Український математичний вісник
Том 11 **(2014)**, № 3, 291 – 311

Initial-boundary value problems for anisotropic elliptic-parabolic-pseudoparabolic equations with variable exponents of nonlinearity

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Abstract. Existence and uniqueness of weak solutions of initial-boundary value problems for anisotropic elliptic-parabolic-pseudoparabolic equations with variable exponents of nonlinearity are proved. Estimates of the weak solutions of this problems are received. This estimates implies continuous dependence on the input data for the weak solutions of considered problems.

2010 MSC. 35K70.

Key words and phrases. Initial-boundary value problem, degenerated pseudoparabolic equation, elliptic-parabolic-pseudoparabolic equation.

1. Introduction

The pseudoparabolic equations are a kind of Sobolev–Galpern type equations. They are characterized by mixed time and space derivatives appearing in the highest order terms of this equations. Such equations were first studied by S. L. Sobolev in the linear case [1]. Pseudoparabolic equations arise in numerous physical applications, e.g., seepage of fluids through fissured rocks, unsteady flows of second-order fluids, dynamic capillary pressure in unsaturated flow, the theory of thermodynamics involving two temperatures [2, 3].

Mathematical study of pseudoparabolic equations goes back to works of Showalter in the seventies [4]. Since then, a number of interesting results on linear and nonlinear pseudoparabolic equations have been obtained. In particular, existence and uniqueness of solutions to nonlinear pseudoparabolic equations are proved in [5–8].

Received 27.02.2014

ISSN 1810 – 3200.

In this paper, we are interested in degenerated pseudoparabolic equations. Such equations were studied in [5, 9–14]. Let us formulate one of results that is relevant to what we are going to do in this paper.

Let V be a separable reflexive Banach space and V' be its dual. Assume $\mathcal{A}(t;\cdot) : V \to V'$, $0 \le t \le T$, is a measurable family of monotone, hemicontinuous, uniformly bounded and coercive operators. Let $B : V \to V'$ be a linear, continuous, symmetric and monotone operator. Let V_b be the completion of space V with seminorm $\|\cdot\|_b := \langle B\cdot,\cdot\rangle^{1/2}$, and let V_b' be the dual to V_b. It is known that V_b' is a Hilbert space and $\|Bv\|_{V_b'} = \|v\|_{V_b}$ $\forall\, v \in V_b$. The operator B can be extended to an operator acting from V_b to V_b'. Let $p \ge 2$, $p' := p/(p-1)$.

The problem is, given $u_0 \in V_b$ and $f \in L_{p'}(0,T;V')$, to find a function $u \in L_p(0,T;V) \cap C([0,T];V_b)$ which satisfies the equation

$$\frac{d}{dt}\big(Bu(t)\big) + \mathcal{A}(t;u(t)) = f(t) \quad \text{in} \quad L_{p'}(0,T;V') \tag{1.1}$$

and the initial condition

$$\|u(0) - u_0\|_{V_b} = 0. \tag{1.2}$$

As follows from [5, Corollary III.6.3], this problem has a unique solution.

Here is a simple example of problem (1.1),(1.2). For given $l > 0$, let $V = \overset{\circ}{W}{}^1_p(0,l) := \{v \in L_p(0,l) \,|\, v' \in L_p(0,l),\ v(0) = v(l) = 0\}$ be the Sobolev space and $V' = W^{-1}_{p'}(0,l)$ be its dual. Any element $g \in W^{-1}_{p'}(0,l)$ can be written as $g = g_0 - (g_1)'$, where $g_0, g_1 \in L_{p'}(0,l)$ and $(g_1)'$ is the derivative of g_1 in the distribution space $D'(0,l)$. Then $\langle g, v\rangle := \int_0^l (g_0(x)v(x) + g_1(x)v'(x))\,dx$ is the action of g on $v \in \overset{\circ}{W}{}^1_p(0,l)$. Define a family of operators $\mathcal{A}(t;\cdot) : V \to V', 0 \le t \le T$, by $\mathcal{A}(t;v) := (a(x,t)|v'|^{p-2}v')'$, where a is a measurable bounded function such that $\operatorname{ess\,inf}\{a(x,t) | (x,t) \in (0,l) \times (0,T)\} > 0$.

Let b_0, b_1 be measurable bounded functions on $(0,l)$ such that $b_j(x) > 0$ if $x \in (\alpha_j, \beta_j)$ and $b_j(x) = 0$ otherwise $(j = 0,1)$, where $(\alpha_1,\beta_1) \subset (\alpha_0,\beta_0) \subset (0,l)$. Also, assume $\inf\{b_j(x) \,|\, x \in [\alpha_j^*, \beta_j^*]\} > 0$ for all $[\alpha_j^*, \beta_j^*] \subset (\alpha_j,\beta_j)$ $(j = 0,1)$. Define the operator $B : V \to V'$ by $Bv := b_0 v - (b_1 v')'$, $v \in \overset{\circ}{W}{}^1_p(0,l)$. Let $\widetilde{b}_0(x) := b_0(x)$ if $x \in (\alpha_0,\beta_0)$ and $\widetilde{b}_0(x) := 1$ if $x \in (0,l) \setminus (\alpha_0,\beta_0)$. Denote by V_b the space of functions w such that $w = \widetilde{b}_0^{-1/2} v$, where $v \in L_2(0,l)$, and $w' \in L_{2,\mathrm{loc}}(\alpha_1,\beta_1)$, $b_1^{1/2} w' \in L_2(\alpha_1,\beta_1)$. This space is the completion of V by the seminorm $\|v\|_{V_b} := \big(\int_0^l [b_0|v|^2 + b_1|v'|^2]\,dx\big)^{1/2}$.

Now we can formulate the simple example of problem (1.1), (1.2). Given $u_0 \in V_b$ and $f_0, f_1 \in L_2((0,1)\times(0,T))$, find a function $u \in L_p(0,T;\overset{\circ}{W}{}^1_p(0,l))$ such that $b_0^{1/2}u, b_1^{1/2}u_x \in C([0,T];L_2(0,l))$ and

$$(b_0u-(b_1u_x)_x)_t-(a|u_x|^{p-2}u_x)_x=f_0-(f_1)_x \quad \text{in } L_{p'}(0,T;W^{-1}_{p'}(0,l)), \tag{1.3}$$

$$b_0^{1/2}u|_{t=0}=b_0^{1/2}u_0, \quad b_1^{1/2}u_x|_{t=0}=b_1^{1/2}u_0' \quad \text{in } L_2(0,l). \tag{1.4}$$

Note that equation (1.3) is pseudoparabolic in the domain $(\alpha_1,\beta_1)\times(0,T)$, parabolic in the domain $((\alpha_0,\beta_0)\setminus(\alpha_1,\beta_1))\times(0,T)$ and elliptic in $((0,l)\setminus(\alpha_0,\beta_0))\times(0,T)$. Such equations belong to the class of degenerate pseudoparabolic equations, and we believe that the right name for them is elliptic-parabolic-pseudoparabolic equations. By the way, if $b_0>0$ almost everywhere on $(0,l)$ then corresponding equations should be called parabolic-pseudoparabolic equations.

In this paper, we consider anisotropic elliptic-parabolic-pseudoparabolic equations with the variable exponents of nonlinearity that generalize equation (1.3). A typical example is an equation

$$(b_0(x)u-\sum_{i=1}^n\big(b_i(x)u_{x_i}\big)_{x_i})_t-\sum_{i=1}^n(\widetilde{a}_i(x,t)|u_{x_i}|^{p_i(x)-2}u_{x_i})_{x_i} + \widetilde{a}_0(x,t)|u|^{p_0(x)-2}u=0, \quad (x,t)\in Q, \tag{1.5}$$

where $b_j\geq 0$ on Ω (the functions b_j can be zero on subsets of Ω of positive measure) and $\widetilde{a}_j, p_j$ are measurable, nonnegative and bounded functions, moreover, $\operatorname{ess\,inf}_{x\in Q}\widetilde{a}_i(x,t)>0$ $(i=\overline{1,n})$ and $\operatorname{ess\,inf}_{x\in\Omega}p_j(x)>1$ $(j=\overline{0,n})$. The functions p_j are called exponents of nonlinearity.

