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INTEGRABILITY IN AdS/CFT

This work provides a detailed review of recent developments in the field of AdS/CFT correspondence for a particular subject of the correspondence between type IIB superstring on $AdS_5 \times S_5$ and the $\mathcal{N} = 4$ super Yang–Mills theory. Through analyzing the bound states of Bethe roots for the corresponding PSU(2,2|4) spin chain, it is shown how the lattice system of functional equations, called Y-system, appears.

Keywords: AdS/CFT, integrable systems, spin chains.

1. Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence found by Maldacena [1] is an intriguing duality that connects the theory in a strong coupling regime with the theory in a weak coupling regime and *vice versa*. This work focuses on a particular case of the duality between the type-IIB string theory on $AdS_5 \times S_5$ and the $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory. This example is important to be studшув because of the integrability on both sides of the duality.

2. Symmetry Analysis

For the beginning, we should check if the theories on both sides have the same symmetry. This proves, of course, nothing, but gives a general feelling of how the whole thing works. On the AdS side, we have the type-IIB superstring theory on $AdS_5 \times S_5$. It is obvious that one has at least the same symmetry, as its target space, $AdS_5 \times S_5$. The symmetry group for S_5 is $SO(6)$, and the symmetry group for AdS_5 is $SO(4,2)$. These groups are isomorphic (up to a discrete subgroup) to $SU(4)$ and $SU(2,2)$, respectively. But, as usual for the supersymmetry, the symmetry group of the space is extended by fermion generators, and the total symmetry of the theory would be not just $SU(4)\times SU(2,2)$, but the whole supergroup $PSU(2,2|4)$.

On the CFT side, we have $\mathcal{N} = 4$ SYM theory on $R^{3,1}$. The R-symmetry for $\mathcal{N} = 4$ is $SU(4)$. Since $\mathcal{N} = 4$, the SYM theory is superconformal, and the space symmetry is not just $SO(3,1)$, but the full conformal group for $R^{3,1}$, namely $SO(4,2)$. With fermion generators, this gives us the already mentioned $PSU(2,2|4)$.

3. Integrability on the CFT Side

The beauty of our system manifests itself in the integrability on both sides. The integrability of the $\mathcal{N} = 4$ SYM theory was proposed by Minahan and Zarembo [2] and studied in details by Beisret and Staudacher [3]. The idea is quite tricky. First of all, to find the integrability, one should study the anomalous dimensions of long operators, i.e., the correlation functions

$$\langle D_\mu \lambda_\beta D_\nu X^i \lambda^\alpha X^k ...(0)| D_\mu D_\nu X^i \lambda_\beta \lambda_\alpha X^k ...(z)\rangle = z^\Delta. \quad (1)$$

To compute such correlators by means of perturbative field theory, one should consider Feynman graphs (see Fig. 1). These Feynman graphs are ribbon graphs, and they ought to catch the matrix structure of fields. For a symmetry group of the theory such as SU(N), this matrix structure contribution decouples and can be easily computed to be equal to $N^{\#\mathrm{loops}}$. It is not hard to see (Fig. 2) that the planar diagrams have more loops than nonplanar ones on the same level of perturbation theory. So, if we consider the large N limit, only the planar diagrams survive. For the planar diagrams, the interaction is only possible between operators which are close enough. Such physical systems containing a long chain of states with local interactions (i.e., interactions between the operators in sites that are close enough) resemble a well-known class of physical systems that are spin chains.

