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Stochastic impulsive processes on superposition of two renewal processes

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Abstract. Stochastic impulsive processes given by a sum of random variables on superposition of two renewal processes are considered on increasing time intervals.

Algorithms of average, diffusion approximation and large deviation generators are realized in the series scheme with a small series parameter under suitable scalings.

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1. Introduction

The *Stochastic Impulsive Process (SIP)* given by a sum of random variables on the Markov chains is described by the superposition on two *Renewal Processes (RP)*

$$S_n = u + \sum_{k=1}^{n} \alpha_k(x_k), \quad n \geq 0,\ S_0 = u \in \mathbb{R}^d \tag{1.1}$$

The Markov chain $x_k,\ k \geq 0$ is defined by the *Markov Renewal Process (MRP)* [1, ch. 1] on the space $E = \{\pm x;\ x > 0\}$ with the sojourn times $\theta_n^{\pm}(x) := \theta_n^{\pm} \wedge x,\ x \in \mathbb{R}_+ = (0, +\infty)$. Each of the two RP is defined by a sum of positive i.i.d. random variables [2, S. 8.3]:

$$\tau_n^{\pm} = \sum_{k=1}^{n} \theta_k^{\pm}, \quad n \geq 0, \quad \tau_0^{\pm} = 0, \quad P_{\pm}(t) = \mathcal{P}\{\theta_k^{\pm} \leq t\}, \quad t \geq 0. \tag{1.2}$$

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The random variables (impulsive) $\alpha_k^{\pm}(x),\ x \in \mathbb{R}_+$ are given by the distribution functions

$$G_{\pm}(A) = \mathcal{P}\{\alpha_k^{\pm}(x) \in A\}, \quad A \in \mathbb{R}^d.$$

The SIP is a particular case of the *Random Evolution Process (REP)* [2].

In our previous work [3] the SIP was considered on the MRP with the merging phase space $\widehat{E} = \{+, -\}$.

The increments of the SIP (1.1)

$$\Delta S_n = S_n - S_{n-1} = \alpha_n^{\pm}, \quad n \geq 0,\ x \in \mathbb{R}_+$$

may be interpreted as a success α_n^+, or as a failure α_n^-. This is the natural interpretation of the SIP in the risk theory [5].

The asymptotical behaviour of the SIP in the series scheme (average and diffusion approximation [2]) and the scheme of asymptotically small diffusion [4, 6] is considered.

The peculiarity of the MRP on the phase space $E = \{\pm x;\ x > 0\}$ is that the stationary distribution of the Markov chain $x_n,\ n \geq 0$ is given in the explicit form (see Section 2). So, the algorithms of averaging (Proposition 4.1) and diffusion approximation (Proposition 5.1) may be realized effectively. Hence, the simplified models of the SIP may be used in applications to the risk problems [5, S.6.5].

2. Superposition of two renewal processes

The renewal processes are defined by sum of positive valued random variables, independent in common and identically distributed [1] (see also [2, S. 8.3]):

$$\tau_n^{\pm} = \sum_{k=1}^{n} \theta_k^{\pm}, \quad n \geq 0,\ \tau_0^{\pm} = 0, \tag{2.1}$$

$$P_{\pm}(t) = \mathcal{P}\{\theta_k^{\pm} \leq t\}, \quad t \geq 0,\ P_{\pm}(0) = 0.$$

The renewal processes can be given by the counting processes:

$$\nu_{\pm}(t) := \max\{n > 0 :\ \tau_n^{\pm} \leq t\}, \quad t \geq 0.$$

The superposition of two renewal processes (2.1) is defined by the counting process

$$\nu(t) = \nu_+(t) + \nu_-(t), \quad t \geq 0. \tag{2.2}$$

The superposition of two renewal processes (2.2) can be characterized by *the Markov Renewal Process (MRP)* [1, ch. 1]

$$x_n,\ \theta_n, \quad n \geq 0,$$

given on the phase space

$$E = \{\pm x;\ x > 0\}, \tag{2.3}$$

with the sojourn times

$$\theta_n^{\pm}(x) = \theta_n^{\pm} \wedge x, \quad x \in \mathbb{R}_+ = (0, +\infty).$$

The symbols + or − in (2.3) are fixed the renewal moment of one or another renewal processes (2.1). The continuous component x is fixed *the remainder time* up to the renewal moment other renewal process in (2.2).