Nonlinear differential equations with variable exponents of nonlinearity describe many physical processes such us electromagnetic fields, electrorheological fluids, image reconstruction processes, current flow in variable temperature field [15]. Solutions of these problems belong to some generalized Lebesgue and Sobolev spaces. The spaces were first introduced in [16]. The properties of these spaces and their applications to nonlinear differential equations with variable exponents of nonlinearity have been actively studied (see, e.g., [17–24]). But we do not known works where to consider the anisotropic elliptic-parabolic-pseudoparabolic equations with variable exponents of nonlinearity.

In this paper we find sufficient conditions for the existence and uniqueness of the weak solutions to the initial-boundary value problems for the anisotropic elliptic-parabolic-pseudoparabolic equations with variable exponents of nonlinearity. To proof the existence of weak solutions, we apply a combination of approximation and Galerkin methods. The paper is organized as follows. In Section 2, we formulate the problem and the main results. Auxiliary statements are given in Section 3. Finally, in Section 4 we prove main statements.

2. Statement of the problem and the main result

Let $n \in \mathbb{N}$, $T > 0$ be some numbers, $\mathbb{R}^n$ be the Euclidean space with norm $|\cdot|$ defined by $|x| := (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with the piecewise smooth boundary $\partial\Omega$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the closure of an open set on $\partial\Omega$ (in particular, Γ_0 can be $\emptyset$ or $\partial\Omega$), $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, $\nu = (\nu_1, \ldots, \nu_n)$ is a unit, outward pointing normal vector on the $\partial\Omega$. Let $Q := \Omega \times (0, T)$, $\Sigma_0 := \Gamma_0 \times (0, T)$, $\Sigma_1 := \Gamma_1 \times (0, T)$.

In this paper we consider the following problem: to find the function $u : \overline{Q} \to \mathbb{R}$ satisfying (in some sense) the equation

$$\big(b_0(x)u - \sum_{i=1}^{n} \big(b_i(x)u_{x_i}\big)_{x_i}\big)_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, t, u, \nabla u) + a_0(x, t, u, \nabla u) = -\sum_{i=1}^{n} \big(f_i(x, t)\big)_{x_i} + f_0(x, t), \quad (x, t) \in Q, \tag{2.1}$$

the boundary conditions

$$u\Big|_{\Sigma_0} = 0, \qquad \sum_{i=1}^{n} a_i(x, t, u, \nabla u)\, \nu_i\Big|_{\Sigma_1} = 0, \tag{2.2}$$

and the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega_0 := \{x \in \Omega \,|\, b_0(x) > 0\}. \tag{2.3}$$

Here $b_j : \Omega \to \mathbb{R}$, $a_j : Q \times \mathbb{R}^{1+n} \to \mathbb{R}$, $f_j : Q \to \mathbb{R}$ $(j = \overline{0, n})$, $u_0 : \Omega \to \mathbb{R}$ are given functions, moreover, $b_j \geq 0$ on Ω $(j = \overline{0, n})$ and $\{x \in \Omega \,|\, b_i(x) > 0\} =: \Omega_i \subset \Omega_0$ $(i = \overline{1, n})$. Notice that the functions b_j can be zero on subsets of Ω of positive measure.

Next we are going to define a weak solution of the problem (2.1)–(2.3) and formulate the main result of our paper. For this, we need some functional spaces and classes of input data of the given problem.

First we introduce the functional spaces. Let G denote Ω or Q. Suppose that $r \in L_\infty(\Omega)$, $r(x) \geq 1$ for a.e. $x \in \Omega$. Consider a subspace $L_{r(\cdot)}(G)$ of the vector space $L_1(G)$ consisting of all measurable functions v such that $\rho_{G,r}(v) < \infty$, where $\rho_{G,r}(v) := \int_\Omega |v(x)|^{r(x)}\, dx$ if $G = \Omega$ and $\rho_{G,r}(v) := \iint_Q |v(x, t)|^{r(x)}\, dx\, dt$ if $G = Q$. This is a Banach space with respect to the norm $\|v\|_{L_{r(\cdot)}(G)} := \inf\{\lambda > 0 \mid \rho_{G,r}(v/\lambda) \le 1\}$ and it is called a *generalized Lebesgue space.* Note that the set $C(\overline{G})$ is dense in $L_{r(\cdot)}(G)$, and if $r(x) = r_0 = \text{const} \ge 1$ for a.e. $x \in \Omega$, then $||\cdot||_{L_{r(\cdot)}(G)}$ is the standard norm $||\cdot||_{L_{r_0}(G)}$ on the Lebesgue space $L_{r_0}(G)$. If $\operatorname*{ess\,inf}_{x\in\Omega} r(x) > 1$, then the space $L_{r(\cdot)}(G)$ is reflexive and the dual space $[L_{r(\cdot)}(G)]'$ equals $L_{r'(\cdot)}(G)$, where the function r' is defined by $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ for a.e. $x \in \Omega$.

Consider a vector-function $p = (p_0, \dots, p_n) : \Omega \to \mathbb{R}^{n+1}$ satisfying the following condition:

($\mathcal{P}$) for every $j \in \{0, 1, \dots, n\}$ the function $p_j : \Omega \to \mathbb{R}$ is measurable and

$$p_j^- := \operatorname*{ess\,inf}_{x\in\Omega} p_j(x) \ge 2, \quad p_j^+ := \operatorname*{ess\,sup}_{x\in\Omega} p_j(x) < +\infty.$$

Denote by $p' = ({p_0}', \dots, {p_n}') : \Omega \to \mathbb{R}^{n+1}$ the vector-function such that $\frac{1}{p_i(x)} + \frac{1}{{p_j}'(x)} = 1$ for a.e. $x \in \Omega$ $(j = \overline{0,n})$.

Define $W^1_{p(\cdot)}(\Omega)$ to be the space of functions $v \in L_{p_0(\cdot)}(\Omega)$ such that $v_{x_1} \in L_{p_1(\cdot)}(\Omega)$, $\dots$, $v_{x_n} \in L_{p_n(\cdot)}(\Omega)$. This is a Banach space with respect to the norm $\|v\|_{W^1_{p(\cdot)}(\Omega)} := \|v\|_{L_{p_0(\cdot)}(\Omega)} + \sum_{i=1}^n \|v_{x_i}\|_{L_{p_i(\cdot)}(\Omega)}$ and it is called a generalized anisotropic Sobolev space. Let $\widetilde{W}^1_{p(\cdot)}(\Omega)$ be the subspace of $W^1_{p(\cdot)}(\Omega)$ that is the closure of the space $\widetilde{C}^1(\overline{\Omega}) := \{v \in C^1(\overline{\Omega}) \bigm| v|_{\Gamma_0} = 0\}$ with respect to the norm $\|\cdot\|_{W^1_{p(\cdot)}(\Omega)}$.

Denote by $W^{1,0}_{p(\cdot)}(Q)$ the space of functions $w \in L_{p_0(\cdot)}(Q)$ such that $w_{x_1} \in L_{p_1(\cdot)}(Q), \dots, w_{x_n} \in L_{p_n(\cdot)}(Q)$. Endow this space with the norm $\|w\|_{W^{1,0}_{p(\cdot)}(Q)} := \|w\|_{L_{p_0(\cdot)}(Q)} + \sum_{i=1}^n \|w_{x_i}\|_{L_{p_i(\cdot)}(Q)}$. It is a generalized anisotropic Sobolev space as well. Define $\widetilde{W}^{1,0}_{p(\cdot)}(Q)$ to be the subspace of $W^{1,0}_{p(\cdot)}(Q)$ such that if $w \in \widetilde{W}^{1,0}_{p(\cdot)}(Q)$ then $w(\cdot, t) \in \widetilde{W}^1_{p(\cdot)}(\Omega)$ for a. e. $t \in (0, T)$.

