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Kaleidoscopical configurations

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Abstract. Let G be a group and X be a G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$. A subset A of X is called a kaleidoscopical configuration if there is a coloring $\chi : X \to \kappa$ (i.e. a mapping of X onto a cardinal κ) such that the restriction $\chi|_{gA}$ is a bijection for each $g \in G$. We survey some recent results on kaleidoscopical configurations in metric spaces considered as G-spaces with respect to the groups of its isometries and in groups considered as left regular G-spaces.

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1. Introduction

Let X be a set $\mathfrak{F}$ be a family of subsets of X. The pair $(X, \mathfrak{F})$ is called a *hypergraph*. Following [9], we say that a coloring $\chi : X \to \kappa$ (i.e. a mapping of X onto a cardinal κ) is *kaleidoscopical* if $\chi|_F$ is bijective for all $F \in \mathfrak{F}$. A hypergraph $(X, \mathfrak{F})$ is called *kaleidoscopical* if there exists a kaleidoscopical coloring $\chi : X \to \kappa$. The adjective "kaleidoscopical" appeared in definition [13] of an s-regular graph $\Gamma(V, E)$ (each vertex $v \in V$ has degree s) admitting a vertex $(s+1)$-colloring such that each unit ball $B(v, 1) = \{u \in V : d(u, v) = 1\}$ has the vertices of all colors (d is the path metric on V). These graphs define the kaleidoscopical hypergraphs $(V, \{B(v, 1) : v \in V\})$ and can be considered as the graph counterparts of the Hamming codes [10].

In this paper we survey some recent results and open problems on kaleidoscopical configurations in G-spaces.

Let G be a group. A *G-space* is a set X endowed with an action $G \times X \to X$, $(g, x) \mapsto gx$. All G-spaces are suppose to be *transitive*: for any $x, y \in X$, there exists $g \in G$ such that $gx = y$. For a subset $A \subseteq X$, we denote $G[A] = \{gA : g \in G\}$ where $gA = \{ga : a \in A\}$.

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A subset $A \subseteq X$ is called a *kaleidoscopical configuration* if the hypergraph $(X, G[A])$ is kaleidoscopical, in words, if there exists a coloring $\chi : X \to |A|$ such that $\chi|_{gA}$ is bijective for every $g \in G$.

We note that finite kaleidoscopical configurations in a sense are antipodal to monochromatizable configurations defined and studied in [9, Chapter 8]: a subset A of a G-space X is called *monochromatizable* if, for any finite coloring of X, there is $g \in G$ such that gA is monochrome.

In Section 2 we discus a relationship between the kaleidoscopical configurations in a G-space X and transversals of the family $\{gA : g \in G\}$, $A \subseteq G$. We present also an effective method (namely, the splitting), of construction of kaleidoscopical configurations in a G-space X from the finite chains of G-invariant equivalence relations on X.

The main results of Section 3 are about kaleidoscopical configurations in $\mathbb{R}^n$ considered as a G-space with respect to the group $G = \mathrm{Iso}(\mathbb{R}^n)$ of all Euclidean isometries. For $n = 1$, it is easy to find a kaleidoscopical configuration in $\mathbb{R}$ of any size $\leq$ the cardinality of the continuum. The problem is much more difficult for $n \geq 2$. Surprisingly, the subsets $\mathbb{Z} \times \{0\}$, $\mathbb{Q} \times \{0\}$, $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Z} \times \mathbb{Z}$ are kaleidoscopical in $\mathbb{R}^2$. The most intriguing open problem: for $n \geq 2$, does there exist a finite kaleidoscopical configuration K, $|K| \geq 2$ in $\mathbb{R}^n$. We show that if such a K exists in $\mathbb{R}^2$ then $|K| \geq 5$.

Each group G can be considered as a (left) regular G-space $X = G$, where $(g, x) \longmapsto gx$ is the group product. In Section 4 we show that kaleidoscopical configurations in G are tightly connected with factorizations of $G = AB$ by subsets A, B. The factorizations were introduced by Hajoś [5] to solve the famous Minkowsky's problem on tiling of $\mathbb{R}^n$ by the copies of a cube. For modern state of factorizations see [17, 18]. Also we establish a connection between kaleidoscopical configurations and T-sequences from [12].

2. Transversality and factorization

Let $(X, \mathfrak{F})$ be a hypergraph. A subset $T \subseteq X$ is called an $\mathfrak{F}$-*transversal* if $|F \bigcap T| = 1$ for each $F \in \mathfrak{F}$. All results of this section are from [1].

Theorem 2.1. *A hypergraph $(X, \mathfrak{F})$ is kaleidoscopical if and only if X can be partitioned into $\mathfrak{F}$-transversals.*

For a cardinal κ, $cf\kappa$ denotes the cofinality of κ.