4. Integrability on the String Side

The idea of integrability of a IIB superstring on $\mathrm{AdS}_5 \times \mathrm{S}_5$ comes from considering it as a σ-model on the coset space

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \tag{2}$$

with the bosonic part

$$\frac{SO(4,2) \times SO(6)}{SO(4,1) \times SO(5)} = \mathrm{AdS}_5 \times \mathrm{S}_5. \tag{3}$$

The action of the model can be rewritten through the Z_4 graded current

$$J = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)} = -g^{-1}dg \tag{4}$$

as

$$S = \frac{\sqrt{\lambda}}{2\pi} \int \mathrm{STr}\, J^{(2)} \wedge * J^{(2)} - J^{(1)} \wedge J^{(3)} + \Lambda \wedge J^{(2)}. \tag{5}$$

As was shown in [4], this graded current allows one to construct a one-parameter family of flat connections:

$$L(x) = J^{(0)} + \frac{x^2+1}{x^2-1} J^{(2)} - \frac{2x}{x^2-1} \left(* J^{(2)} - \Lambda \right) + \\ + \sqrt{\frac{x+1}{x-1}} J^{(1)} + \sqrt{\frac{x-1}{x+1}} J^{(3)}. \tag{6}$$

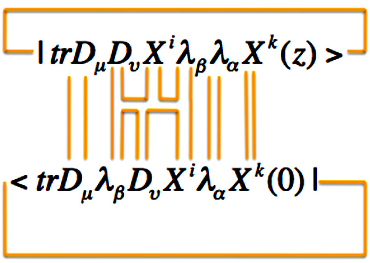

Fig. 1. Planar ribbon Feynman graph

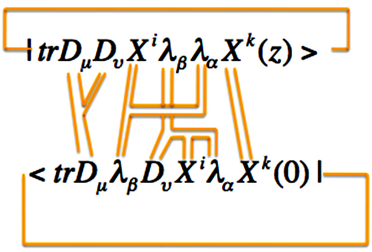

Fig. 2. Non-planar ribbon Feynman graph

The monodromy $\Omega(x) = P \exp\left(\int_\gamma L_x\right)$ of this connection over compact string directions gives us the 1-parameter set of integrals of motions. The equation for the eignvalues of $\Omega(x)$,

$$S \det(\lambda - \Omega(x)), \tag{7}$$

gives an algebraic curve with eight sheets. The cuts on this curve are quantum excitations of the string.

5. GL $(N|M)$ Spin Chain

In this section and the following ones, we will closely follow [5] and [6].

The spin chain is a physical system consisting of a large number of (possibly different) representations of (possibly different) Lie (super)groups organized as the sites of a closed or open chain. In what follows, we will only consider the case of the same representation of the Lie supergroup $GL(N|M)$ on each site. In particular, we are interesting in $PSU(2,2|4)$ supergroup, as a real representation of $GL(4|4)$. As $PSU(2,2|4)$ is a symmetry group for SYM, the SYM fields form a defining representation of it. The dilatation operator makes a Hamiltonian for this particular spin chain. So, the eigenstates for the spin chain would be the primary fields for SYM.

The usual tool to analyze spin chains is the so-called Bethe ansatz. First, we have to find the vacuum, i.e., the state with minimal energy. For a ferromagnetic, this would be the same state at each site. Then we construct excitations. For a one-particle excitation, we change the state at one particular site. A particle with defined momentum would be the superposition

$$|p\rangle = |baaaa...\rangle + e^{ip}\,|abaaa...\rangle + \\ + e^{2ip}\,|aabaa...\rangle + \ldots . \tag{8}$$

Two-particle excitations are a bit trickier. When two particles are far one from another, the state is just like a composition of two one-particle states. When they are close, the interaction [1] comes to play. When they are far away in another direction, they are else a composition of one-particle states, but with some

[1] To find the exact behaviour of a state in the interaction zone, one should use the more advanced technics of the algebraic Bethe ansatz.

additional phase. We will call this phase as the S-matrix. Explicitly, the two-particle excitations state are

$$|p_1p_2\rangle = \exp(ip_1n_1 + ip_2n_2)\left|..\underset{n_1}{a\,b}\,a..\underset{n_2}{a\,b}\,a...\right\rangle + \\ + S(p_1,p_2)\exp(ip_1n_1 + ip_2n_2)\left|..\underset{n_2}{a\,b}\,a..\underset{n_1}{a\,b}\,a...\right\rangle. \quad (9)$$

