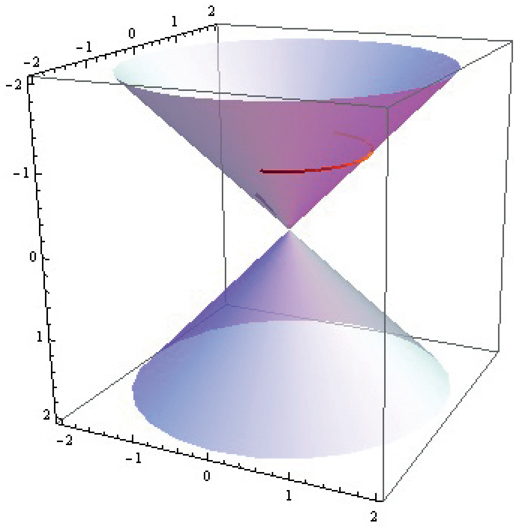

Fig. 3

$$\frac{\partial}{\partial t} = -\frac{\partial}{\partial x_0} - \frac{2u}{\sqrt{1+4u^2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{1+4u^2}}\frac{\partial}{\partial x_2}.$$

Then we calculate

$$n^T + \xi = -\frac{\partial}{\partial x_0} - \frac{2u}{\sqrt{1+4u^2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{1+4u^2}}\frac{\partial}{\partial x_2} = \frac{\partial}{\partial t}.$$

It can be seen that the lightcone Gauss image $LG(n^T)\,(u,\xi) = n^T(u) + \xi$ is in the direction of the lightlike vector $Y_t \in \operatorname{Rad} T\tilde{M} = T\tilde{M}^{\perp}$. Figures 2 and 3 show the lightcone Gauss image of the spacelike curve $\alpha(u)$.

We can rewrite $n^T + \xi = \left((n^T+\xi)^1, (n^T+\xi)^2\right) = (0,1)$ according to the base $\left\{\dfrac{\partial}{\partial u}, \dfrac{\partial}{\partial t}\right\}$. Since $a_1 = u$, $a_2 = t$ we have

$$\frac{\partial (n^T+\xi)^1}{\partial a_2} = \frac{\partial (n^T+\xi)^2}{\partial a_1} = 0.$$

According to Theorem 2, the lightlike hypersurface $Y(u,t)$ is screen conformal.

***Example* 3.** Let we take the spacelike curve $\beta(u) = (0, \cos u, \sin u)$. Then the Frenet frame is

$$T = (0, -\sin u, \cos u),$$

$$N = (0, \cos u, \sin u),$$

$$B = (-1, 0, 0).$$

We can choose $B = n^T$ and $N = \xi$, then $n^T + \xi = (-1, \cos u, \sin u)$. Hence, the lightlike hypersurface along β is

$$Y(u,t) = (-t, (t+1)\cos u, (t+1)\sin u)\,.$$

It can be calculated that $n^T + \xi = (t+1)^2 \dfrac{\partial}{\partial u} = \big((t+1)^2, 0\big)$ and this vector field is in the direction of the lightlike vector Y_t too. Since $\dfrac{\partial (n^T + \xi)^1}{\partial a_2} \neq 0$, assertions of Theorem 2 do not hold.

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