The embedded Markov chain $x_n,\ n \geq 0$, is given by the matrix of the transition probabilities

$$P(x, dy) = \begin{bmatrix} P_+(x - dy) & P_+(x + dy) \\ P_-(x + dy) & P_-(x - dy) \end{bmatrix} \tag{2.4}$$

The specific property of the embedded Markov chain with the transition probabilities (2.4) is existence of *the stationary distribution* with the densities

$$\begin{aligned} &\rho_\pm(x) = \rho \overline{P}_\mp(x), \quad \overline{P}_\mp(x) := 1 - P_\mp(x), \\ &\rho = (p_+ + p_-)^{-1}, \quad p_\pm := \int_0^\infty \overline{P}_\pm(x)\, dx. \end{aligned} \tag{2.5}$$

The stationary distribution on merged phase space $\widehat{E} = \{+, -\}$, is given by

$$\rho_\pm = \rho p_\mp = \lambda_\pm / \lambda, \quad \lambda_\pm = 1/p_\pm, \quad \lambda = \lambda_+ + \lambda_-.$$

3. Storage Impulsive Process

The SIP on superposition of two renewal processes (2.2) is defined by the sum of random variables take values in Euclidean space $\mathbb{R}^d,\ d \geq 1$

$$S_n = u + \sum_{k=1}^{n} \alpha_k(x_k), \quad n \geq 0,\ S_0 = u \in \mathbb{R}^d. \tag{3.1}$$

The random variables $\alpha_k^{\pm}(x), k \geq 1, x \in \mathbb{R}_+$, are given by the distribution functions on $(R^d, \mathcal{R}^d)$

$$G_\pm(A) = \mathcal{P}\{\alpha_k^{\pm}(x) \in A\}, \quad A \in \mathcal{R}^d.$$

Example 3.1. *The risk process* (3.1) constructed by the (positive) *input* random variables $\alpha_k^+ > 0$, in the renewal moments of the renewal process $\nu_+(t), t \geq 0$, and by (negative) *output* random variables $-\alpha_k^- > 0$, in the renewal moments of the renewal process $\nu_-(t), t \geq 0$ that is

$$S(t) = u + \sum_{k=1}^{\nu_+(t)} \alpha_k^+ - \sum_{k'=1}^{\nu_-(t)} \alpha_{k'}^-.$$

The SIP (3.1) can be characterized by the generator of the two components Markov chain

$$S_n, \ x_n, \quad n \geq 0. \tag{3.2}$$

Lemma 3.1. *The two component Markov chain* (3.2) *is characterized by the generator given on the vector test function* $\varphi(u,x) = (\varphi_+(u,x), \varphi_-(u,x))$*:*

$$\mathbf{L}\varphi(u,x) = \mathbf{PG}\varphi(u,x) - \varphi(u,x), \tag{3.3}$$

where the operator $\mathbf{P}$ *is given by the matrix* (2.4) *and*

$$\begin{aligned} \mathbf{G}\varphi(u) &= \begin{bmatrix} \mathbf{G}_+ & 0 \\ 0 & \mathbf{G}_- \end{bmatrix} \begin{pmatrix} \varphi_+(u) \\ \varphi_-(u) \end{pmatrix} = (\mathbf{G}_+\varphi_+(u), \mathbf{G}_-\varphi_-(u)), \\ \mathbf{G}_\pm\varphi_\pm(u) &:= \int_{\mathbb{R}^d} G_\pm(dv)\varphi_\pm(u+v) \end{aligned} \tag{3.4}$$

Remark 3.1. The generator (3.3) can be represented in scalar form:

$$\begin{aligned} \mathbf{L}_\pm\varphi(u,x) = &\int_R G_\pm(dv) \int_0^x P_\pm(dt)\varphi_\pm(u+v, x-t) \\ &+ \int_R G_\mp(dv) \int_x^\infty P_\pm(dt)\varphi_\mp(u+v, t-x) - \varphi_\pm(u,x). \end{aligned} \tag{3.5}$$

