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Keller–Osserman a priori estimates and removability result for the anisotropic porous medium equation with absorption term

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(Presented by A. E. Shishkov)

Abstract. In this article we obtained the removability result for quasilinear equations model of which is

$$u_t - \sum_{i=1}^{n} \left(u^{m_i-1} u_{x_i}\right)_{x_i} + f(u) = 0,\ u \geq 0.$$

and prove a priori estimates of Keller–Osserman type.

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1. Introduction and main results

In this paper we study solutions to quasilinear parabolic equation in the divergent form

$$u_t - divA(x,t,u,\nabla u) + a_0(u) = 0,\ (x,t) \in \Omega_T = \Omega \times (0,T), \qquad (1.1)$$

satisfying a initial condition

$$u(x,0) = 0,\ \ x \in \Omega \setminus \{0\}, \qquad (1.2)$$

where Ω is a bounded domain in $R^n, n \geq 2, 0 < T < \infty$.

We suppose that the functions $A = (a_1, \dots, a_n)$ and a_0 satisfy the Caratheodory conditions and the following structure conditions hold

$$A(x,t,u,\xi)\xi \geq \nu_1 \sum_{i=1}^{n} |u|^{m_i-1} |\xi_i|^2,$$

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$$|a_i(x,t,u,\xi)| \leq \nu_2 u^{(m_i-1)\frac{1}{2}} \left(\sum_{j=1}^{n} |u|^{m_j-1}|\xi_j|^2 \right)^{\frac{1}{2}}, \ i = \overline{1,n}, \tag{1.3}$$

$$a_0(u) \geq \nu_1 f(u),$$

with positive constants ν_1, ν_2 and continuous, positive function $f(u)$ and

$$\min_{1\leq i\leq n} m_i > 1 - \frac{2}{n}, \ \max_{1\leq i\leq n} m_i \leq m + \frac{2}{n}, \tag{1.4}$$

where $m = \frac{1}{n}\sum_{i=1}^{n} m_i$. Without loss of generality we will assume later that $m_1 \leq m_2 \leq \ldots \leq m_n$.

Many authors studied problems of singularities of solutions of second order quasilinear elliptic and parabolic equations. Review of these results can be found in the monograph of Veron [19]. Brezis and Veron [2] proved that for $q \geq \frac{n}{n-2}$ the isolated singularities of solutions to the elliptic equation

$$-\triangle u + u^q = 0,$$

are removable. In [3] Brézis and Friedman proved that for $q \geq \frac{n+2}{n}$ the isolated singularities of solutions for the following parabolic equation

$$\frac{\partial u}{\partial t} - \triangle u + |u|^{q-1}u = 0, \ \ (x,t) \in \Omega_T \setminus \{(0,0)\}$$

are removable. The removability of isolated singularity for solutions of the nonanisotropic porous medium equation $(m = m_1 = \ldots = m_n,)$

$$u_t - \triangle \left(|u|^{m-1}u\right) + |u|^{q-1}u = 0,$$

has been proved under the assumption $q \geq m + \frac{2}{n}$ by Kamin and Peletier [5].

Development of the qualitative theory of second order quasilinear elliptic and parabolic equations with nonstandart growth conditions has been observed in recent decades. Some results of [4, 6, 7, 9, 10, 12, 15–18] we mention here. One of the example of such equations is

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|u|^{(m_i-1)(p_i-2)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 0, p_i \geq 2, m_i \geq 1, i = \overline{1,n}.$$

The removability result of isolated singularity and a priori estimates of Keller–Osserman type for this equation was obtained in [9, 12].

We now define a weak solution of the problem (1.1), (1.2) with singularity at the point $(0,0)$. We will write $V_{2,m}(\Omega_T)$ for the class of functions $\varphi \in C_{loc}(0,T,L_{loc}^{1+m^-}(\Omega))$ with $\sum_{i=1}^{n} \iint_{\Omega_T} |\varphi|^{m_i+m^- -2} |\varphi_{x_i}|^2 \,dxdt < \infty$, where $m^- = \min(m_n,1)$. By a weak solution of the problem (1.1), (1.2) we mean a function $u(x,t) \geq 0$ satisfying the inclusion $u\psi \in V_{2,m}(\Omega_T) \cap L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega))$ and the integral identity

$$\int_\Omega u(x,\tau)\psi\varphi dx$$

$$+\int_0^\tau\int_\Omega \{-u(\psi\varphi)_t + A(x,t,u,\nabla u)\nabla(\psi\varphi) + a_0(u)\psi\varphi\}\,dx\,dt = 0 \quad (1.5)$$

holds for any testing function $\varphi \in W^{1,2}_{loc}(0,T;L^2_{loc}(\Omega)) \cap L^2_{loc}(0,T;\overset{o}{W}{}^{1,2}_{loc}(\Omega))$, any $\psi \in C^1(\overline{\Omega}_T)$ vanishing in the neighborhood of $\{(0,0)\}$ and for all $\tau \in (0,T)$.