Consider functions $b_j : \Omega \to \mathbb{R}$ $(j = \overline{0,n})$ such that the following condition holds:

($\mathcal{B}$) for every $j \in \{0, 1, \dots, n\}$ the function $b_j : \Omega \to \mathbb{R}$ is measurable and bounded, $b_j(x) \ge 0$ for a.e. $x \in \Omega$, the set $\Omega_j := \{x \in \Omega \bigm| b_j(x) > 0\}$ is open, and $\operatorname{ess\,sup}_{x\in\Omega'} b_j(x) > 0$ for every open set Ω' such that $\overline{\Omega'} \subset \Omega_j$; moreover, $\Omega_i \subset \Omega_0$ and $\partial\Omega_i \cap \Gamma_1 = \emptyset$ for each $i \in \{1, \dots, n\}$.

Let $\widetilde{b}_0(x) = b_0(x)$ if $x \in \Omega_0$ and $\widetilde{b}_0(x) = 1$ if $x \in \Omega \setminus \Omega_0$. We denote by $\widetilde{H}^b(\Omega)$ the vector space of functions of the form $w = \widetilde{b}_0^{-1/2} v$, where $v \in L_2(\Omega)$, such that for every $i \in \{1, \dots, n\}$ the restriction of w to Ω_i admits a generalized derivative $w_{x_i} \in L_{2,\mathrm{loc}}(\Omega_i)$ and, moreover, $b_i^{1/2} w_{x_i} \in L_2(\Omega_i)$ (to simplify notation, we regard $b_i^{1/2} w_{x_i}$ as an element of

$L_2(\Omega)$). We introduce a seminorm on $\widetilde{H}^b(\Omega)$ by $|||w||| := \big(\|b_0^{1/2}w\|^2_{L_2(\Omega)} + \sum_{i=1}^n \|b_i^{1/2}w_{x_i}\|^2_{L_2(\Omega)}\big)^{1/2}$. It is easy to check that $\widetilde{H}^b(\Omega)$ is the completion of $\widetilde{W}^1_{p(\cdot)}(\Omega)$ with respect to the seminorm $|||\cdot|||$ (see [5]).

Let us introduce vector space $C([0,T];\widetilde{H}^b(\Omega))$ consisting of those functions $h : [0,T] \to \widetilde{H}^b(\Omega)$ for which $b_0^{1/2}h \in C([0,T];L_2(\Omega))$ and $b_i^{1/2}h_{x_i} \in C([0,T];L_2(\Omega))$ $(i=\overline{1,n})$. We endow this space with a seminorm

$$\|h\|_{C([0,T];\widetilde{H}^b(\Omega))} := \max_{t\in[0,T]} \|b_0^{1/2}(\cdot)h(\cdot,t)\|_{L_2(\Omega)} + \sum_{i=1}^n \max_{t\in[0,T]} \|b_i^{1/2}(\cdot)h_{x_i}(\cdot,t)\|_{L_2(\Omega)}.$$

Set by definition

$$\mathbb{V}_p := \widetilde{W}^1_{p(\cdot)}(\Omega), \quad \mathbb{U}^b_p := \widetilde{W}^{1,0}_{p(\cdot)}(Q) \cap C\big([0,T];\widetilde{H}^b(\Omega)\big).$$

Clearly, for every $w \in \mathbb{U}^b_p$ we have $w(\cdot,t) \in \mathbb{V}_p$ for a.e. $t \in [0,T]$.

Finally, denote by $\mathbb{F}_{p'}$ the space of vector-functions $(f_0, f_1, \dots, f_n)$ such that $f_i \in L_{p_i'(\cdot)}(Q)$, and $f_i = 0$ a.e. in some neighborhood of the surface Σ_1 $(i=\overline{1,n})$.

Now let us introduce classes of the data of the problem (2.1)–(2.3).

Define $\mathbb{A}_p(1{-}3)$ to be the set of functions $(a_0, a_1, \dots, a_n)$ that satisfy the following assumptions:

($\mathcal{A}$**1**) for every $j \in \{0,1,\dots,n\}$, the function

$$Q \times \mathbb{R}^{1+n} \ni (x,t,s,\xi) \mapsto a_j(x,t,s,\xi) \in \mathbb{R}$$

is Caratheodory, i.e., $a_j(x,t,\cdot,\cdot) : \mathbb{R}^{1+n} \to \mathbb{R}$ is a continuous function for a.e. $(x,t) \in Q$, and $a_j(\cdot,\cdot,s,\xi) : Q \to \mathbb{R}$ is a measurable function for every $(s,\xi) \in \mathbb{R}^{1+n}$;

($\mathcal{A}$**2**) for every $j \in \{0,1,\dots,n\}$, every $(s,\xi) \in \mathbb{R}^{1+n}$, and a.e. $(x,t) \in Q$, we have

$$|a_j(x,t,s,\xi)| \le C_1\big(|s|^{p_0(x)/p_j'(x)} + \sum_{l=1}^n |\xi_l|^{p_l(x)/p_j'(x)}\big) + h_j(x,t),$$

where $C_1 = \mathrm{const} > 0$, $h_j \in L_{p_j'(\cdot)}(Q)$;

($\mathcal{A}$**3**) for every (s_1,ξ^1), $(s_2,\xi^2)\in\mathbb{R}^{1+n}$ and a.e. $(x,t)\in Q$,

$$\sum_{i=1}^{n}\big(a_i(x,t,s_1,\xi^1)-a_i(x,t,s_2,\xi^2)\big)(\xi_i^1-\xi_i^2) \\ +\big(a_0(x,t,s_1,\xi^1)-a_0(x,t,s_2,\xi^2)\big)(s_1-s_2)\ge 0. \quad (2.4)$$

For simplicity of notations, we denote

$$\partial_0 v := v,\quad \partial_i v := v_{x_i}\ (i=\overline{1,n})$$

for any function v from Ω or Q to $\mathbb{R}$.

Definition 2.1. *Let p_j, b_j $(j=\overline{0,n})$ satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$), respectively, $u_0\in\widetilde{H}^b(\Omega)$, $(f_0,f_1,\dots,f_n)\in\mathbb{F}_{p'}$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1-3)$. The function $u\in\mathbb{U}_p^b$ is called a weak solution of problem* (2.1)–(2.3) *if u satisfies the initial condition* (*see* (2.3))

$$|||u(\cdot,0)-u_0(\cdot)|||=0, \quad (2.5)$$

and the integral equality

$$\iint_Q\Big\{\Big(\sum_{j=0}^{n}a_j(x,t,u,\nabla u)\,\partial_j v\Big)\varphi-\Big(\sum_{j=0}^{n}b_j\partial_j u\,\partial_j v\Big)\varphi'\Big\}\,dx\,dt \\ =\iint_Q\Big\{\sum_{j=0}^{n}f_j\,\partial_j v\Big\}\varphi\,dx\,dt \quad (2.6)$$

holds for every $v\in\mathbb{V}_p$ and $\varphi\in C_0^1(0,T):=\{\varphi\in C^1([0,T])\mid \operatorname{supp}\varphi\subset(0,T)\}$.

Denote by $\mathbb{A}_p(1–3,3^*)$ the set of functions $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1–3)$ satisfying an extra condition:

($\mathcal{A}$**3***) if $s_1\ne s_2$ then for a.e. $(x,t)\in(\Omega\setminus\Omega_0)\times(0,T)$ the sign "$\ge$" can be replaced by the sign "$>$" in the inequality (2.4).

Theorem 2.1. *If p_j, b_j $(j=\overline{0,n})$ satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$), respectively, and $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1–3,3^*)$, then the weak solution of problem* (2.1)–(2.3) *is unique.*

Denote by $\mathbb{A}_p(1-4)$ the set of functions $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p(1-3)$ satisfying a condition

($\mathcal{A}4$) for every $(s,\xi)\in\mathbb{R}^{1+n}$ and for a.e. $(x,t)\in Q$,

$$\sum_{j=0}^{n} a_i(x,t,s,\xi)\xi_i + a_0(x,t,s,\xi)s \geq K_1\Big(\sum_{i=1}^{n}|\xi_i|^{p_i(x)} + |s|^{p_0(x)}\Big) - g(x,t),$$

where $K_1 = \text{const} > 0$, $g\in L_1(Q)$.