Proof of Lemma 3.1*.* The conditional expectation

$$\mathbf{L}\varphi(u,x) = E[\varphi(S_{n+1}, x_{n+1}) - \varphi(u,x) | S_n = u, x_n = \pm x]$$

is calculated directly:

$$\begin{aligned} \mathbf{L}_\pm\varphi(u,x) &= E[\varphi(u+\alpha_{n+1}, x_{n+1}) - \varphi(u,x)] \\ &= \mathbf{G}_\pm\mathbf{P}_\pm\varphi_\pm(u,x) + \mathbf{G}_\mp\overline{\mathbf{P}}_\pm\varphi_\mp(u,x), \end{aligned}$$

where by definition

$$\mathbf{P}_{\pm}\varphi(x) := \int_0^x P_{\pm}(dt)\varphi(x-t), \quad \overline{\mathbf{P}}_{\pm}\varphi(x) := \int_x^\infty P_{\pm}(dt)\varphi(t-x).$$

□

Remark 3.2. The generator (3.3) can be transformed as follows:

$$\mathbf{L}\varphi(u,x) = Q\varphi(\cdot,x) + \mathbf{P}[\mathbf{G} - I]\varphi(u,x),$$

where by definition

$$Q := \mathbf{P} - I,$$

is the generator of the embedded Markov chain $x_n,\ n \geq 0$, given by the transition probabilities (2.4).

The two component Markov chain, given by the generator (3.3), is characterized by *the martingale* with respect to the standard σ-algebras $\mathcal{F}_n := \sigma\{(S_k, x_k),\ 0 \leq k \leq n\}$

$$\mu_{n+1} = \varphi(S_{n+1}, x_{n+1}) - \varphi(u,x) - \sum_{k=1}^{n} \mathbf{L}\varphi(S_k, x_k). \tag{3.6}$$

The martingale characterization (3.6) of the SIP (3.1) will be used in asymptotical analysis on increasing time intervals in the series scheme with the small parameter series $\varepsilon \to 0$ ($\varepsilon > 0$).

4. SIP in the average scheme

The SIP in the series scheme with the small parameter series $\varepsilon \to 0$ ($\varepsilon > 0$), is considered in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon \sum_{k=1}^{[t/\varepsilon]} \alpha_k(x_k), \quad t \geq 0,\ \varepsilon > 0,\ u \in \mathbb{R}^d. \tag{4.1}$$

The averaging behavior of the SIP is analyzed by using a martingale characterization (3.6).

Lemma 4.1. *The normalized SIP* (4.1) *can be characterized by the martingale*

$$\mu^\varepsilon(t) = \varphi(S^\varepsilon(t), x^\varepsilon(t)) - \varphi(S^\varepsilon(0), x^\varepsilon(0)) - \int_0^{\varepsilon[t/\varepsilon]} \mathbf{L}^\varepsilon\varphi(S^\varepsilon(h), x^\varepsilon(h))\, dh.$$

The compensating generator is such that

$$\mathbf{L}^{\varepsilon}\varphi(u,x) = \varepsilon^{-1}Q\varphi(\cdot,x) + \mathbf{P}(x)\mathbf{G}^{\varepsilon}\varphi(u,x), \tag{4.2}$$

where

$$\begin{aligned}
&\mathbf{P}(x) = \begin{bmatrix}\mathbf{P}_+(x) & \overline{\mathbf{P}}_+(x)\\ \overline{\mathbf{P}}_-(x) & \mathbf{P}_-(x)\end{bmatrix}, \quad \overline{\mathbf{P}}_{\pm}(x) = 1 - P_{\pm}(x)\\
&\mathbf{G}^{\varepsilon} = \begin{bmatrix}\mathbf{G}^{\varepsilon}_+ & 0\\ 0 & \mathbf{G}^{\varepsilon}_-\end{bmatrix}, \ \mathbf{G}^{\varepsilon}_{\pm}\varphi(u) = \varepsilon^{-1}\int_{\mathbb{R}^d}\mathbf{G}_{\pm}(dv)[\varphi(u+\varepsilon v) - \varphi(u)].
\end{aligned} \tag{4.3}$$