The result of this paper is the removability of isolated singularities for solutions of the anisotropic porous medium equation with absorption term. The proof of this result is based on a priori estimates of Keller–Osserman type of the solution to the equation (1.1). The main difficulty lies in the fact that part of $m_i < 1$ (singular case), and another part of $m_i > 1$ (degenerate case).

Theorem 1.1. *Let the conditions* (1.3), (1.4) *be fulfilled and u be a nonnegative weak solution to the problem* (1.1), (1.2). *Assume also that $f(u) = u^q$ and*

$$q \geq m + \frac{2}{n}, \quad (1.6)$$

then the singularity at the point $\{(0,0)\}$ is removable.

Let $(x^{(0)},t^{(0)}) \in \Omega_T$, for any $\tau, \theta_1, \theta_2, \ldots, \theta_n > 0$, $\theta = (\theta_1,\ldots,\theta_n)$ we define $Q_{\theta,\tau}(x^{(0)},t^{(0)}) := \{(x,t) : |t-t^{(0)}| < \tau,\ |x_i - x_i^{(0)}| < \theta_i,\ i = \overline{1,n}\}$ and set $M(\theta,\tau) := \sup\limits_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} u$, $F(\theta,\tau) := \sup\limits_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} F(u)$, $F(u) = \int_0^u s^{m^- -1} f(s)ds$, $m^+ = max(m_n,1)$.

Theorem 1.2. *Let the conditions* (1.3), (1.4) *be fulfilled and u be a nonnegative weak solution to equation* (1.1), *assume also that $f \in C^1(R^1_+)$ and $f'(u) \geq 0$. Let $(x^{(0)},t^{(0)}) \in \Omega_T$, fix $\sigma \in (0,1)$, let $Q_{8\theta,8\tau}(x^{(0)},t^{(0)}) \subset$*

Ω_T. *Set* $\rho = \begin{cases} \theta_n, \textit{if } m_n > 1, \\ \tau^{\frac{1}{2}}, \textit{if } m_n < 1, \end{cases}$, *then there exist positive number* c_1, c_2 *depending only on* $n, \nu_1, \nu_2, m_1, \dots, m_n$ *such that either*

$$u(x^{(0)}, t^{(0)}) \leq \left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}} + \sum_{i=1}^{n-1} \left(\frac{\rho}{\theta_i}\right)^{\frac{2}{m^+-m_i}}, \tag{1.7}$$

or

$$(M(\sigma\theta, \sigma\tau))^{1-m^- + \frac{n(m-m^-)}{2}} F(M(\sigma\theta, \sigma\tau))$$
$$\leq c_1(1-\sigma)^{-\gamma} \rho^{-2} (M(\theta, \tau))^{m^+ + 1 + \frac{n(m-m^-)}{2}} \tag{1.8}$$

holds true.

We also have, in particular, if

$$F(\varepsilon u) \leq \varepsilon^{m^+ + m^- + \beta} F(u), \; \beta > 0, \tag{1.9}$$

then

$$F(M(\theta, \tau)) \leq c_2(1-\sigma)^{-\gamma} M^{m^+ + m^-}(\theta, \tau) \rho^{-2}, \tag{1.10}$$

An example of the function f, which satisfies the conditions (1.9) is $f(u) = u^q, q \geq m + \frac{2}{n}$. Assuming for simplicity that $dist(x^{(0)}, \partial\Omega) = |x^{(0)}|$, and choosing $\tau,\ \theta_i,$ from the conditions

- $m_n > 1$: $\left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}} = \theta_n^{-\frac{2}{q-m_n}}$, i.e. $\tau = \theta_n^{\frac{2(q-1)}{q-m_n}}$,

  $\left(\frac{\theta_n}{\theta_i}\right)^{\frac{2}{m_n-m_i}} = \theta_n^{-\frac{2}{q-m_n}}$, i.e. $\theta_i = \theta_n^{\frac{q-m_i}{q-m_n}}$,

- $m_n < 1$: $\left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}} = \tau^{-\frac{1}{q-m_n}}$, i.e. $\tau = \theta_n^{\frac{2(q-1)}{q-m_n}}$,

  $\left(\frac{\tau^{\frac{1}{2}}}{\theta_i}\right)^{\frac{2}{1-m_i}} = \tau^{-\frac{1}{q-1}}$, i.e. $\theta_i = \tau^{\frac{q-m_i}{2(q-1)}}$,

from (1.7), (1.10) we obtain an estimate

$$u(x^{(0)}, t^{(0)}) \leq c \left(\sum_{i=1}^{n} |x_i^{(0)}|^{\frac{2}{q-m_i}} + (t^{(0)})^{\frac{1}{q-1}}\right)^{-1}. \tag{1.11}$$

2. Keller–Osserman a priori estimates

2.1. Auxiliary propositions

Let $E(2\rho) = \{(x,t) \in \Omega_T : u(x,t) > M(2\rho)\}$, $u^{(\rho)}(x,t) = \min\left(M\left(\frac{\rho}{2}\right) - M(2\rho),\ u(x,t) - M(2\rho)\right)$.