Note that the function g above satisfies $g(x,t)\geq 0$ for a.e. $(x,t)\in Q$. This follows from the inequality in condition ($\mathcal{A}4$) when $\xi_1=\cdots=\xi_n=0$ and $s=0$.

Theorem 2.2. *If p_j, b_j $(j=\overline{0,n})$ satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$), respectively, $u_0\in\widetilde{H}^b(\Omega)$, $(f_0,f_1,\ldots,f_n)\in\mathbb{F}_{p'}$ and $(a_0,a_1,\ldots,a_n)\in\mathbb{A}_p(1{-}4)$, then problem* (2.1)–(2.3) *has a weak solution u. Moreover, any weak solution u of this problem satisfies the following estimate:*

$$\max_{t\in[0,T]}|\!|\!|u(\cdot,t)|\!|\!|^2 + \iint_Q \Big\{\sum_{j=0}^{n}\big|\partial_j u(x,t)\big|^{p_j(x)}\Big\}\,dx\,dt \leq C_2|\!|\!|u_0(\cdot)|\!|\!|^2 + C_3\iint_Q \Big\{\sum_{j=0}^{n}\big|f_j(x,t)\big|^{{p_j}'(x)} + g(x,t)\Big\}\,dx\,dt, \quad (2.7)$$

where C_2, C_3 are positive constants depending only on K_1 and p_j^- $(i=\overline{0,n})$.

Finally, let $\mathbb{A}_p(1{-}3,3^*,4)$ be the set of functions $(a_0,a_1,\ldots,a_n)\in\mathbb{A}_p(1{-}3)$ satisfying both conditions ($\mathcal{A}3^*$) and ($\mathcal{A}4$).

Corollary 2.1. *If p_j, b_j $(j=\overline{0,n})$ satisfy conditions ($\mathcal{P}$), ($\mathcal{B}$), respectively, $u_0\in\widetilde{H}^b(\Omega)$, $(f_0,f_1,\ldots,f_n)\in\mathbb{F}_{p'}$ and $(a_0,a_1,\ldots,a_n)\in\mathbb{A}_p(1{-}3,3^*,4)$, then problem* (2.1)–(2.3) *has a unique weak solution, and one satisfies* (2.7)*.*

3. Auxiliary statements

In this section, we prove some technical statements, that will be important for the proof of the main results.

Let $\omega_1\in C_0^\infty(\mathbb{R})$ be a standard mollifier (see [26, p. 629]), i.e., $\operatorname{supp}\omega_1\subset[-1,1]$, $\omega_1(z)\geq 0$, $\omega_1(-z)=\omega_1(z)$ if $z\in\mathbb{R}$, $\int_{\mathbb{R}}\omega_1(z)\,dz=1$. Consider a family of functions $\{\omega_\rho:\mathbb{R}\to\mathbb{R}\,|\,\rho>0\}$ defined by $\omega_\rho(z):=(1/\rho)\,\omega_1(z/\rho)$ for all $z\in\mathbb{R}$ and $\rho>0$.

For every $\rho > 0$ we define the mollification of any $\psi \in L_1(Q)$ by the rule

$$\psi_\rho(x,t) := \int\limits_{\mathbb{R}} \psi^*(x,\tau)\omega_\rho(\tau - t)\,d\tau \quad \text{for a.e.} \quad (x,t) \in Q,$$

where $\psi^*(x,t) := \psi(x,t)$ if $x \in \Omega,\ t \in (0,T)$, and $\psi^*(x,t) := 0$ if $x \in \Omega,\ t \notin (0,T)$.

The following statement is well known for standard Lebesgue spaces (see [26]). For the generalized Lebesgue spaces it was proved in [23] (see also the proof of Lemma 1 in [22]).

Lemma 3.1. *If $r \in L_\infty(\Omega)$, $r(x) \geq 1$ for a.e. $x \in \Omega$, then for every function $f \in L_{r(\cdot)}(Q)$ we have*

$$f_\rho \underset{\rho \to 0}{\longrightarrow} f \quad \textit{strongly in} \quad L_{r(\cdot)}(Q).$$

Lemma 3.2. *Suppose that b_j $(j = \overline{0,n})$ satisfy condition ($\mathcal{B}$), and functions $w \in \widetilde{W}^{1,0}_{p(\cdot)}(Q)$ and $g_j \in L_{p_j'(\cdot)}(Q)$ $(j = \overline{0,n})$ satisfy an identity*

$$\iint\limits_Q \Big\{ \Big(\sum_{j=0}^n g_j \partial_j v \Big)\varphi - \Big(\sum_{j=0}^n b_j \partial_j w \partial_j v \Big)\varphi' \Big\}\, dx\, dt = 0,$$

$$v \in \mathbb{V}_p^b,\ \varphi \in C_0^1(0,T). \quad (3.1)$$

Then $w \in C\big([0,T]; \widetilde{H}^b(\Omega)\big)$ and for every $\theta \in C^1([0,T])$, $v \in \mathbb{V}_p$, and $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\theta(t_2) \int\limits_\Omega \Big\{ \sum_{j=0}^n b_j(x)\partial_j w(x,t_2)\partial_j v(x) \Big\} dx - \theta(t_1) \int\limits_\Omega \Big\{ \sum_{j=0}^n b_j(x)\partial_j w(x,t_1)\partial_j v(x) \Big\} dx + \int\limits_{t_1}^{t_2} \int\limits_\Omega \Big\{ \Big(\sum_{j=0}^n g_j \partial_j v \Big)\theta - \Big(\sum_{j=0}^n b_j \partial_j w \partial_j v \Big)\theta' \Big\}\, dx\, dt = 0, \quad (3.2)$$

$$\frac{1}{2}\theta(t_2)\|w(\cdot,t_2)\|^2 - \frac{1}{2}\theta(t_1)\|w(\cdot,t_1)\|^2 - \frac{1}{2}\int\limits_{t_1}^{t_2} \|w(\cdot,t)\|^2\theta'(t)\,dt + \int\limits_{t_1}^{t_2}\int\limits_\Omega \Big(\sum_{j=0}^n g_j \partial_j w \Big)\theta\, dx\, dt = 0. \quad (3.3)$$

Proof. Let us construct functions $\widehat{w}, \widehat{g}_j : \Omega \times (-T, 2T)$ $(j = \overline{0,n})$ by

$$\widehat{w}(x,t) := \begin{cases} w(x,-t) & \text{if } -T < t < 0, \\ w(x,t) & \text{if } 0 \le t \le T, \\ w(x, 2T - t) & \text{if } T < t < 2T, \end{cases}$$

$$\widehat{g}_j(x,t) := \begin{cases} -g_j(x,-t) & \text{if } -T < t < 0, \\ g_j(x,t) & \text{if } 0 \le t \le T, \\ -g_j(x, 2T - t) & \text{if } T < t < 2T. \end{cases}$$

It is easy to check that the equality

$$\int_{-T}^{2T}\int_{\Omega} \Big\{ \Big(\sum_{j=0}^{n} \widehat{g}_j\, \partial_j v \Big)\varphi - \Big(\sum_{j=0}^{n} b_j \partial_j \widehat{w} \partial_j v \Big) \varphi' \Big\}\, dx\, dt = 0 \tag{3.4}$$

holds for every $v \in \mathbb{V}_p$, $\varphi \in C_0^1(-T, 2T)$.

Now let $\{\omega_\rho \mid \rho > 0\}$ be the functions introduced earlier in this section. Choose a number $k_0 \in \mathbb{N}$ such that $1/k_0 < T/2$. By definition, for each $k \ge k_0$ we set

$$\widehat{w}_k(x,\tau) := \int_{\mathbb{R}} \widehat{w}(x,t)\, \omega_{1/k}(t - \tau)\, dt,$$

$$\widehat{g}_{j,k}(x,\tau) := \int_{\mathbb{R}} \widehat{g}_j(x,t)\, \omega_{1/k}(t - \tau)\, dt, \quad j = \overline{0,n},$$

for every $\tau \in [-T/2, T]$ and for a.e. $x \in \Omega$.