Lemma 4.2. *The generator* (4.2)–(4.3) *admits the following asymptotic representation*

$$\mathbf{L}^{\varepsilon}\varphi(u,x) = \varepsilon^{-1}Q\varphi(\cdot,x) + \mathbf{P}(x)\mathbf{G}\varphi(u,x) + \delta^{\varepsilon}_l(x)\varphi(u,x) \tag{4.4}$$

with the negligible term

$$|\delta^{\varepsilon}_l(x)\varphi(u)| \to 0, \ \varepsilon \to 0, \ \varphi(u) \in C^2(R^d).$$

The operator is defined as follows:

$$\mathbf{G} = \begin{bmatrix}\mathbf{G}_+ & 0\\ 0 & \mathbf{G}_-\end{bmatrix}, \quad \mathbf{G}_{\pm}\varphi(u) = g_{\pm}\varphi'(u), \ g_{\pm} := \int_{R^d} vG_{\pm}(dv). \tag{4.5}$$

Proof of Lemma 4.2 follows from the Taylor expansion applied to the formula (4.3).

Proposition 4.1. *The finite-dimensional distributions of the SIP* (4.1) *in the average scheme converges to*

$$S^{\varepsilon}(t) \Rightarrow S^0(t) = u + \widehat{g}t, \quad t \ge 0, \ \varepsilon \to 0, \tag{4.6}$$

where the velocity

$$\widehat{g} = \rho_+ g_+ + \rho_- g_- = (\lambda_+ g_+ + \lambda_- g_-)/\lambda.$$

Proof. The generator of the normalized SIP (4.4)–(4.5) has the singular perturbation form. The singular perturbed operator $Q = P - I$ is reducible invertible [2, ch. 5] with the projector on the null-space given by the stationary distribution (2.5):

$$\Pi\varphi(x) = \rho\left[\int_0^{\infty}\overline{P}_-(x)\varphi_+(x)\,dx + \int_0^{\infty}\overline{P}_+(x)\varphi_-(x)\,dx\right].$$

To get the limit operator the algorithm of solving of perturbation problem [2, ch. 5] can be used.

The generator (4.4)–(4.5) is considered on the perturbed test function

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x), \quad \varphi(u) := (\varphi(u), \varphi(u)).$$

Let's calculate:

$$\mathbf{L}^\varepsilon \varphi^\varepsilon(u,x) = \varepsilon^{-1} Q\varphi(u) + [Q\varphi_1 + \mathbf{P}(x)\mathbf{G}\varphi(u)] + \delta_l^\varepsilon(x)\varphi(u), \tag{4.7}$$

with the negligible term

$$|\delta_l^\varepsilon(x)\varphi(u)| \to 0, \quad \varepsilon \to 0,\ \varphi(u) \in C^2(\mathbb{R}^d).$$

The first term in (4.7) equals zero, because $\varphi(u)$ is a constant for the generator Q. The next term in (4.7) gives us a problem of singular perturbation

$$Q\varphi_1(u,x) + \mathbf{P}(x)\mathbf{G}\varphi(u) = \mathbf{L}^0\varphi(u). \tag{4.8}$$

The limit operator $\mathbf{L}^0$ is determined by using solvability condition for the equation (4.8) (see [2, ch. 5])

$$\mathbf{L}^0\Pi = \Pi\mathbf{P}(x)\mathbf{G}\Pi. \tag{4.9}$$

Let's calculate (4.9) taking in mind (4.4)–(4.5):

$$\begin{aligned}
\mathbf{L}^0 = \rho\Bigg[& G_+ \int_0^\infty \overline{P}_-(x)P_+(x)\,dx + G_- \int_0^\infty \overline{P}_-(x)\overline{P}_+(x)\,dx \\
& + G_- \int_0^\infty \overline{P}_+(x)P_-(x)\,dx + G_+ \int_0^\infty \overline{P}_+(x)\overline{P}_-(x)\,dx \Bigg] \\
& = \rho[Gg_+p_- + G_-p_+] = \rho_+G_+ + \rho_-G_- \\
& = (\lambda_+G_+ + \lambda_-G_-)/\lambda = \rho_+G_+ + \rho_-G_- = \widehat{G}.
\end{aligned}$$