5.1. Yang–Baxter relations

Up to this moment, the procedure has worked for any spin chain. However, to construct higher excitations, we must set an additional constraint on the spin chain. For example, if we have three excitations, there are six different asymptotic states given by permutations. But we can reach any given state by different sets of pair collisions. So, the higher excitations can be constructed if the total acquired phase for a given state does not depend on the specific set of pair collisions resulting in that permutation. Naturally, this could be achieved, if and only if we could resolve triple permutations:

$$\hat{S}\,(p_1,p_2)\,\hat{S}\,(p_1,p_3)\,\hat{S}\,(p_2,p_3) = \\ = \hat{S}\,(p_2,p_3)\,\hat{S}\,(p_1,p_2)\,\hat{S}\,(p_1,p_3). \quad (10)$$

This equation is noting but the famous Yang–Baxter relation. If it is satisfied, we can easily construct any particle number excitation as a superposition over all possible permutations σ:

$$|p_1p_2...p_k\rangle = \sum_\sigma S(\sigma)\exp\left(i\sum_{j=1} p_kn_k\right) \times \\ \times \left|\underset{n_{\sigma_1}}{a\ b}\ a..\underset{n_{\sigma_2}}{a\ b}\ a...\underset{n_{\sigma_k}}{a\ b}\ a\right\rangle. \quad (11)$$

5.2. Nested Bethe ansatz

If the dimension of a representation is greater than two, all states we build yet do not cover the whole state space of the spin chain. However, they do cover a whole part containing the state which we called vacuum on the previous step. So, our problem is reduced by one dimension. Then we can use the Bethe ansatz once more, by considering excitations from the previous step as sites of a new spin chain, the previous excited states as a new vacuum, and making new excited states, by using the next generator. We should repeat this procedure, until we use all generators, and this will also cover the whole state space of the system. This procedure is called the nested Bethe ansatz.

6. Bethe Equations

For a closed spin chain, the momenta of excitations cannot be chosen arbitrary. Constraints on them arise from the particles virtually moving along the whole circle. If the particles would not interact, this give the well-known momentum quantization law. However, the interactions make this condition more interesting. We have

$$e^{ip_jN} = \prod_{k\neq j}\hat{S}(p_j,p_k). \quad (12)$$

As usual, it is useful to introduce a new variable u instead of the momentum:

$$p = \frac{1}{i}\log\left(\frac{u+i/2}{u-i/2}\right). \quad (13)$$

For our particular case of PSU(2,2|4) (and the particular Dynkin diagram ⊗–⊙–⊗–⊙–⊗–⊙–⊗), the Bethe equations read

$$\prod_k \frac{v_i-u_k-i/2}{v_i-u_k+i/2} = 1, \quad (14)$$

$$\prod_{i\neq j}\frac{u_i-u_j+i}{u_i-u_j-i}\prod_k\frac{u_i-v_k-i/2}{u_i-v_k+i/2} = 1, \quad (15)$$

$$\prod_k \frac{w_i-u_k-i/2}{w_i-u_k+i/2}\prod_m\frac{w_i-z_m+i/2}{w_i-z_m-i/2} = 1, \quad (16)$$

$$\prod_{i\neq j}\frac{z_i-z_j+i}{z_i-z_j-i}\prod_k\frac{z_i-w_k-i/2}{z_i-w_k+i/2} \times \\ \times\prod_m\frac{z_i-\tilde{w}_m-i/2}{z_i-\tilde{w}_m+i/2} = \left(\frac{z_i+i/2}{z_i-i/2}\right)^L, \quad (17)$$

$$\prod_k \frac{\tilde{w}_i-\tilde{u}_k-i/2}{\tilde{w}_i-\tilde{u}_k+i/2}\prod_m\frac{\tilde{w}_i-z_m+i/2}{\tilde{w}_i-z_m-i/2} = 1, \quad (18)$$

$$\prod_{i\neq j}\frac{\tilde{u}_i-\tilde{u}_j+i}{\tilde{u}_i-\tilde{u}_j-i}\prod_k\frac{\tilde{u}_i-\tilde{v}_k-i/2}{\tilde{u}_i-\tilde{v}_k+i/2} = 1, \quad (19)$$

$$\prod_k \frac{\tilde{v}_i-\tilde{u}_k-i/2}{\tilde{v}_i-\tilde{u}_k+i/2} = 1. \quad (20)$$