According to Lemma 3.1, we have

$$\partial_j \widehat{w}_k \underset{k\to\infty}{\longrightarrow} \widehat{\partial_j w} \quad \text{in} \quad L_{p_j(\cdot)}\big(\Omega \times (-T/2, T)\big), \quad j = \overline{0,n}, \tag{3.5}$$

$$b_j^{1/2} \partial_j \widehat{w}_k \underset{k\to\infty}{\longrightarrow} b_j^{1/2} \partial_j \widehat{w} \quad \text{in} \quad L_2\big(\Omega \times (-T/2, T)\big), \quad j = \overline{0,n}, \tag{3.6}$$

$$\widehat{g}_{j,k} \underset{k\to\infty}{\longrightarrow} \widehat{g}_j \quad \text{in} \quad L_{p_j{}'(\cdot)}\big(\Omega \times (-T/2, T)\big), \quad j = \overline{0,n}. \tag{3.7}$$

Note that $b_j^{1/2} \partial_j \widehat{w}_k \in C([-T/2, T]; L_2(\Omega))$ $(k \ge k_0,\ j = \overline{0,n})$.

For each $\tau \in [T/2, T]$ and $k \ge k_0$ we substitute $\omega_{1/k}(\cdot - \tau)$ for $\varphi(\cdot)$ in (3.4), which yields

$$\int_{\Omega} \Big\{ \sum_{j=0}^{n} b_j(x) \frac{\partial}{\partial \tau} \partial_j \widehat{w}_k(x,\tau)\, \partial_j v(x) + \sum_{j=0}^{n} \widehat{g}_{j,k}(x,\tau)\, \partial_j v(x) \Big\} dx = 0. \tag{3.8}$$

Let $k, l \in \mathbb{N}$ be arbitrary numbers such that $k, l \geq k_0$. Set $\widehat{w}_{kl} := \widehat{w}_k - \widehat{w}_l$, $\widehat{g}_{j,kl} := \widehat{g}_{j,k} - \widehat{g}_{j,l}$ $(j = \overline{0,n})$. Then it follows from (3.8) that

$$\int\limits_{\Omega} \Big\{ \sum_{j=0}^{n} b_j(x) \frac{\partial}{\partial \tau} \partial_j \widehat{w}_{kl}(x,\tau)\, \partial_j v(x) + \sum_{j=0}^{n} \widehat{g}_{j,kl}(x,\tau)\, \partial_j v(x) \Big\} dx = 0, \quad (3.9)$$

where $v \in \mathbb{V}_p$, $\tau \in [-T/2,\, T]$.

Take a function $\theta \in C^1(\mathbb{R})$. For every $\tau \in [-T/2,\, T]$, the functions $\widehat{w}_{kl}(\cdot,\tau)\,\theta(\tau)$ belong to $\mathbb{V}_p$. Substituting $\widehat{w}_{kl}(\cdot,\tau)\,\theta(\tau)$ for of $v(\cdot)$ in (3.9) and integrating the obtained equality for τ from t_1 to t_2 $(-T/2 \leq t_1 < t_2 \leq T)$, we get

$$\frac{1}{2} \int\limits_{\Omega} \Big(\sum_{j=0}^{n} b_j(x) |\partial_j \widehat{w}_{kl}(x,\tau)|^2 \Big) \theta(\tau) \Big|_{\tau=t_1}^{\tau=t_2} dx - \frac{1}{2} \int\limits_{t_1}^{t_2} \int\limits_{\Omega} \Big(\sum_{j=0}^{n} b_j(x) |\partial_j \widehat{w}_{kl}(x,\tau)|^2 \Big) \theta'(\tau)\, dx\, d\tau + \int\limits_{t_1}^{t_2} \int\limits_{\Omega} \Big\{ \sum_{j=0}^{n} \widehat{g}_{j,kl}(x,\tau) \partial_j \widehat{w}_{kl}(x,\tau) \Big\} \theta(\tau)\, dx\, d\tau = 0. \quad (3.10)$$

Now suppose

$$0 \leq \theta(\tau) \leq 1 \quad \text{if} \quad \tau \in \mathbb{R}, \qquad \theta(\tau) = 0 \quad \text{if} \quad \tau \leq -T/2,$$
$$\theta(\tau) = 1 \quad \text{if} \quad \tau \geq 0, \qquad |\theta'(\tau)| \leq 4/T \quad \text{if} \quad \tau \in [-T/2,\, 0].$$

Taking $t_1 = -T/2$ and $t_2 = t \in [0,T]$ in (3.10), we obtain

$$\|\widehat{w}_{kl}\|_{C([0,T];\widetilde{H}^b(\Omega))} \equiv \max_{t \in [0,T]} \|\widehat{w}_{kl}(\cdot,t)\|^2 \leq \frac{4}{T} \int\limits_{-T/2}^{0} \|\widehat{w}_{kl}(\cdot,\tau)\|^2\, d\tau + 2 \int\limits_{-T/2}^{T} \int\limits_{\Omega} \Big\{ \sum_{j=0}^{n} |\widehat{g}_{j,kl}(x,\tau)|\, |\partial_j \widehat{w}_{kl}(x,\tau)| \Big\} dx\, d\tau. \quad (3.11)$$

In view of (3.5)–(3.7), it follows from (3.11) that

$$b_j^{1/2}\, \partial_j \widehat{w}_{kl} \underset{k,\, l \to +\infty}{\longrightarrow} 0 \quad \text{in} \quad C([0,T]; L_2(\Omega)), \quad j = \overline{0,n}.$$

Therefore $\{b_j^{1/2} \partial_j \widehat{w}_k\}_{k=1}^{\infty}$ $(j = \overline{0,n})$ are Cauchy sequences in the space $C([0,T];\, L_2(\Omega))$ and hence

$$b_j^{1/2} \partial_j \widehat{w}_k \underset{k \to +\infty}{\longrightarrow} b_j^{1/2} \partial_j \widehat{w} \quad \text{in} \quad C([0,T]; L_2(\Omega)), \quad j = \overline{0,n}. \quad (3.12)$$

Hence $b_j^{1/2}\partial_j w \in C([0,T];L_2(\Omega))$ $(j=\overline{0,n})$ and $w \in C([0,T];\widetilde{H}^b(\Omega))$.

Take an arbitrary function $\theta \in C^1([0,T])$ and any points $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$. For each $\tau \in [0,T]$, we multiply both sides of (3.8) by $\theta(\tau)$ and integrate for τ over $[t_1,t_2]$:

$$\int_{t_1}^{t_2}\int_{\Omega}\Big(\sum_{j=0}^{n} b_j(x)\Big[\frac{\partial}{\partial\tau}\,\partial_j\widehat{w}_k(x,\tau)\Big]\,\partial_j v(x)\Big)\theta(\tau)\,dxd\tau + \int_{t_1}^{t_2}\int_{\Omega}\Big\{\sum_{j=0}^{n}\widehat{g}_{j,k}(x,\tau)\partial_j v(x)\theta(\tau)\Big\}\,dx\,d\tau = 0. \quad (3.13)$$

Then we integrate by parts the first term in the left-hand side of equality (3.13), and let $k \to +\infty$. In view of (3.7), (3.12), we get (3.2).