That is the limit generator

$$\mathbf{L}^0\varphi(u) = \widehat{g}\varphi'(u),$$

defines the deterministic drift in (4.6)

$$S^0(t) = u + \widehat{g}t, \quad t \geq 0.$$

□

5. SIP in the diffusion approximation scheme

The SIP in the series scheme with the small parameter series $\varepsilon \to 0$ ($\varepsilon > 0$), is considered in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon \sum_{k=1}^{[t/\varepsilon^2]} \alpha_k(x_k), \quad t \geq 0, \ u \in \mathbb{R}^d, \tag{5.1}$$

under the additional *Balance Condition (BC)*:

$$\widehat{g} = \rho_+ g_+ + \rho_- g_- = 0. \tag{5.2}$$

Proposition 5.1. *The SIP* (5.1) *under BC* (5.2) *converges weakly*

$$S^\varepsilon(t) \Rightarrow w_\sigma(t), \quad \varepsilon \to 0.$$

The limit Brownian motion $w_\sigma(t)$, $t \geq 0$, is defined by the variance

$$\sigma^2 = \sigma_b^2 + \sigma_g^2, \tag{5.3}$$

$$\sigma_b^2 = \rho_+ B_+ + \rho_- B_-, \quad B_\pm = \int_{\mathbb{R}^d} v^2 G_\pm(dv), \tag{5.4}$$

$$\sigma_g^2 = 2\Pi \mathbf{P}(x) \mathbf{G} \mathbf{R}_0 \mathbf{P}(x) \mathbf{G} \Pi. \tag{5.5}$$

The potential operator $\mathbf{R}_0$ is defined by a solution of the equation

$$Q\mathbf{R}_0 = \mathbf{R}_0 Q = \Pi - I.$$

Remark 5.1. The component of variance (5.5) can be calculated by using the reducible inverse operator $\mathbf{R}_0$ to the generator Q of the embedded Markov chain. That is an open problem (see [3]).

Proof of Proposition 5.1. As in previous section 3 the martingale characterization of the SIP (5.1) is a starting point in asymptotic analysis. □

Lemma 5.1. *The normalized SIP* (5.1) *can be characterized by the martingale*

$$\mu^\varepsilon(t) = \varphi(S^\varepsilon(t), x^\varepsilon(t)) - \varphi(u, x) - \int_0^{\varepsilon^2[t/\varepsilon^2]} \mathbf{L}^\varepsilon \varphi(S^\varepsilon(h), x^\varepsilon(h))\, dh. \tag{5.6}$$

The generator $\mathbf{L}$ of the two component Markov chain

$$S_n^\varepsilon = S^\varepsilon(\tau_n^\varepsilon), \quad x_n^\varepsilon = x^\varepsilon(\tau_n^\varepsilon), \quad n \geq 0,$$

is represented by

$$\mathbf{L}^\varepsilon\varphi(u,x) = [\varepsilon^{-2}Q + \mathbf{P}(x)\mathbf{G}^\varepsilon]\varphi(u,x), \tag{5.7}$$

$$\mathbf{G}^\varepsilon = \begin{bmatrix} \mathbf{G}^\varepsilon_+ & 0 \\ 0 & \mathbf{G}^\varepsilon_- \end{bmatrix}, \quad \mathbf{G}^\varepsilon_\pm\varphi(u) = \int_{R^d} \mathbf{G}_\pm(dv)[\varphi(u+\varepsilon v) - \varphi(u)]. \tag{5.8}$$

Lemma 5.2. *The generator* (5.7)–(5.8) *admit the asymptotic representation on the smooth enough test function* $\varphi(u) \in C^3(\mathbb{R}^d)$:

$$\mathbf{L}^\varepsilon\varphi(u,x) = \varepsilon^{-2}Q\varphi(\cdot,x) + \varepsilon^{-1}\mathbf{P}(x)\mathbf{G}\varphi(u,x) \\ + \mathbf{P}(x)\mathbf{B}\varphi(u,x) + \delta^\varepsilon_e(x)\varphi(u,x), \tag{5.9}$$

with the negligible term

$$|\delta^\varepsilon_e(x)\varphi(u,\cdot)| \to 0, \quad \varepsilon \to 0, \ \varphi(u,\cdot) \in C^3(\mathbb{R}^d).$$