Lemma 2.1. [11] *Under the assumptions of Theorem 1.1 following inequality holds*

$$\iint\limits_{E(2\rho)} u^{(\rho)} u^q \psi_r^l dxdt \le \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho) \right)$$

$$\times \left\{ F_3(r,\lambda) + (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda) \right\}, \tag{2.1}$$

where

$$F_1(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{\frac{q-2}{q-1}} \dfrac{1}{r}, \ \lambda = 0, \ q > 2, \\ \ln\ln \dfrac{1}{r}, \ \lambda = 0, \ q = 2, \\ \ln^{-\frac{2-q}{q-1}} \dfrac{1}{r}, \ \lambda = 0, \ q < 2 \end{cases}$$

$$F_2(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{\frac{q-2m_1}{q-m_1}} \dfrac{1}{r}, \ \lambda = 0, \ q > 2m_1, \\ \ln\ln \dfrac{1}{r}, \ \lambda = 0, \ q = 2m_1, \\ \ln^{-\frac{2m_1-q}{1-m_1}} \dfrac{1}{r}, \ \lambda = 0, \ q < 2m_1. \end{cases}$$

$$F_3(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{-\frac{1}{q-1}} \dfrac{1}{r}, \ \lambda = 0, \end{cases} \quad F_4(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{-1} \dfrac{1}{R(r)}, \ \lambda = 0, \end{cases}$$

where $\lambda = n - \frac{2}{q-m}, 0 < r < R_0$.

Lemma 2.2. [1] *Let* $\Omega \subset R^n$, $n \ge 2$ *be a bounded domain,* $v \in \overset{o}{W}{}^{1,1}(\Omega)$ *and*

$$\sum_{i=1}^{n} \int\limits_{\Omega} |v|^{\alpha_i} |v_{x_i}|^{p_i} dx < \infty, \ \alpha_i \ge 0, \ p_i > 1. \tag{2.2}$$

If $1 < p < n$*, then* $v \in L^q(\Omega), q = \frac{np}{n-p}\left(1 + \frac{1}{n}\sum\limits_{i=1}^{n} \frac{\alpha_i}{p_i}\right)$, $\frac{1}{p} = \frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{p_i}$ *and the following inequality holds*

$$\|v\|_{L^q(\Omega)} \le \gamma \prod_{i=1}^{n} \left(\int\limits_{\Omega} |v|^{\alpha_i} |v_{x_i}|^{p_i} dx \right)^{\frac{1}{np_i\left(1+\frac{1}{n}\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}\right)}}, \tag{2.3}$$

where the positive constant γ *depends only on* $n, p_i, \alpha_i, i = \overline{1,n}$.

Lemma 2.3. [8, chap. 2] *Let* $\{y_j\}_{j\in N}$ *be a sequence of nonnegative numbers such that for any* $j = 0, 1, 2, \dots$ *the inequality*

$$y_{j+1} \le C b^j y_j^{1+\varepsilon}$$

holds with positive $\varepsilon, C > 0, b > 1$. *Then the following estimate is true*

$$y_j \le C^{\frac{(1+\varepsilon)^j-1}{\varepsilon}} b^{\frac{(1+\varepsilon)^j-1}{\varepsilon^2}-\frac{j}{\varepsilon}} y_0^{(1+\varepsilon)^j}.$$

Particulary, if $y_0 \le C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$, *then* $\lim\limits_{j\to\infty} y_j = 0$.