Finally, for each $\tau \in [0,T]$ and $k \geq k_0$ we substitute $\widehat{w}_k(\cdot,\tau)\theta(\tau)$ for $v(\cdot)$ in (3.8), then integrate for τ over $[t_1,t_2]$. Similarly to (3.10), we get

$$\frac{1}{2}\int_{\Omega}\Big(\sum_{j=0}^{n} b_j(x)|\partial_j\widehat{w}_k(x,\tau)|^2\Big)\theta(\tau)\Big|_{\tau=t_1}^{\tau=t_2}dx - \frac{1}{2}\int_{t_1}^{t_2}\int_{\Omega}\Big(\sum_{j=0}^{n} b_j(x)|\partial_j\widehat{w}_k(x,\tau)|^2\Big)\theta'(\tau)\,dx\,d\tau + \int_{t_1}^{t_2}\int_{\Omega}\Big\{\sum_{j=0}^{n}\widehat{g}_{j,k}(x,\tau)\partial_j\widehat{w}_k(x,\tau)\Big\}\theta(\tau)\,dx\,d\tau = 0. \quad (3.14)$$

Letting $k \to +\infty$ in (3.14) and using (3.5), (3.7), (3.12), we get (3.3). □

4. Proof of the main results

For an arbitrary function $w \in L_1(Q)$ such that $w_{x_1},\dots,w_{x_n} \in L_1(Q)$, we denote

$$a_j(w)(x,t) := a_j(x,t,w(x,t),\nabla w(x,t)), \quad (x,t)\in Q,\ j=\overline{0,n}.$$

Proof of Theorem 2.1. We assume the contrary. Let u_1, u_2 be two weak solutions of the problem (2.1)–(2.3). Let us subtract equality (2.6) with $u = u_2$ from the same equality with $u = u_1$. Using Lemma 3.2 with

$w = u_1 - u_2$, $\theta \equiv 1$, $t_1 = 0$, $t_2 = \tau \in (0, T]$, we get (see (3.3))

$$\frac{1}{2}\int\limits_{\Omega}\Big\{\sum_{j=0}^{n} b_j(x)|\partial_j u_1(x,\tau) - \partial_j u_2(x,\tau)|^2\Big\}\,dx + \int\limits_0^{\tau}\int\limits_{\Omega}\Big\{\sum_{j=0}^{n}(a_j(u_1) - a_j(u_2))(\partial_j u_1 - \partial_j u_2)\Big\}\,dx\,dt = 0, \quad \tau \in (0, T]. \tag{4.1}$$

This equality and $(\mathcal{A}3)$ yield $\sum_{j=0}^{n} b_j|\partial_j u_1 - \partial_j u_2|^2 = 0$ a.e. on Q, and $\sum_{j=0}^{n}(a_j(u_1) - a_j(u_2))(\partial_j u_1 - \partial_j u_2) = 0$ a.e. on Q. The first equality implies that $w(x,t) = 0$ for a.e. $(x,t) \in \Omega_0 \times (0,T)$. The second equality and condition $(\mathcal{A}3^*)$ imply that $w(x,t) = 0$ for a.e. $(x,t) \in (\Omega\backslash\Omega_0) \times (0,T)$. Therefore $w(x,t) = 0$ for a.e. $(x,t) \in Q$, that is, $u_1 = u_2$. We have arrived at a contradiction, which proves the theorem. □

Proof of Theorem 2.2. We use Galerkin's method. Let $\{w_j \mid j \in \mathbb{N}\}$ be a set of linearly independent functions from $\mathbb{V}_p$ that is complete in $\mathbb{V}_p$ and in $\widetilde{H}^b(\Omega)$. For each $m \in \mathbb{N}$, let $U_m := \big\{d_1 w_1 + \cdots + d_m w_m \mid d_1, \ldots, d_m \in \mathbb{R}\big\}$ be the span of $\{w_1, \ldots, w_m\}$. Obviously, the closure of $\bigcup_{m\in\mathbb{N}} U_m$ by the norm $W^1_{p(\cdot)}(\Omega)$ coincides with $\mathbb{V}_p$ while the closure of $\bigcup_{m\in\mathbb{N}} U_m$ by the seminorm $|||\cdot|||$ coincides with $\widetilde{H}^b(\Omega)$.

We take a sequence $\big\{u_{0,m}\big\}_{m=1}^{\infty}$ such that $u_{0,m} \in U_m$ for all $m \in \mathbb{N}$ and

$$|||u_0 - u_{0,m}||| \underset{m\to+\infty}{\longrightarrow} 0. \tag{4.2}$$

Notice that for every $\eta \in (0,1]$ and for a.e. $x \in \Omega$, we have

$$\Big|b_j^{1/2}(x) - \Big(b_j(x) + \eta\Big)^{1/2}\Big|^2\Big|\partial_j u_{0,m}(x)\Big|^2 \le 4(b_j(x) + 1)\Big|\partial_j u_{0,m}(x)\Big|^2, \quad j = \overline{0,n}.$$

By the Dominated Convergence Theorem, for every $m \in \mathbb{N}$ we get

$$\Big\|b_j^{1/2}\partial_j u_{0,m} - \Big(b_j + \eta\Big)^{1/2}\partial_j u_{0,m}\Big\|_{L_2(\Omega)} \underset{\eta\to0+}{\longrightarrow} 0, \quad j = \overline{0,n}.$$

Therefore there exist sequences of positive numbers $\{\eta_{j,m}\}_{m=1}^{\infty}$ $(j = \overline{0,n})$ such that $\eta_{j,m} \underset{m\to+\infty}{\longrightarrow} 0$ $(j = \overline{0,n})$ and

$$\Big\|b_j^{1/2}\partial_j u_{0,m} - \Big(b_j + \eta_{j,m}\Big)^{1/2}\partial_j u_{0,m}\Big\|_{L_2(\Omega)} \underset{m\to+\infty}{\longrightarrow} 0, \quad j = \overline{0,n}. \tag{4.3}$$

Set by definition

$$b_{j,m}(x) := b_j(x) + \eta_{j,m}, \quad x \in \Omega,\ j = \overline{0,n},\ m \in \mathbb{N}. \tag{4.4}$$

Therefore, by (4.2)–(4.4), we have

$$\left\| b_j^{1/2} \partial_j u_0 - b_{j,m}^{1/2} \partial_j u_{0,m} \right\|_{L_2(\Omega)} \underset{m\to+\infty}{\longrightarrow} 0, \quad j = \overline{0,n}. \tag{4.5}$$

According to Galerkin's method, for every $m \in \mathbb{N}$ we set

$$u_m(x,t) := \sum_{k=1}^{m} c_{m,k}(t) w_k(x), \quad (x,t) \in \overline{Q}, \tag{4.6}$$

where $(c_{m,1},\dots,c_{m,m})$ are solutions of the Cauchy problem for the system of ordinary differential equations

$$\int_\Omega \left\{ \sum_{j=0}^{n} b_{j,m} \Big[\frac{\partial}{\partial t} \partial_j u_m \Big] \partial_j w_l \right\} dx + \int_\Omega \left\{ \sum_{j=0}^{n} (a_j(u_m) - f_j) \partial_j w_l \right\} dx = 0,$$
$$t \in [0,T],\ l = \overline{1,m}, \tag{4.7}$$

$$u_m \Big|_{t=0} = u_{0,m}. \tag{4.8}$$

The system (4.7) can be transformed into the normal form. Hence, according to the theorems of existence, uniqueness and extension of the solution to this problem (see [25]), there exists a unique global solution $(c_{1,m},\dots,c_{m,m})$ of problem (4.7), (4.8). This solution is defined on an interval $[0,T_m\rangle$, where $T_m \le T$ and $\rangle$ means either) or]. Further we will get estimates that imply the equality $[0,T_m\rangle = [0,T]$.

For each $l \in \{1,\dots,m\}$, we multiply equality with number l of (4.7) by $c_{m,l}$, then sum up over l. Next we integrate for t over an interval $[0,\tau] \subset [0,T_m\rangle$. Integrating by parts and using (4.6), (4.8), we obtain

$$\frac{1}{2} \int_\Omega \left\{ \sum_{j=0}^{n} b_{j,m}(x) |\partial_j u_m(x,\tau)|^2 \right\} dx - \frac{1}{2} \int_\Omega \left\{ \sum_{j=0}^{n} b_{j,m}(x) |\partial_j u_{0,m}(x)|^2 \right\} dx$$
$$+ \int_0^\tau \int_\Omega \left\{ \sum_{j=0}^{n} a_j(u_m) \partial_j u_m \right\} dx\, dt = \int_0^\tau \int_\Omega \left\{ \sum_{j=0}^{n} f_j \partial_j u_m \right\} dx\, dt. \tag{4.9}$$

Now we need the following form of Young's inequality:

$$ab \le \varepsilon |a|^{r(x)} + \varepsilon^{-\frac{1}{r^- - 1}} |b|^{r'(x)}, \quad a,b \in \mathbb{R}, \quad \varepsilon \in (0,1), \tag{4.10}$$

for a.e. $x \in \Omega$, where $r \in L_\infty(\Omega)$, $r^- := \operatorname{ess\,inf}_{x\in\Omega} r(x) > 1$, $r'(x) := r(x)/(r(x)-1)$ for a.e. $x \in \Omega$.