6.1. Bound states

In the limit of large L, these equations yield an important property of having bound states. If one of z_i has a positive imaginary part, the rhs of (17) is large. So, the lhs should have a pole, which implies that either there is some $z_j = z_i - i$ or some $w_k = z_i + i/2$. The first possibility leads us to a string of z's, spaced by i, and second leads to a more complex set of boson and

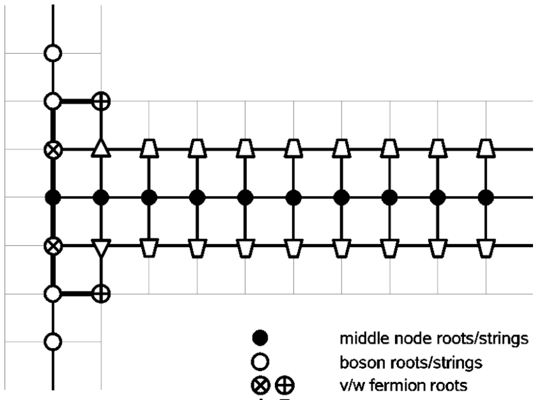

Fig. 3

fermion roots with the same real part. Actually, there are important conditions on the number of roots

$$L > 2\#z > 4\#w > 8\#u > 16\#v$$
$$\#z > 2\#\tilde{w} > 4\#\tilde{u} > 8\#\tilde{v}. \quad (21)$$

Under these conditions, the Bethe equations imply the following set of bound states: strings of u, $\tilde{u}$, and z and trapezia consisting of N-strings of w, $N-1$ strings of u, and $N-2$ strings of v. The set of equations for the real part of bound states would look quite complicated (we omit the tilded variables to avoid further complications):

$$\prod_{j,n} \frac{v_i - u_j^n + ni/2}{v_i - u_j^n - ni/2} \frac{v_i - t_j^n + i(n-1)/2}{v_i - t_j^n - i(n-1)/2} = 1, \quad (22)$$

$$\prod_{j\neq i,m} \prod_{l=|n-m|/2}^{n+m} \frac{u_i^n - u_j^m + li}{u_n - u_j^m - li} \frac{u_i^n - u_j^m + (l+1)i}{u_i^n - u_j^m - (l+1)i} \times$$
$$\times \prod_j \frac{u_i^n - t_j^1 + in/2}{u_i - t_j^1 - in/2} \frac{u_i^n - v_j + in/2}{u_i - v_j - in/2} = 1, \quad (23)$$

$$\prod_{j\neq i,m} \prod_{l=|n-m|/2}^{(n+m)/2} \frac{t_i^n - t_j^m + li}{t_n - t_j^m - li} \frac{t_i^n - t_j^m + (l+1)i}{t_i^n - t_j^m - (l+1)i} \times$$
$$\times \prod_j \frac{t_i^n - t_j^1 + i(n-1)/2}{t_i^n - t_j^1 - i(n-1)/2} \frac{t_i^n - v_j + i(n-1)/2}{t_i^n - v_j - i(n-1)/2} \times$$
$$\times \prod_{j,m} \prod_{l=\frac{|n-m|+1}{2}}^{\frac{n+m-1}{2}} \frac{t_i^n - z_j^m + li}{t_i^n - z_j^m - li} = 1, \quad (24)$$

$$\prod_{j\neq i,m} \prod_{l=|n-m|/2}^{(n+m)/2} \frac{z_i^n - z_j^m + li}{z_n - z_j^m - li} \frac{z_i^n - z_j^m + (l+1)i}{z_i^n - z_j^m - (l+1)i} \times$$
$$\times \prod_{j,m} \prod_{l=\frac{|n-m|+1}{2}}^{\frac{n+m-1}{2}} \frac{z_i^n - t_j^m + li}{z_i^n - t_j^m - li} = \left(\frac{z_i^n + in}{z_i^n - in}\right)^L. \quad (25)$$