2.2. Integral estimates of solutions

Consider a cylinder $Q_{\theta,\tau}(x^{(0)}, t^{(0)})$ and let $(\bar{x}, \bar{t})$ be an arbitrary point in $Q_{\sigma\theta,\sigma\tau}(x^{(0)}, t^{(0)})$. If $u(x^{(0)}, t^{(0)}) \ge \left(\frac{\theta_n^2}{\tau}\right)^{\frac{1}{m_n-1}} + \sum\limits_{i=1}^{n-1}\left(\frac{\rho}{\theta_i}\right)^{\frac{2}{m^+-m_i}}$ then $M(\theta,\tau){=}\max(M(\theta,\tau),\delta(\theta,\tau)){\ge}(\tau^{-1}\theta_n)^{\frac{1}{m_n-1}} + \sum\limits_{i=1}^{n}(\theta_i^{-1}\rho)^{\frac{2}{m^+-m_i}}$, and hence $Q_{\eta,s}(\bar{x},\bar{t}){\subset}Q_{\theta,\tau}(x^{(0)}, t^{(0)})$, where $s = (1-\sigma)M^{1-m^+}(\theta,\tau)\rho^2$, $\eta_i = (1-\sigma)M^{\frac{m_i-m^+}{2}}(\theta,\tau)\rho$, $i = \overline{1,n}$. For fixed $k > 0$ and $l, j = 0, 1, 2\dots$ set $\alpha_l = \frac{1}{4}(1 + 2^{-1} + \dots + 2^{-l})$, set $k_j = k(1-2^{-j})$, $\eta_{i,j,l} = (\alpha_l + \frac{1}{4}2^{-j-l-1})\eta_i$, $i = \overline{1,n}$, $\eta_{j,l} = (\eta_{1,j,l}, \dots, \eta_{n,j,l})$, $s_{j,l} = (\alpha_l + \frac{1}{4}2^{-j-l-1})s$, $Q_{j,l} = Q_{\eta_{j,l},s_{j,l}}(\bar{x},\bar{t})$, $A_{k_j,j,l} = \{x \in Q_{j,l}(\bar{x},\bar{t}) : F(u) > k_j\}$. Let $\xi_j \in C_0^\infty(Q_{j,l}(\bar{x},\bar{t}))$, $0 \le \xi_j \le 1$, $\xi_j = 1$ in $Q_{j+1,l}(\bar{x},\bar{t})$, $\left|\frac{\partial\xi_j}{\partial t}\right| \le \gamma 2^{j+l}s^{-1}$, $\left|\frac{\partial\xi_j}{\partial x_i}\right| \le \gamma 2^{j+l}\eta_i^{-1}, i = \overline{1,n}$.

In what follows γ stands for a constant depending only n, ν_1, ν_2, $m_1, \dots, m_n$ which may vary from line to line.

Lemma 2.4. *Let* u *be a nonnegative weak solution to equation* (1.1) *and let conditions* (1.3), (1.4) *hold. Then for any* $j \ge 0$ *the following inequality holds true*

$$l_j^{1-m^-}\int\limits_{A_{k_j,j,l}(t)}(F(u)-k_j)_+^2\xi_j^2dx + \sum_{i=1}^{n} l_j^{m_i-m^-}\iint\limits_{A_{k_j,j,l}}\left|\nabla((F(u)-k_j)_+)\right|^2\xi_j^2dxdt$$

$$+\iint\limits_{A_{k_j,j,l}}(F(u)-k_j)_+f^2(u)\xi_j^2dxdt \le \gamma M^{m^+-m^-}(\theta,\tau)\rho^{-2}\iint\limits_{A_{k_j,j,l}}(F(u)-k_j)_+^2dxdt \tag{2.4}$$

where $l_j = F^{-1}(k_j), j = 0, 1, 2, \dots$

Proof. Testing identity (1.5) by $\varphi = (F(u)-k_j)_+ f(u)\xi_j^2$, using conditions (1.3) we obtain

$$\iint\limits_{A_{k_j,j,l}} u_t f(u)(F(u)-k_j)_+\xi_j^2 dxdt$$

$$+\sum_{i=1}^{n}\iint\limits_{A_{k_j,j,l}} u^{m_i+m^- -2}|u_{x_i}|^2 f^2(u)\xi_j^2 dxdt + \iint\limits_{A_{k_j,j,l}} (F(u)-k_j)_+ f^2(u)\xi_j^2 dxdt$$

$$\leq \gamma\sum_{i=1}^{n}\iint\limits_{A_{k_j,j,l}} u^{\frac{m_i-1}{2}}\left(\sum_{l=1}^{n} u^{m_l-1}|u_{x_l}|^2\right)^{\frac{1}{2}} (F(u)-k_j)_+ f(u)\xi_j\left|\frac{\partial \xi_j}{\partial x_i}\right| dxdt.$$

From this, using the Young inequality and the evident inequality $l_j < u(x,t) < M(\theta,\tau)$ on $A_{k_j,j,l}$ we arrive at the required (2.4). □