Using condition $(\mathcal{A}4)$ and inequality (4.10) with small enough $\varepsilon \in (0,1)$, for example, $\varepsilon = \frac{1}{2}\min\{1, K_1\} > 0$, we derive from (4.9) that

$$\int\limits_\Omega \Big\{ \sum_{j=0}^n b_{j,m}(x)\partial_j |u_m(x,\tau)|^2 \Big\}\, dx + K_1 \int\limits_0^\tau \int\limits_\Omega \Big\{ \sum_{j=0}^n |\partial_j u_m(x,t)|^{p_j(x)} \Big\}\, dx\, dt$$
$$\le C_5 \int\limits_0^\tau \int\limits_\Omega \sum_{j=0}^n |f_j(x,t)|^{p_j{}'(x)}\, dx\, dt + 2\int\limits_0^\tau \int\limits_\Omega g(x,t)\, dx\, dt + \int\limits_\Omega \Big\{ \sum_{j=0}^n b_{j,m}(x)|\partial_j u_{0,m}(x)|^2 \Big\}\, dx, \quad \tau \in (0, T_m\rangle. \tag{4.11}$$

It follows from (4.5) that the sequences $\big\{ \int_\Omega \sum_{j=0}^n b_{j,m}(x)|\partial_j u_{0,m}(x)|^2 \times dx \big\}_{m=1}^{+\infty}$ are bounded. Hence (4.11) implies the estimates

$$\int\limits_\Omega \Big\{ \sum_{j=0}^n b_{j,m}(x)|\partial_j u_m(x,\tau)|^2 \Big\}\, dx \le C_8, \tag{4.12}$$

$$\int\limits_0^\tau \int\limits_\Omega \Big\{ \sum_{j=0}^n |\partial_j u_m(x,t)|^{p_j(x)} \Big\}\, dx\, dt \le C_9, \tag{4.13}$$

where $\tau \in (0, T_m\rangle$ is arbitrary and C_8, C_9 are positive constants independent of τ and m.

Note that (4.12) implies that $[0, T_m\rangle = [0, T]$. Therefore the estimates (4.12), (4.13) hold for each $\tau \in [0, T]$.

Condition $(\mathcal{A}2)$ and estimates (4.13) yield

$$\iint\limits_Q |a_j(u_m)(x,t)|^{p_j{}'(x)}\, dx\, dt \le C_{10}, \quad j = \overline{0,n}, \tag{4.14}$$

where $C_{10} > 0$ is independent of m.

Since the spaces $L_{p_j(\cdot)}(Q)$, $L_{p_j{}'(\cdot)}(Q)$ $(j = \overline{0,n})$ are reflexive (see [17, p. 600]), it follows from (4.12), (4.13) and (4.14) that there exists a subsequence of the sequence $\{u_m\}$ (which will be denoted by $\{u_m\}_{m\in\mathbb{N}}$ for simplicity), functions $u \in \widetilde{W}^{1,0}_{p(\cdot)}(Q)$, $\widetilde{u}_j \in L_\infty(0,T; L_2(\Omega))$ and $\chi_j \in$

$L_{p_j{}'(\cdot)}(Q)$ $(j=\overline{0,n})$ such that

$$b_{j,m}^{1/2}\partial_j u_m \underset{m\to\infty}{\longrightarrow} \widetilde{u}_j \quad *\text{-weakly in} \quad L_\infty(0,T;L_2(\Omega)), \quad j=\overline{0,n}, \tag{4.15}$$

$$u_m \underset{m\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad \widetilde{W}^{1,0}_{p(\cdot)}(Q), \tag{4.16}$$

$$a_j(u_m) \underset{m\to\infty}{\longrightarrow} \chi_j \quad \text{weakly in} \quad L_{p_j{}'(\cdot)}(Q), \quad j=\overline{0,n}. \tag{4.17}$$

Let us prove that u is a weak solution of problem (2.1)–(2.3). First note that

$$b_{j,m}^{1/2} \underset{m\to\infty}{\longrightarrow} b_j^{1/2} \quad \text{strongly in} \quad L_2(\Omega) \quad \text{and a.e. on} \quad \Omega \quad (j=\overline{0,n}). \tag{4.18}$$

Now let us show that

$$\tilde{u}_j = b_j^{1/2}\partial_j u \quad (j=\overline{0,n}) \quad \text{a.e. on} \quad Q. \tag{4.19}$$

Indeed, take an arbitrary function $\psi\in C(\overline{Q})$. Then (4.15) yield that

$$\iint\limits_Q b_{j,m}^{1/2}\partial_j u_m \psi\,dx\,dt \underset{m\to+\infty}{\longrightarrow} \iint\limits_Q \tilde{u}_j\psi\,dx\,dt, \quad j=\overline{0,n}. \tag{4.20}$$

Using the Dominated Convergence Theorem and (4.18), it is easy to show that $b_{j,m}^{1/2}\psi \underset{m\to+\infty}{\longrightarrow} b_j^{1/2}\psi$ in $L_{p_j{}'(\cdot)}(Q)\,(j=\overline{0,n})$. By (4.16), we obtain

$$\iint\limits_Q \partial_j u_m b_{j,m}^{1/2}\psi\,dxdt \underset{m\to+\infty}{\longrightarrow} \iint\limits_Q \partial_j u b_j^{1/2}\psi\,dx\,dt, \quad j=\overline{0,n}. \tag{4.21}$$

Relations (4.20), (4.21) imply that for every $\psi\in C(\overline{Q})$ the equalities

$$\iint\limits_Q \tilde{u}_j\psi\,dxdt = \iint\limits_Q b_j^{1/2}\partial_j u\psi\,dxdt, \quad j=\overline{0,n},$$

hold, which implies equalities (4.19).

Fix number $l\in\mathbb{N}$ and $m\in\mathbb{N}$ such that $m\ge l$. We multiply equality with number l of (4.7) by a function $\theta\in C^1([0,T])$ such that $\theta(T)=0$, then integrate for t over $[0,T]$. Next we integrate by parts and let $m\to\infty$. In view of (4.5), (4.8), (4.15)–(4.19), we get

$$-\theta(0)\int\limits_Q\left\{\sum_{j=0}^n b_j(x)\partial_j u_0(x)\partial_j w_l(x)\right\}dx - \iint\limits_Q\left(\sum_{j=0}^n b_j\partial_j u\partial_j w_l\right)\theta'\,dx\,dt$$
$$+\iint\limits_Q\left(\sum_{j=0}^n(\chi_j-f_j)\partial_j w_l\right)\theta\,dx\,dt=0. \tag{4.22}$$

The latter equality implies that for every $v \in \mathbb{V}_p$ and $\theta \in C^1([0,T])$, $\theta(T)=0$,

$$-\theta(0)\int\limits_Q \Big\{\sum_{j=0}^n b_j(x)\partial_j u_0(x)\partial_j v(x)\Big\}\,dx - \iint\limits_Q \Big(\sum_{j=0}^n b_j\partial_j u\partial_j v\Big)\theta'\,dx\,dt + \iint\limits_Q \Big(\sum_{j=0}^n (\chi_j - f_j)\partial_j v\Big)\theta\,dx\,dt = 0. \quad (4.23)$$

Notice that if we take $\theta=\varphi\in C_0^1(0,T)$ in (4.23), then

$$\iint\limits_Q \Big\{\sum_{j=0}^n \big((\chi_j-f_j)\partial_j v\big)\varphi - \Big(\sum_{j=0}^n b_j\partial_j u\partial_j v\Big)\varphi'\Big\}\,dx\,dt = 0 \quad (4.24)$$

for every $v\in\mathbb{V}_p$ and $\varphi\in C_0^1(0,T)$. According to Lemma 3.2, (4.24) implies that

$$u\in C\big([0,T];\widetilde{H}^b(\Omega)\big) \quad (4.25)$$

and for every $v\in\mathbb{V}_p$ and $\theta\in C^1([0,T])$, $\theta(T)=0$, the equality

$$-\theta(0)\int\limits_\Omega \Big\{\sum_{j=0}^n b_j(x)\partial_j u(x,0)\partial_j v(x)\Big\}\,dx - \iint\limits_Q \Big(\sum_{j=0}^n b_j\partial_j u\partial_j v\Big)\theta'\,dx\,dt + \iint\limits_Q \Big(\sum_{j=0}^n (\chi_j - f_j)\partial_j v\Big)\theta\,dx\,dt = 0 \quad (4.26)$$

holds.