Here

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_+ & 0 \\ 0 & \mathbf{B}_- \end{bmatrix}, \quad \mathbf{B}_\pm\varphi(u) := \frac{1}{2}B_\pm\varphi''(u).$$

Now a solution of singular perturbation problem [2, *ch.* 5, *Proposition* 5.2] *is used on the perturbed test function*

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x). \tag{5.10}$$

Lemma 5.3. *The generator* (5.9) *on the perturbed test function* (5.10) *admit the asymptotic representation*

$$\mathbf{L}^\varepsilon\varphi^\varepsilon(u,x) = \mathbf{L}^0\varphi(u) + \delta^\varepsilon_e(x)\varphi(u),$$

with the negligible term

$$|\delta^\varepsilon_e(x)\varphi(u)| \to 0, \quad \varepsilon \to 0, \ \varphi(u) \in C^3(\mathbb{R}^d).$$

The limit generator

$$\mathbf{L}^0\varphi(u) = \frac{1}{2}\sigma^2\varphi''(u),$$

is the generator of the Brownian motion $w_\sigma(t), \ t \geq 0$, *with the variance* (5.3)–(5.5).

Proof. Let's calculate

$$\mathbf{L}^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon^{-2}Q\varphi(u) + \varepsilon^{-1}[Q\varphi_1(u,x) + \mathbf{P}(x)\mathbf{G}\varphi(u)] \\ + \Big[Q\varphi_2 + \mathbf{P}(x)\mathbf{G}\varphi_1(u,x) + \frac{1}{2}\mathbf{B}(x)\varphi(u)\Big] + \delta^\varepsilon_e(x)\varphi(u),$$

with the negligible term

$$|\delta_e^\varepsilon(x)\varphi(u)| \to 0, \quad \varepsilon \to 0, \ \varphi(u) \in C^3(\mathbb{R}^d).$$

The BC (5.2) can be represented as follows:

$$\Pi \mathbf{P}(x)\mathbf{G}\Pi = 0.$$

Hence there exists solution of the equation

$$Q\varphi_1(u,x) + \mathbf{P}(x)\mathbf{G}\varphi(u) = 0,$$

that is

$$\varphi_1(u,x) = \mathbf{R}_0\mathbf{G}(x)\varphi(u).$$

Now the next equation

$$Q\varphi_2 + \Big[\mathbf{P}(x)\mathbf{G}R_0\mathbf{P}(x)\mathbf{G} + \frac{1}{2}\mathbf{B}(x)\Big]\varphi(u) = \mathbf{L}^0\varphi(u),$$

gives the limit generator

$$\mathbf{L}^0\varphi(u) = \frac{1}{2}\sigma^2\varphi''(u),$$

by using solvability condition:

$$\mathbf{L}^0\varphi(u) = \Big[\Pi\mathbf{P}(x)\mathbf{G}R_0\mathbf{P}(x)\mathbf{G}\Pi + \frac{1}{2}\Pi\mathbf{B}(x)\Pi\Big]\varphi(u) = \frac{1}{2}\sigma^2\varphi''(u).$$

□

6. Large deviation problem

The SIP in the scheme of asymptotically small diffusion is considered in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon^2 \sum_{k=1}^{[t/\varepsilon^3]} \alpha_k(x_k), \quad t \geq 0, \ u \in \mathbb{R}^d \tag{6.1}$$

under additional BC (5.2).

Proposition 6.1. *The large deviation problem for SIP* (6.1) *under the BC* (5.2) *is realized by the exponential generator of asymptotically small diffusion*

$$H\varphi(u) = \frac{1}{2}\sigma^2[\varphi'(u)]^2. \tag{6.2}$$

The variance is defined in (5.3)–(5.5).