2.3. Proof of Theorem 1.2

By Lemma 2.2 and the Hölder inequality we obtain

$$Y_{j+1,l} = \iint\limits_{A_{k_{j+1},j+1,l}} (F(u)-k_{j+1})_+^2 dxdt$$

$$\leq \left|A_{k_{j+1},j+1,l}\right|^{\frac{2}{n+2}}\left(\iint\limits_{A_{k_{j+1},j+1,l}} ((F(u)-k_{j+1})_+\xi_j)^{\frac{2(n+2)}{n}} dxdt\right)^{\frac{n}{n+2}}$$

$$\leq \left|A_{k_{j+1},j+1,l}\right|^{\frac{2}{n+2}} \operatorname*{ess\,sup}_{0<t<T}\left(\int\limits_{A_{k_{j+1},j+1,l}(t)} (F(u)-k_{j+1})_+^2\xi_j^2 dx\right)^{\frac{2}{n+2}}$$

$$\times\left(\int_0^T\prod_{i=1}^{n}\left(\int\limits_{A_{k_{j+1},j+1,l}(t)} |((F(u)-k_{j+1})_+\xi_j)_{x_i}|^2\, dx\right)^{\frac{1}{n}} dt\right)^{\frac{n}{n+2}}$$

$$\leq \left|A_{k_{j+1},j+1,l}\right|^{\frac{2}{n+2}} \operatorname*{ess\,sup}_{0<t<T}\left(\int\limits_{A_{k_{j+1},j+1,l}(t)} (F(u)-k_{j+1})_+^2\xi_j^2 dx\right)^{\frac{2}{n+2}}$$

$$\times\left(\int_0^T\prod_{i=1}^n\left(\int_{A_{k_{j+1},j+1,l}(t)}\left|((F(u)-k_{j+1})_+)_{x_i}\right|^2\xi_j^2dx\right.\right.$$

$$\left.\left.+\int_{A_{k_{j+1},j+1,l}(t)}(F(u)-k_{j+1})_+^2\left|\frac{\partial\xi_j}{\partial x_i}\right|^2dx\right)^{\frac{1}{n}}dt\right)^{\frac{n}{n+2}}.$$

Denote $Q_l=Q_{\alpha_l\eta,\alpha_l s}$, $M_l=\sup\limits_{Q_l}u$, using (2.4), it follows from Lemma 2.3 that $y_{j,l}\to 0$ as $j\to\infty$, provided k is chosen to satisfy

$$k^2=\gamma 2^{l\gamma}l_j^{m^--1+\frac{n(m^--m)}{2}}M_{l+1}^{\frac{(n+2)(m^+-m^-)}{2}}(\theta,\tau)\rho^{-n-2}\iint\limits_{Q_{l+1}}F^2(u)dxdt.$$

From this we obtain

$$M_l^{1-m^-+\frac{n(m-m^-)}{2}}F^2(M_l)$$

$$\le\gamma(1-\sigma)^{-\gamma}2^{l\gamma}M_{l+1}^{\frac{(n+2)(m^+-m^-)}{2}}\rho^{-n-2}\iint\limits_{Q_{l+1}}F^2(u)dxdt.$$

Denoting $M_l^{\frac{1-m^-}{2}+\frac{n(m-m^-)}{4}}F(M_l)=M_l^{\frac{a}{2}}F(M_l)=\Psi_l$, we have

$$\Psi_l^2\le\gamma(1-\sigma)^{-\gamma}2^{l\gamma}\Psi_{l+1}M_{l+1}^{\frac{(n+2)(m^+-m^-)}{2}-\frac{a}{2}}\rho^{-n-2}\iint\limits_{Q_{l+1}}F(u)dxdt$$

$$\le\varepsilon\Psi_{l+1}^2+\frac{1}{\varepsilon}(1-\sigma)^{-\gamma}\gamma 2^{l\gamma}(M(\theta,\tau))^{(n+2)(m^+-m^-)-a}$$

$$\times\rho^{-2(n+2)}\left(\iint\limits_{Q_{l+1}}F(u)dxdt\right)^2.$$

From this by iteration

$$\Psi^2(u(\bar{x},\bar{t}))\le\Psi_0^2\le\varepsilon^l\Psi_l^2+\frac{1}{\varepsilon}\gamma(1-\sigma)^{-\gamma}\sum_{i=0}^{l-1}(\varepsilon 2^\gamma)^i$$

$$\times(M(\theta,\tau))^{(n+2)(m^+-m^-)-a}\rho^{-2(n+2)}\left(\iint\limits_{Q_{l+1}}F(u)dxdt\right)^2.$$

We choose $\varepsilon = 2^{-\gamma-1}$ and passing to the limit as $l \to \infty$, we obtain

$$(u(\bar{x},\bar{t}))^{1-m^-+\frac{n(m-m^-)}{2}} F(u(\bar{x},\bar{t})))$$
$$\leq \gamma(1-\sigma)^{-\gamma}\rho^{-n-2}(M(\theta,\tau))^{\frac{(n+2)(m^+-m^-)}{2}} \iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} f(u)u^{m^-}dxdt. \quad (2.5)$$