7. Thermodynamic Bethe Ansatz

We now know that all solutions of the Bethe equation are either real or sets of roots with fixed imaginary parts, parametrized by real parameters. So we can introduce the densities for roots and each type of bound states. To obtain the equation for densities, one should introduce counting functions. For example, for Eq. (14), the counting function would be

$$N(x) = \frac{1}{2\pi i} \sum_k \log \frac{x - u_k - i/2}{x - u_k + i/2}. \quad (26)$$

If $N(x)$ is an integer for some value of x, Eq. (14) is satisfied. It could be that x is one of v_i's or not. In the second case, we call such x holes. If we introduce a density for holes $\bar{\rho}$, it is obvious that

$$\frac{dN}{dx} = \rho_v + \bar{\rho}_v = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dy}{1/4 + (x-y)^2} \rho_u. \quad (27)$$

If we apply this technique to (22)–(25), we get the following set of integral equations:

$$\nu + \bar{\nu} = K_n * s_n + K_n * \tau_n, \quad (28)$$
$$s_n + \bar{s}_n = K_{nm} * s_m + K_n * \nu + K_n * \tau_1, \quad (29)$$
$$\tau_n + \bar{\tau}_n = K_{nm} * \tau_m + + K_{n-1} * \nu +$$
$$+ K_{n-1} * \tau_1 + \tilde{K}_{nm} \sigma_m, \quad (30)$$
$$\sigma_n + \bar{\sigma}_n = K_{nm} \sigma_m + \tilde{K}_{nm} \tau_m \quad (31)$$

with integral kernels

$$K_n = \frac{n/2}{n^2/4 + x^2}, \quad (32)$$

$$K_{nm} = \sum_{l=|n-m|/2}^{(n+m)/2} (1 - \delta_{l0}) K_{2l} + K_{2l+2}, \quad (33)$$

$$\tilde{K}_{nm} = \sum_{l=\frac{|n-m|+1}{2}}^{\frac{n+m-1}{2}} K_{2l}. \quad (34)$$

The set of equations (28)–(31) is not complete, since we lost some information, while changing to the density formalism. However, they possess enough information to consider the system at finite temperatures. The partition function of the system is

$$Z = \sum_{\text{states}} \exp\left(\beta \sum_i \frac{1}{1/4 + z_i^2}\right) =$$
$$= \int \prod \mathcal{D}s^n \mathcal{D}\tau^n \mathcal{D}\nu \mathcal{D}\sigma^n \exp\left(\sum_n \int \frac{\beta\sigma_n}{n^2/4 + x^2} +\right.$$
$$+ s_n \log\left(1 + \frac{\bar{s}_n}{s_n}\right) + \bar{s}_n \log\left(1 + \frac{s_n}{\bar{s}_n}\right) +$$
$$+ \tau_n \log\left(1 + \frac{\bar{\tau}_n}{\tau_n}\right) + \bar{\tau}_n \log\left(1 + \frac{\tau_n}{\bar{\tau}_n}\right) +$$
$$+ \nu \log\left(1 + \frac{\bar{\nu}}{\nu}\right) + \bar{\nu} \log\left(1 + \frac{\nu}{\bar{\nu}}\right) +$$
$$\left. + \sigma_n \log\left(1 + \frac{\bar{\sigma}_n}{\sigma_n}\right) + \bar{\sigma}_n \log\left(1 + \frac{\sigma_n}{\bar{\sigma}_n}\right) dx\right). \quad (35)$$