Proof. The SIP (6.1) can be characterized by the exponential martingale [4, Part I]. □

Lemma 6.1. *The two component Markov process* $S^\varepsilon(t),\ x^\varepsilon(t) := x_{[t/\varepsilon^3]}$, $t \geq 0$, *is characterized by the exponential martingale*

$$\exp\{\varphi(S^\varepsilon(t), x^\varepsilon(t))/\varepsilon - \varphi(S^\varepsilon(0), x^\varepsilon(0))/\varepsilon - \varepsilon^{-1} \int_0^{\varepsilon^3[t/\varepsilon^3]} H^\varepsilon(S^\varepsilon(h), x^\varepsilon(h))\,dh\} = \mu^\varepsilon(t)$$

is $\mathcal{F}_t(S, x)$*-martingale.*

The exponential generator

$$H^\varepsilon \varphi(u,x) = \varepsilon^{-2} \ln[e^{-\varphi(u,x)/\varepsilon} \mathbf{L}^\varepsilon e^{\varphi(u,x)/\varepsilon} + 1]. \tag{6.3}$$

The compensative generator

$$\mathbf{L}^\varepsilon \varphi(u,x) = [Q + \mathbf{P}(x)\mathbf{G}^\varepsilon]\varphi(u,x),$$

$$\mathbf{P}(x) = \begin{bmatrix} P_+(x) & \overline{P}_+(x) \\ \overline{P}_-(x) & P_-(x) \end{bmatrix}, \quad \mathbf{G}^\varepsilon = \begin{bmatrix} \mathbf{G}^\varepsilon_+(x) & 0 \\ 0 & \mathbf{G}^\varepsilon_-(x) \end{bmatrix}, \tag{6.4}$$

$$\mathbf{G}^\varepsilon_\pm \varphi(u) = \int_{\mathbb{R}^d} G_\pm(dv)[\varphi(u + \varepsilon^2 v) - \varphi(u)].$$

Note that the generators $\mathbf{G}^\varepsilon_\pm$ admit the asymptotic representation

$$\mathbf{G}^\varepsilon_\pm \varphi(u) = \varepsilon^2 g_\pm \varphi'(u) + \varepsilon^2 \delta^\varepsilon_g \varphi(u), \tag{6.5}$$

with the negligible form $|\delta^\varepsilon_g \varphi| \to 0,\ \varepsilon \to 0,\ \varphi \in C^2(\mathbb{R}^d)$.

Note the exponential generator (6.3)–(6.5) is considered on the perturbing test function

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon \ln[1 + \varepsilon\varphi_1(u,x) + \varphi^2(u,x)]. \tag{6.6}$$

Lemma 6.2. *The compensating operator* (6.4) *on the perturbing test function* (6.6) *admits the asymptotic representation*

$$H^\varepsilon_L \varphi^\varepsilon(u,x) := e^{-\varphi^\varepsilon/\varepsilon} L^\varepsilon e^{\varphi^\varepsilon/\varepsilon} = \varepsilon[Q\varphi_1 + P(x)\mathbf{G}\varphi] + \varepsilon^2[Q\varphi_2 - \varphi_1 Q\varphi_1 + H_b(x)\varphi] + \delta^\varepsilon_l \varphi, \tag{6.7}$$

with the negligible term $|\delta^\varepsilon_l \varphi| \to 0,\ \varepsilon \to 0,\ \varphi \in C^3(\mathbb{R}^d)$.

Proof. The asymptotic representation (6.7) is the consequence of the following asymptotic relations:

$$e^{-\varphi^\varepsilon/\varepsilon}Qe^{\varphi^\varepsilon/\varepsilon} = e^{-\varphi/\varepsilon}[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]^{-1}Q[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]e^{\varphi/\varepsilon}$$
$$= \varepsilon Q\varphi_1 + \varepsilon^2[Q\varphi_2 - \varphi_1 Q\varphi_1] + \varepsilon^2\delta_q^\varepsilon\varphi,$$

$$e^{-\varphi^\varepsilon/\varepsilon}\mathbf{P}(x)\mathbf{G}e^{\varphi^\varepsilon/\varepsilon} = e^{-\varphi/\varepsilon}[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]^{-1}\mathbf{P}(x)\mathbf{G}[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]e^{\varphi/\varepsilon}$$
$$= \varepsilon\mathbf{P}(x)\mathbf{G}\varphi + \varepsilon^2[\mathbf{P}(x)\mathbf{G}\varphi_1 + P(x)H_b(x)\varphi] + \delta_g^\varepsilon\varphi,$$

with the negligible terms $\delta_q^\varepsilon\varphi$ and $\delta_g^\varepsilon\varphi$.