To estimate the integral on the right-hand side of (2.5) we test integral identity by $\varphi = u^{m^-}\zeta^2$, using conditions (1.4) and the Hölder inequality, we obtain

$$\iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} f(u)u^{m^-}\zeta^2 dxdt \leq \gamma \iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} u^{m^-+1}|\zeta_t|\zeta dxdt + \gamma\sum_{i=1}^{n} \iint\limits_{Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})} u^{m_i+m^-}|\zeta_{x_i}|^2 dxdt$$

$$\leq \gamma\rho^{-2}M^{m^++m^-}(\theta,\tau)|Q_{\frac{\eta}{2},\frac{s}{2}}(\bar{x},\bar{t})| \leq \gamma\rho^n M^{m^-+1+\frac{m-m^+}{2}n}(\theta,\tau). \quad (2.6)$$

Since $(\bar{x},\bar{t})$ is an arbitrary point in $Q_{\sigma\theta,\sigma\tau}(x^{(0)},t^{(0)})$ from (2.5), (2.6) we arrive at

$$(M(\sigma\theta,\sigma\tau))^{1-m^-+\frac{n(m-m^-)}{2}} F(M(\sigma\theta,\sigma\tau))$$
$$\leq \gamma(1-\sigma)^{-\gamma}\rho^{-2}(M(\theta,\tau))^{m^++1+\frac{n(m-m^-)}{2}}. \quad (2.7)$$

For $j = 0,1,2\ldots$ define the sequences $\{\sigma_j\},\{\theta_j\},\{\tau_j\},\{M_j\}$ by $\sigma_j := \frac{1-2^{-j-1}}{1-2^{-j-2}}$, $\theta_j := (\theta_{1j},\theta_{2j},\ldots,\theta_{nj})$, $\theta_{ij} = \theta_i\left(1+\frac{1}{2}+\cdots+\frac{1}{2^j}\right)$, $i = \overline{1,n}$, $\tau_j = \tau\left(1+\frac{1}{2}+\cdots+\frac{1}{2^j}\right)$, $M_j := \sup\limits_{Q_{\theta_j,\tau_j}(x^{(o)})}$, $\Gamma(M_j) = \left[\frac{F(M_j)}{M_j^{m^++m^-}}\right]^{\frac{1}{m^++1+\frac{n(m-m^-)}{2}}}$.

We write (2.7) for the pair of boxes $Q_{\theta_j,\tau_j}(x^{(0)},t^{(0)})$ and $Q_{\theta_{j+1},\tau_{j+1}}(x^{(0)},t^{(0)})$. This gives

$$M_j\Gamma(M_j) \leq \gamma(1-\sigma)^{-\gamma}2^{j\gamma}\rho^{\frac{-2}{m^++1+\frac{n(m-m^-)}{2}}}M_{j+1}.$$

Using the following inequality which is an immediate consequence of our choice of Γ

$$\Gamma(u)v \leq \varepsilon^{-1}\Gamma(u)u + \Gamma(\varepsilon v)v, \ \varepsilon,u,v > 0, \quad (2.8)$$

indeed if $v \leq \varepsilon^{-1}u$, then $\Gamma(u)v \leq \varepsilon^{-1}\Gamma(u)u$, and if $v \geq \varepsilon^{-1}u$, then $\Gamma(u)v \leq \Gamma(\varepsilon v)v$, and in both cases (2.8) holds.

If $\varepsilon \in (0,1), \mu = \frac{\beta}{m^+ + 1 + \frac{n(m-m^-)}{2}}$ then

$$\Gamma(M_l) \le \Gamma(\varepsilon M_{l+1}) + \frac{1}{\varepsilon}\frac{\Gamma(M_l)M_l}{M_{l+1}}$$

$$\le \Gamma(\varepsilon M_{l+1}) + \varepsilon^{-1}\gamma(1-\sigma)^{-\gamma}2^{l\gamma}\rho^{\overline{m^+ + 1 + \frac{n(m-m^-)}{2}}^{-2}}$$

$$\le \varepsilon^{\mu}\Gamma(M_{l+1}) + \varepsilon^{-1}\gamma(1-\sigma)^{-\gamma}2^{l\gamma}\rho^{\overline{m^+ + 1 + \frac{n(m-m^-)}{2}}^{-2}}.$$

From this by iteration

$$\Gamma(M_0) \le \varepsilon^{l\mu}\Gamma(M_{i+1}) + \varepsilon^{-1}\gamma(1-\sigma)^{-\gamma}\sum_{i=0}^{l}(\varepsilon^{i\mu}2^{i\gamma})\rho^{\overline{m^+ + 1 + \frac{n(m-m^-)}{2}}^{-2}}.$$

We choose $\varepsilon^{\mu} = 2^{-\gamma-1}$ and passing to the limit as $l \to \infty$, we obtain

$$\Gamma(u(x^{(0)}, t^{(0)})) \le \gamma(1-\sigma)^{-\gamma}\rho^{\overline{m^+ + 1 + \frac{n(m-m^-)}{2}}^{-2}}.$$