Theorem 2.2. *Let κ be an infinite cardinal, $(X, \mathfrak{F})$ be a hypergraph such that $|\mathfrak{F}| = \kappa$ and $|F| = \kappa$ for each $F \in \mathfrak{F}$. If $|F \cap F'| < cf\kappa$ for all distinct $F, F' \in \mathfrak{F}$ then there is a disjoint family $\mathfrak{T}$ of $\mathfrak{F}$-transversals such that $|\mathfrak{T}| = \kappa$ and $|T| = \kappa$ for each $T \in \mathfrak{T}$.*

For a hypergraph $(X, \mathfrak{F})$, $x \in X$ and $A \subseteq X$, we put

$$St(x, \mathfrak{F}) = \bigcup\{F \in \mathfrak{F} : x \in F\},$$

$$St(A, \mathfrak{F}) = \bigcup\{St(a, F) : a \in A\}.$$

Theorem 2.3. *A hypergraph $(X, \mathfrak{F})$ is kaleidoscopical provided that, for some infinite cardinal κ, the following two conditions are satisfied:*

(i) $|\mathfrak{F}| \leq \kappa$ and $|F| = \kappa$ for each $F \in \mathfrak{F}$;

(ii) for any subfamily $\mathfrak{A} \subset \mathfrak{F}$ of cardinality $|\mathfrak{A}| < \kappa$ and any subset $B \subset X \setminus (\bigcup \mathfrak{A})$ of cardinality $|B| < \kappa$ the intersection $St(B, \mathfrak{F}) \cap (\bigcup \mathfrak{A})$ has cardinality less than κ.

Now we present some construction of kaleidoscopical configurations in arbitrary G-space, called the splitting. The kaleidoscopical configurations obtained in this way will be called *splittable*.

Given an equivalence relation $E \subseteq X \times X$ on a set X, let $X/E = \{[x]_E : x \in X\}$ be the quotient space consisting of the equivalence classes $[x]_E = \{y \in X : (x, y) \in E\}$, $x \in X$. Denote by $q_E : X \to X/E$, $q_E(x) = [x]_E$, the quotient mapping. For a subset K of X, let $K/E = \{[x]_E : x \in K\} \subseteq X/E$ and $[K]_E = \bigcup_{x \in K}[x]_E \subseteq X$.

Let E be an equivalence relation on a set X. A subset $K \subseteq X$ is defined to be

- *E-parallel* if $K \cap [x]_E = [x]_E$ for all $x \in K$;
- *E-orthogonal* if $K \cap [x]_E = \{x\}$ for all $x \in K$.

Given two equivalence relations E, F on X such that $F \subseteq E$, we generalize these two notions defining $K \subseteq X$ to be

- *E/F-parallel* if $[K]_F \cap [x]_E = [x]_E$ for all $x \in K$;
- *E/F-orthogonal* if $[K]_F \cap [x]_E = [x]_F$ for all $x \in K$.

We observe that $K \subseteq X$ is E-parallel (E-orthogonal) if it is E/Δ_X-parallel (E/Δ_X-orthogonal), where $\Delta_X = \{(x, x) : x \in X\}$.

An equivalence relation E on a G-space X is called *G-invariant* if, for each $(x, y) \in E$ and every $g \in G$ we have $(gx, gy) \in E$. For a G-invariant equivalence relation E on X, the quotient space X/E is a G-space under the induced action

$$G \times X/E \to X/E, \quad (g, [x]_E) \mapsto [gx]_E$$

of the group G.

Theorem 2.4. *Let $\Delta_X = E_0 \subset E_1 \subset \cdots \subset E_m = \{X \times X\}$ be a chain of G-invariant equivalence relations on a G-space X. A subset K of X is kaleidoscopical provided that, for every $i \in \{0, \ldots, m-1\}$, K is either E_{i+1}/E_i-parallel or E_{i+1}/E_i-orthogonal.*

A subset K of a G-space X is called *splittable* if there is a chain $\Delta_X = E_0 \subset E_1 \subset \cdots \subset E_m = \{X \times X\}$ of G-invariant equivalence relations on X such that, for each $i \in \{0, \ldots, m-1\}$, K is either E_{i+1}/E_i-parallel or E_{i+1}/E_i-orthogonal. By Theorem 2.4, each splittable subset of X is a kaleidoscopical configuration.

Some partial answers to the following general question are in the next sections.

Question 2.1. *Given a G-space X, how one can detect whether each kaleidoscopical configuration in X is splittable?*

For motivation of the following definition see [1, Section 4].

A G-space has the *semi-Hajós property* if, for every kaleidoscopical subset $K \subset X$, there is an equivalence relation E on X, $E \neq \Delta_X$ such that K is E-parallel or E-orthogonal and K/E is kaleidoscopical in the G-space K/E.