Now (4.23) and (4.26) imply (2.5). In view of (4.16) and (4.25), we conclude that $u\in\mathbb{U}_p^b$.

According to (4.24), to prove (2.6) it is enough to show that the equality

$$\int\limits_\Omega \Big\{\sum_{j=0}^n \chi_j\partial_j v\Big\}\,dx = \int\limits_\Omega \Big\{\sum_{j=0}^n a_j(u)\partial_j v\Big\}\,dx \quad (4.27)$$

is valid for every $v\in\mathbb{V}_p$ and for a.e. $t\in(0,T)$. To this end, we use the monotonicity method (see [27]). Take an arbitrary function $w\in W^{1,0}_{p(\cdot)}(Q)$. Using condition $(\mathcal{A}3)$ for every $m\in\mathbb{N}$, we obtain

$$W_m := \iint\limits_Q \Big\{\sum_{j=0}^n (a_j(u_m)-a_j(w))(\partial_j u_m - \partial_j w)\Big\}\theta\,dx\,dt \ge 0,$$

where $\theta(t) = 1 - t/T,\ t \in \mathbb{R}$.

Hence

$$W_m = \iint\limits_Q \Big\{ \sum_{j=0}^n a_j(u_m)\partial_j u_m \Big\} \theta \, dx \, dt - \iint\limits_Q \Big\{ \sum_{j=0}^n \big[a_j(u_m)\partial_j w + a_j(w)(\partial_j u_m - \partial_j w) \big] \Big\} \theta \, dx \, dt \ge 0, \quad m \in \mathbb{N}. \tag{4.28}$$

For each $l \in \{1, \ldots, m\}$, we multiply equality with number l of (4.7) by $c_{m,l}\theta$ and then sum up over l. Next we integrate for t over $[0, T]$, then integrate by parts and use (4.6) and (4.8). We obtain

$$\iint\limits_Q \Big\{ \sum_{j=0}^n a_j(u_m)\partial_j u_m \Big\} \theta \, dx \, dt = \iint\limits_Q \Big\{ \sum_{j=0}^n f_j \partial_j u_m \Big\} \theta \, dx \, dt - \frac{1}{2T} \iint\limits_Q \Big(\sum_{j=0}^n b_{j,m} |\partial_j u_m|^2 \Big) dx \, dt + \frac{1}{2} \int\limits_\Omega \Big\{ \sum_{j=0}^n b_{j,m} |\partial_j u_{0,m}|^2 \Big\} dx, \quad m \in \mathbb{N}. \tag{4.29}$$

By (4.28) and (4.29), we get

$$W_m = \iint\limits_Q \Big\{ \sum_{j=0}^n f_j \partial_j u_m \Big\} \theta \, dx \, dt - \frac{1}{2T} \iint\limits_Q \Big\{ \sum_{j=0}^n b_{j,m} |\partial_j u_m|^2 \Big\} dx \, dt + \frac{1}{2} \int\limits_\Omega \Big\{ \sum_{j=0}^n b_{j,m} |\partial_j u_{0,m}|^2 \Big\} dx - \iint\limits_Q \Big\{ \sum_{j=0}^n \big[a_j(u_m)\partial_j w + a_j(w)(\partial_j u_m - \partial_j w) \big] \Big\} \theta \, dx \, dt \ge 0, \quad m \in \mathbb{N}. \tag{4.30}$$

The relations (4.15), (4.19) imply

$$\lim_{m\to\infty} \inf \iint\limits_Q \Big\{ \sum_{j=0}^n b_{j,m} |\partial_j u_m|^2 \Big\} dx \, dt \ \ge \ \iint\limits_Q \Big\{ \sum_{j=0}^n b_j |\partial_j u|^2 \Big\} dx \, dt. \tag{4.31}$$

Using (4.5), (4.16), (4.17) and (4.31), we derive from (4.30) that

$$0 \leq \lim_{m\to\infty} \sup W_m \leq \iint\limits_Q \Big\{ \sum_{i=1}^n f_j \partial_j u \Big\} \theta \, dx \, dt$$
$$- \frac{1}{2T} \iint\limits_Q \Big\{ \sum_{j=0}^n b_j |\partial_j u|^2 \Big\} \, dx \, dt + \frac{1}{2} \int\limits_\Omega \Big\{ \sum_{j=0}^n b_j |\partial_j u_0|^2 \Big\} \, dx$$
$$- \iint\limits_Q \Big\{ \sum_{j=0}^n \big[\chi_j \partial_j w + a_j(w)(\partial_j u - \partial_j w) \big] \Big\} \, \theta \, dx \, dt. \quad (4.32)$$

Using Lemma 3.2 and (2.5), we derive from (4.24) that

$$\iint\limits_Q \Big\{ \sum_{j=0}^n \chi_j \partial_j u \Big\} \theta \, dx \, dt = \iint\limits_Q \Big\{ \sum_{j=0}^n f_j \partial_j u \Big\} \theta \, dx \, dt$$
$$- \frac{1}{2T} \iint\limits_Q \Big\{ \sum_{j=0}^n b_j |\partial_j u|^2 \Big\} \, dx \, dt + \frac{1}{2} \int\limits_\Omega \Big\{ \sum_{j=0}^n b_j |\partial_j u_0|^2 \Big\} \, dx. \quad (4.33)$$

Now (4.32) and (4.33) imply that

$$\iint\limits_Q \Big\{ \sum_{j=0}^n (\chi_j - a_j(w))(\partial_j u - \partial_j w) \Big\} \, \theta \, dx \, dt \geq 0. \quad (4.34)$$

In the case $w = u - \lambda v \varphi$, where $v \in \mathbb{V}_p$, $\varphi \in C_0^1(0,T)$ and $\lambda > 0$, inequality (4.34) implies that

$$\iint\limits_Q \Big\{ \sum_{j=0}^n (\chi_j - a_j(u - \lambda v \varphi)) \partial_j v \Big\} \theta \varphi \, dx \, dt \geq 0. \quad (4.35)$$

Letting $\lambda \to 0+$ in (4.35), using conditions $(\mathcal{A}1)$, $(\mathcal{A}2)$, and the Dominated Convergence Theorem (see [26, p. 648]), we obtain

$$\iint_Q \Big\{ \sum_{j=0}^n (\chi_j - a_j(u)) \partial_j v \Big\} \theta \varphi \, dx \, dt = 0$$

for all $v \in \mathbb{V}_p$ and $\varphi \in C_0^1(0,T)$. Therefore, (4.27) holds.

From (4.24), taking into account (4.27), we get (2.6). It follows that u is a weak solution of problem (2.1)–(2.3).

Finally, let us prove estimate (2.7). Take an arbitrary weak solution u of problem (2.1)–(2.3). Using Lemma 3.2 with $\theta \equiv 1$, $t_1 = 0$, $t_2 = \tau \in (0, T]$ (see (3.3)), we derive from (2.6) that

$$\frac{1}{2}\int\limits_{\Omega}\Big\{\sum_{j=0}^{n} b_j(x)|\partial_j u(x,\tau)|^2\Big\}\,dx + \int\limits_0^{\tau}\int\limits_{\Omega}\Big\{\sum_{j=0}^{n} a_j(u)\partial_j u\Big\}\,dx\,dt$$
$$= \int\limits_0^{\tau}\int\limits_{\Omega}\Big\{\sum_{j=0}^{n} f_j\partial_j u\Big\}\,dx\,dt + \int\limits_{\Omega}\Big\{\sum_{j=0}^{n} b_j(x)|\partial_j u_0(x)|^2\Big\}\,dx.$$

To complete the proof of (2.7), we proceed in the same way as in the proof of inequality (4.11), this time using ($\mathcal{A}4$) and (4.10). □

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