Return to the previous notation

$$F(u(x^{(0)}, t^{(0)})) \le \gamma(1-\sigma)^{-\gamma}(M(\theta,\tau))^{m^+ + m^-}\rho^{-2}. \tag{2.9}$$

Thus Theorem 1.2 is proved. □

3. Proof of Theorem 1.1

3.1. Pointwise estimates of solutions

Let

$$Q_r = \left\{ (x,t) \in \Omega_T : \left(t^{\frac{\kappa(\lambda)}{\kappa_1(\lambda)}} + \sum_{i=1}^{n} |x_i|^{\frac{\kappa_i(\lambda)}{\kappa_1(\lambda)}} \right)^{\kappa_1(\lambda)} < r, \right\},$$

where $\kappa(\lambda) = \frac{1}{2+(n-\lambda)(m-1)}$, $\kappa_i(\lambda) = \frac{2}{2+(n-\lambda)(m-m_i)}$, $i = \overline{1,n}$, $\lambda = n - \frac{2}{q-m}$. For $0 < r < \rho < \frac{R_0}{2}$ $(R_0 : Q_{R_0} \subset \Omega_T)$ we set $M(r) = \sup\limits_{Q_{R_0} \setminus Q_r} u(x,t)$ and $u_{2\rho} = u(x,t) - M(2\rho) \le M\left(\frac{\rho}{2}\right) - M(2\rho)$ for $(x,t) \in Q_{R_0} \setminus Q_{\frac{\rho}{2}}$. For fixed $k > 0$ and $j = 0, 1, \dots$ set $\rho_j = \frac{\rho}{4}\left(1 + \frac{1}{2^j}\right)$, $k_j = k(1 - 2^{-j})$, $A_{k_j,j} = \{(x,t) \in Q_{\rho_j} : u_{2\rho} > k_j\}$. Let $\zeta_j \in C^{\infty}\left(Q_{\frac{\rho_{j+1}+\rho_j}{2}}\right)$, $0 \le \zeta_j \le 1$, $\zeta_j = 1$ outside Q_{ρ_j}, $\zeta_j = 0$ in $Q_{\rho_{j+1}}$, and $\left|\frac{\partial \zeta_j}{\partial t}\right| \le \gamma 2^{j\gamma}\rho^{-\frac{1}{\kappa(\lambda)}}$, $\left|\frac{\partial \zeta_j}{\partial x_i}\right| \le$

$\gamma 2^{j\gamma}\rho^{-\frac{2}{\kappa_i(\lambda)}},\ i=\overline{1,n}$. Let i_0 be the number such that $m_i\le 1,\ i=1,\dots i_0$ and $m_i>1,\ i=i_0+1,\dots n$, $m'=\frac{1}{n}\sum\limits_{i=1}^{i_0}m_i$, $m''=\frac{1}{n}\sum\limits_{i=i_0+1}^{n}m_i$. Note that $i_0=0$ if $m_i>1,\ i=\overline{1,n}$, and $i_0=n$ if $m_i\le 1,\ i=\overline{1,n}$.

Testing identity (1.4) by $\varphi=(u_{2\rho}-k_j)_+\zeta_j^2$, using conditions (1.4) we obtain

$$\operatorname{ess\,sup}\int\limits_{A_{k_j,j}(t)}(u_{2\rho}-k_j)_+^2\zeta_j dx+\sum_{i=1}^{i_0}M^{m_i-1}\left(\frac{\rho}{2}\right)\iint\limits_{A_{k_j,j}}|u_{x_i}|^2\zeta_j^2 dxdt$$

$$+\sum_{i=i_0+1}^{n}k_j^{m_i-1}\iint\limits_{A_{k_j,j}}|u_{x_i}|^2\zeta_j^2dxdt+\iint\limits_{A_{k_j,j}}(u_{2\rho}-k_j)_+u^q\zeta_j^2dxdt$$

$$\le\gamma\left(M^2\left(\frac{\rho}{2}\right)\rho^{-\frac{1}{\kappa(\lambda)}}+\sum_{i=1}^{n}M^{m_i+1}\left(\frac{\rho}{2}\right)\rho^{-\frac{2}{\kappa_i(\lambda)}}\right)|A_{k_j,j}|. \qquad (3.1)$$