Theorem 2.5. *If each kaleidoscopical subset of a G-space X is splittable, then X has the semi-Hajós property.*

On some partial conversions of Theorem 2.5 see [1, Section 4].

A G-space X is called *primitive* if each G-invariant equivalence relation on X is either Δ_X or $\{X \times X\}$. Clearly, each splittable configuration K in a primitive G-space X is trivial, i.e. either $K = X$ or K is a singleton. It is natural to ask whether every kaleidoscopical configuration in a primitive G-space is trivial? The answer to this question is affirmative if X is *2-transitive*: for any $(x, y), (x', y') \in X^2 \setminus \Delta_X$, there is $g \in G$ such that $(x', y') = (gx, gy)$. But for $n \geq 2$, the primitive space $\mathbb{R}^n$ endowed with the action of its group of all Euclidean isometries has a plenty of infinite kaleidoscopical configurations, see Section 3.

Question 2.2. *Is every finite kaleidoscopical configuration in a (finite) primitive G-space trivial?*

3. Kaleidoscopical configurations in metric spaces

Here we consider each metric space (X, d) as a G-space endowed with the natural action of its isometry group $G = Iso(X)$. If this action is transitive, X is called *isometrically homogeneous.*

Let us recall that a metric space (X, d) is *ultrametric* if the metric d satisfies the strong triangle inequality

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for all x, y, z. In this case, for every $\varepsilon \geq 0$, the relation

$$E_\varepsilon = \{(x, y) \in X^2 : d(x, y) \leq \varepsilon\}$$

is an invariant equivalence relation on X.

Theorem 3.1 ([1]). *Let (X, d) be an isometrically homogeneous ultrametric space with the finite distance scale $d(X \times X) = \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n\}$ where $0 = \varepsilon_0 < \varepsilon_1 < \dots < \varepsilon_n$. Then every kaleidoscopical configuration in X is $(E_{\varepsilon_0}, E_{\varepsilon_1}, \dots, E_{\varepsilon_n})$-splittable.*

Let (X, d) be a metric space. By $S(x, r) = \{y \in X : d(x, y) = r\}$, we denote the sphere of radius r centered at x.

A subset K of X is called *rigid* if, for any distinct points $x, y, z \in K$ and numbers $r_x, r_y, r_z \in d(K \times K)$ the spheres $S(x, r_x)$, $S(y, r_y)$, $S(z, r_z)$ have no common points in $X \setminus K$. A proof of the following theorem uses Theorem 2.3.

Theorem 3.2 ([1]). *Let X be a metric space and let $G \subseteq Iso(X)$ be a group of isometries of X. Then each infinite rigit subset K of X of cardinality $|K| \geq |G|$ is kaleidoscopical.*

Now we consider the Euclidean space $\mathbb{R}^n$ as a G-space with respect to the group $G = Iso(\mathbb{R}^n)$ of all isometries of $\mathbb{R}^n$. Given a cardinal $\kappa \leq \mathfrak{c}$, it is easy to find a kaleidoscopical configurations of cardinality κ in $\mathbb{R}$, but the problem is much more delicate for R^n, $n \geq 2$.

Theorem 3.3 ([1]). *Any algebraically independent over $\mathbb{Q}$ subset A of an affine line (identified with $\mathbb{R}$) in the Euclidean space $\mathbb{R}^n$ is rigid. For any $n \geq 2$, $\mathbb{R}^n$ contains $2^{\mathfrak{c}}$ kaleidoscopical configurations of cardinality $\mathfrak{c}$.*

Following [8], we say that a subset A of $\mathbb{R}^n$ has the Steinhaus property if the family $\{gA : g \in \mathrm{Iso}(\mathbb{R}^n)\}$ has a transversal B. In this case, B is a transversal of the family $\{x + A : x \in \mathbb{R}^n\}$. By Theorem 4.1 $\{B - a : a \in A\}$ is a partition of $\mathbb{R}^n$. Since each subset $B - a$ is a transversal of the family $\{gA : g \in \mathrm{Iso}(\mathbb{R}^n)\}$, by Theorem 2.1, A is a kaleidoscopical configuration.

Theorem 3.4 ([6, 7]). *The subsets $\mathbb{Z} \times \{0\}$, $\mathbb{Q} \times \{0\}$, $\mathbb{Q}$ of $\mathbb{R}$ have the Steinhaus property and hence are kaleidoscopical configurations.*

Theorem 3.5 ([2–4]). *The subset* $\mathbb{Z}\times\mathbb{Z}$ *of* $\mathbb{R}^2$ *has a Steinhaus property and hence is a kaleidoscopical configuration.*

Theorem 3.6 ([15]). *The subset* $\mathbb{Z}^m \times \{0\}^{n-m}$ *does not have the Steinhaus property for* $4 \le m < n$.