Now the singular perturbation problems are used for the equations

$$Q\varphi_1 + \mathbf{P}(x)\mathbf{G}\varphi = 0, \quad Q\varphi_2 - \varphi_1 Q\varphi_1 + H_b(x)\varphi = H\varphi. \tag{6.8}$$

The first equation (6.8) under the BC (5.2) has the solution

$$\varphi_1 = R_0\mathbf{P}(x)\mathbf{G}\varphi, \quad Q\varphi_1 = -\mathbf{P}(x)\mathbf{G}\varphi.$$

Hence

$$-\varphi_1 Q\varphi_1 = H_g(x)\varphi(u) := \frac{1}{2}\sigma_g^2(x)[\varphi'(u)]^2. \tag{6.9}$$

The second equation (6.8) with (6.9) is transformed to

$$Q\varphi_2 + [H_g(x) + H_b(x)]\varphi(u) = H\varphi(u). \tag{6.10}$$

The right part side in (6.10) is determined by the solvability condition:

$$H = \Pi[H_g(x) + H_b(x)]\Pi. \tag{6.11}$$

□

Corollary 6.1. *The exponential generator* (6.3) *admits the asymptotic representation*

$$H^\varepsilon\varphi^\varepsilon(u,x) = H\varphi(u) + \delta_h^\varepsilon\varphi(u), \tag{6.12}$$

with the negligible term $|\delta_h^\varepsilon\varphi| \to 0,\ \varepsilon \to 0,\ \varphi \in C^3(\mathbb{R}^d)$.

The proof of Proposition 6.1 is finished as follows. We consider (see (6.7))

$$H^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon^{-2}\ln[1 + H_L^\varepsilon\varphi^\varepsilon(u,x)]$$
$$= \varepsilon^{-2}\ln[1+\varepsilon^2 H\varphi(u) + \varepsilon^2\delta_l^\varepsilon\varphi(u)] = H\varphi(u) + \delta_h^\varepsilon\varphi,$$

with the negligible term $\delta_h^\varepsilon\varphi,\ \varphi \in C^3(\mathbb{R}^d)$. The limit exponential generator H is calculated in (6.11) using the representation (see (5.3)–(5.5))

$$H_g(x)\varphi(u) = \frac{1}{2}\sigma_g^2(x)[\varphi'(u)]^2, \quad \sigma_g^2 = 2\Pi\mathbf{G}R_0\mathbf{P}(x)\mathbf{G1},$$

$$H_b(x)\varphi(u) = \frac{1}{2}\sigma_b^2(x)[\varphi'(u)]^2, \quad \sigma_b^2 = \Pi\mathbf{P}(x)\mathbf{B1}.$$

7. Conclusion

This paper contains the three simplified approximation schemes for the SIP given by a sum (1.1) of random variables defined on the Markov renewal process: average (Section 4), diffusion approximation (Section 5) and the scheme of asymptotically small diffusion (Section 6).

The considered simplification schemes may be effectively used in applications, partially in financial mathematics [7]. The initial SIP (1.1) or (3.1), defined by two distribution functions $G_{\pm}(A)$ of jumps and two distribution functions $P_{\pm}(t)$, given the renewal processes, may be simplified in the average scheme by the deterministic drift process (4.6), given by the four constants $\rho_{\pm}$ and $g_{\pm}$ which are the first moments of the renewal times ($\rho_{\pm}$) and the average jumps ($g_{\pm}$).

The fluctuations of the SIP on increasing time intervals in diffusion approximation scheme (Section 4) is described by the limit Brownian motion (Proposition 5.1) given by the variance (5.3)–(5.5) defined by the first two moments of jumps ($g_{\pm}$ and $B_{\pm}$). The scheme of asymptotically small diffusion (Section 6) represents the exponential generator of large deviations used in the analysis of asymptotically small probabilities (see [4]).

The initial definition of the SIP given in Section 2 may be easily interpreted as the logistic problem. The positive jumps $\alpha_k^+(x)$, $k \geq 0$ are interpreted as a profits and the negative jumps $\alpha_k^-(x)$, $k \geq 0$ are the losses.

The SIP on increasing time intervals may be approximated by the well-known emery drift (Section 3) and in addition by the Brownian motion process of fluctuation.

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