By Lemma 2.2, the Hölder inequality and estimate (3.1), we obtain

$$Y_{j+1}=\iint\limits_{A_{k_{j+1},j+1}}(u_{2\rho}-k_{j+1})_+^2dxdt$$

$$\le\left|A_{k_{j+1},j+1}\right|^{\frac{2}{n+2}}\left(\iint\limits_{A_{k_{j+1},j+1}}((u_{2\rho}-k_{j+1})_+\zeta_j)^{2+\frac{4}{n}}dxdt\right)^{\frac{n}{n+2}}$$

$$\le\left|A_{k_{j+1},j+1}\right|^{\frac{2}{n+2}}\operatorname*{ess\,sup}_{0<t<T}\left(\int\limits_{A_{k_{j+1},j+1}(t)}(u_{2\rho}-k_{j+1})_+^2\zeta_j^2dx\right)^{\frac{2}{n+2}}$$

$$\times\left(\int\limits_0^t\prod_{i=1}^{n}\left(\int\limits_{A_{k_{j+1},j+1}(t)}\left|((u_{2\rho}-k_{j+1})_+\zeta_j)_{x_i}\right|^2dx\right)^{\frac{1}{n}}d\tau\right)^{\frac{n}{n+2}}$$

$$\le\gamma M^{\frac{(1-m')i_0}{n+2}}\left(\tfrac{\rho}{2}\right)k_{j+1}^{\frac{(1-m'')(n-i_0)}{n+2}}$$

$$\times\left(M^2\left(\tfrac{\rho}{2}\right)\rho^{-\frac{1}{\kappa(\lambda)}}+\sum_{i=1}^{n}M^{m_i+1}\left(\tfrac{\rho}{2}\right)\rho^{-\frac{2}{\kappa_i(\lambda)}}\right)|A_{k_{j+1},j+1}|^{1+\frac{2}{n+2}}.$$

From this by the evident inequality $(u_{2\rho}-k_j)_+\ge\frac{k}{2^{j+1}}$ on $A_{k_{j+1},j}$, we obtain estimate

$$Y_{j+1} \le \gamma 2^{j\gamma} M^{\frac{(1-m^{'})i_0}{n+2}} \left(\tfrac{\rho}{2}\right) k_{j+1}^{\frac{(1-m^{''})(n-i_0)}{n+2}} \Bigg(M^2 \left(\tfrac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}}$$

$$+ \sum_{i=1}^{n} M^{m_i+1} \left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \Bigg) Y_j^{1+\frac{2}{n+2}}. \qquad (3.2)$$

It follows from Lemma 2.3 that

$$(M(\rho) - M(2\rho))^{\frac{(m^{''}-1)(n-i_0)}{2}+n+4} \le \gamma 2^{j\gamma} M^{\frac{(1-m^{'})i_0}{n+2}} \left(\frac{\rho}{2}\right)$$

$$\times \left(M^2 \left(\frac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i+1} \left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right) \iint\limits_{Q_{\frac{\rho}{2}}} u_{2\rho}^2 dxdt. \qquad (3.3)$$

By the Hölder inequality and Lemma 2.1 we get

$$(M(\rho) - M(2\rho))^{\frac{(m^{''}-1)(n-i_0)}{2}+n+4} \le \gamma 2^{j\gamma} M^{\frac{(1-m^{'})i_0}{n+2}} \left(\frac{\rho}{2}\right)$$

$$\times \left(M^2 \left(\frac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i+1} \left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right)$$

$$\times \left\{ F_3(r,\lambda) + (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda) \right\} |Q_{\frac{\rho}{2}}|^{\frac{q-1}{q+1}}. \qquad (3.4)$$

Similarly to [11], we obtain the following estimate

$$M(\rho) - M(2\rho) \le 0,$$

iterating last inequality we get for any $\rho \le \frac{R_0}{2}$

$$M(\rho) \le M(R_0),$$

this proves the boundedness of solutions.

3.2. End of the proof of Theorem 1.1

Let K be a compact subset in Ω, and $\xi = 0$ in $\partial\Omega \times (0,T)$, such that $\xi = 1$ for $(x,t) \in K \times (0,T)$. Testing (1.5) by $\varphi = u^{m^-} \xi^2 \psi_r$, $\psi = \psi_r$, using conditions (1.3), the Young inequality, the boundedness of u and passing to the limit $r \to 0$ we get

$$\sup_{0<t<T} \int\limits_K u^{m^-+1} dx + \sum_{i=1}^{n} \int\limits_0^T \int\limits_K u^{m_i+m^--2} |u_{x_i}|^2 dxdt + \int\limits_0^T \int\limits_K u^{q+m^-} dxdt \le \gamma. \qquad (3.5)$$

Testing (1.5) by $\varphi\psi_r$, using (1.3), the boundedness of solution, and passing to the limit $r \to 0$, we obtain the integral identity (1.5) with an arbitrary $\varphi \in W^{1,2}_{loc}(0,T; L^2_{loc}(\Omega)) \cap L^2_{loc}(0,T; \overset{o}{W}{}^{1,2}_{loc}(\Omega))$ and $\psi \equiv 1$. Thus Theorem 1.1 is proved. □

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