Question 3.1. *Does there exist a non-trivial finite kaleidoscopical configuration in* $\mathbb{R}^n$ *for* $n \ge 2$*?*

We put $k(\mathbb{R}^n) = \min\{|F| : |F| > 1$ and F is a kaleidoscopical configuration in $\mathbb{R}^n\}$. It is easy to see that $\kappa(\mathbb{R}^n) \ge \chi(\mathbb{R}^n)$, where $\chi(\mathbb{R}^n)$ is a chromatic number of $\mathbb{R}^n$. We recall that $\chi(\mathbb{R}^n)$ is the smallest number of colors for which there is a coloring of $\mathbb{R}^n$ without monochrome points at the distance 1. It is well known that $4 \le \chi(\mathbb{R}^2) \le 7$ and there is a conjecture that $\chi(\mathbb{R}^n) = 2^{n+1} - 1$, see [16, §47]. Thus, $\kappa(\mathbb{R}^2) \ge 4$. We show that $\kappa(\mathbb{R}^2) \ge 5$.

For $n \ge 1$ and $d > 0$, a *rather red coloring* of $\mathbb{R}^n$ with respect to d is a 2-coloring of $\mathbb{R}^n$, with red and blue, such that no two blue points are a distance d apart. Let $m_c = \min\{|F| : F \subset \mathbb{R}^2$ and each isometric copy of F is forbidden for red by some rather red coloring of $\mathbb{R}^2\}$. By [14, p. 102], $5 \le m_c \le 8$.

Now assume that there is a kaleidoscopical configuration K in $\mathbb{R}^2$ of cardinality $|K| = 4$. Let $\chi : \mathbb{R}^2 \longrightarrow \{1,2,3,4\}$ be the corresponding kaleidoscopical coloring. We recolor $\chi' : \mathbb{R}^2 \longrightarrow \{red, blue\}$ by the following rule $\chi'(x)$ is blue if and only if $\chi'(x) = 4$. Let d be a distance between some two points of K. Since χ is kaleidoscopical, we conclude that χ' is rather red and each isometric copy of F is forbidden for red, contradicting $m_c \ge 5$.

4. Kaleidoscopical configurations in groups

A subset A of a group G is defined to be *complemented* if there exists a subset B of G such that the multiplication mapping $\mu : A \times B \to G$, $(a, b) \mapsto ab$, is bijective. Following [18], we say that B is a *complementer factor* to A and $G = AB$ is a *factorization* of G. In this case, we have the partitions

$$G = \bigsqcup_{a\in A} aB = \bigsqcup_{b\in B} Ab.$$

A subset $A \subseteq G$ is called doubly complemented if there are factorizations $G = AB = BC$ for some subsets B, C of G.

The following interrelations between kaleidoscopical configurations and factorizations are observed in [1].

Theorem 4.1. *Let A, B be subsets of a group G. Then B is $G[A]$-transversal if and only if $G = AB^{-1}$ is a factorization of G. In particular, each kaleidoscopical configuration in G is complemented.*

Theorem 4.2. *A subset A of an Abelian group G is a kaleidoscopical configuration if and only if A is complemented.*

Question 4.1. *Is each complemented subset of a (finite) group kaleidoscopical?*

The remaining results of this section are from [11]. We say that a subset A of a group G is *rigid* if, for each $g \in G \setminus A$, the set $g^{-1}A \cap A^{-1}A$ is finite. Applying Theorem 2.3 we get:

Theorem 4.3. *If A is a countable rigid subset of a group G then A is a kaleidoscopical configuration.*

An injective sequence $(a_n)_{n\in\omega}$ in a group G is called a T-sequence [12] if there exists a Hausdorff group topology in which $(a_n)_{n\in\omega}$ converges to the identity e of G.

Theorem 4.4. *For every T-sequence $(a_n)_{n\in\omega}$ in a group G, the set $A = \{e, a_n, a_n^{-1} : n \in \omega\}$ is a kaleidoscopical configuration. In particular, A is complemented and G can be partitioned into right translations of A.*

Theorem 4.5. *Every infinite subset S of an Abelian group G contains an infinite kaleidoscopical configuration.*

Corollary 4.1. *If S is an infinite subset of an Abelian group, then S contains an infinite complemented subset.*

Let G be a group defined by the following generators and relations

$$\langle x_m, y_m : x_m^2 = y_m^2 = e,\ x_n x_m x_n = y_m,\ m < n < \omega\rangle.$$

Then the subset $\{x_n : n \in \omega\}$ has no infinite rigid subsets.

Question 4.2. *Does every infinite subset of an arbitrary infinite group contains an infinite kaleidoscopical (complemented) subset?*

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