For a large number of particles, one usually uses the saddle-point method. The appropriate equations are called the thermodynamical Bethe ansatz equations. In our case, they are

$$\log\left(\frac{\nu}{\bar{\nu}}\right) = K_n * \log\left(1 + \frac{s_n}{\bar{s}_n}\right) + K_n * \log\left(1 + \frac{\tau_n}{\bar{\tau}_n}\right), \quad (36)$$
$$\log\left(\frac{s_n}{\bar{s}_n}\right) = K_{nm} * \log\left(1 + \frac{s_m}{\bar{s}_m}\right) + K_n * \log\left(1 + \frac{\nu}{\bar{\nu}}\right) +$$
$$+ K_n * \log\left(1 + \frac{\tau_1}{\bar{\tau}_1}\right), \quad (37)$$
$$\log\left(\frac{\tau_n}{\bar{\tau}_n}\right) = K_{nm} \log\left(1 + \frac{\tau_m}{\bar{\tau}_m}\right) + K_{n-1} * \log\left(1 + \frac{\nu}{\bar{\nu}}\right) +$$
$$+ K_n * \log\left(1 + \frac{\tau_1}{\bar{\tau}_1}\right) + \tilde{K}_{nm} \log\left(1 + \frac{\sigma_m}{\bar{\sigma}_m}\right), \quad (38)$$
$$\log\left(\frac{\sigma_n}{\bar{\sigma}_n}\right) = K_{nm} * \log\left(1 + \frac{\sigma_m}{\bar{\sigma}_m}\right) +$$
$$+ \tilde{K}_{nm} \log\left(1 + \frac{\tau_m}{\bar{\tau}_m}\right). \quad (39)$$

Multiplying the equations with free index n by K_{nm}^{-1} and the equations with no free index by $K_1^{-1} + K_1$, we get equations without infinite sums. Then, introducing the notation

$$Y_0^{n+1} = \frac{s_n}{\bar{s}_n}, \quad Y_0^{-n-1} = \frac{\tilde{s}_n}{\bar{\tilde{s}}_n}, \quad (40)$$

$$Y_1^2 = \frac{\nu}{\bar{\nu}}, \quad Y_{-1}^2 = \frac{\tilde{\nu}}{\bar{\tilde{\nu}}}, \quad (41)$$

$$Y_{n-1}^1 = \frac{\bar{\tau}_n}{\tau_n}, \quad Y_{n-1}^{-1} = \frac{\bar{\tilde{\tau}}_n}{\tilde{\tau}_n}, \quad (42)$$

$$Y_{n-1}^0 = \frac{\bar{\sigma}_n}{\sigma_n}, \quad (43)$$

we get the famous Y-system equation

$$Y_s^a\left(u + \frac{i}{2}\right) Y_s^a\left(u - \frac{i}{2}\right) = \frac{1 + Y_{s+1}^a(u) Y_{s-1}^a(u)}{1 + \left(Y_s^{a+1}(u) Y_s^{a-1}(u)\right)^{-1}}. \quad (44)$$

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ІНТЕГРОВНІСТЬ У AdS/CFT ВІДПОВІДНОСТІ

Р е з ю м е

У роботі виконаний огляд сучасного стану розвитку теорії AdS/CFT відповідності для часткового випадку відповідності між суперструнами типу IIB у просторі $AdS_5 \times S_5$ та $\mathcal{N} = 4$ супертеорії Янга–Міллса. Шляхом аналізу зв'язаних станів коренів Бете для відповідного спінового ланцюжку із групою симетрій $PSU(2,2|4)$, показано, як саме виникає система функціональних рівнянь на ґратці, що називається Y-системою.

А.В. Козак

ИНТЕГРИРУЕМОСТЬ В AdS/CFT СООТВЕТСТВИИ

Р е з ю м е

В работе выполнен обзор современного состояния развития теории AdS/CFT соответствия для частного случая соответствия между суперструнами типа IIB в пространстве $AdS_5 \times S_5$ и $\mathcal{N} = 4$ супертеорией Янга–Миллса. Путем анализа связанных состояний корней Бете для соответствующей спиновой цепочки с группой симметрий $PSU(2,2|4)$, показано, как именно возникает система функциональных уравнений на решетке, называющаяся Y-системой.