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UDC 539

AREA QUANTIZATION OF THE PARAMETER SPACE OF RIEMANN SURFACES IN GENUS TWO

We consider a model of compact Riemann surfaces in genus two represented geometrically by two-parametric hyperbolic octagons with an order $\pi/2$ automorphism. We compute the generators of the Fuchsian group and give a real-analytic description of the corresponding Teichmüller space parametrized by the Fenchel–Nielsen variables in terms of geometric data. We state the structure of the parameter space by computing the Weil–Petersson (WP) symplectic two-form and analyzing the isoperimetric orbits. Combining these results, the WP area in the parameter space and the canonical action–angle variables for the orbits are found. Using the ideas from the loop quantum gravity, we apply our formalism to the description of the classical geometrodynamics of Riemann surfaces and the WP area quantization. The results of the paper may be interesting due to their applications to the quantum geometry, chaotic systems, and low-dimensional gravity.

Keywords: Riemann surfaces in genus two, geometrodynamics, area quantization.

1. Introduction

The modern concepts of the area quantization of any manifold is based preferably on the results of quantum gravity/geometry. In both cases, we deal with the principle when any geometric measurable quantities (length and/or angle) are influenced by quantum fluctuations. On the Planckian scale, one cannot mention the standards of length and angle nowadays without accounting for quantum effects. Although the particle interactions and the temperature effects play an essential role in our world, the geometry is universal in the sense that it predicts the trajectory without time and the Hamiltonian. It reflects a purely gauge nature of this science.

The Riemann surface in genus two serves as a geometry carrier in the great number of models[1] of string theory [1], statistical physics [2,3], chaology [4–6], and low-dimensional gravity [7, 8]. Problems, in which the surface geometry is not fixed and is developing in time, are of a special interest. The changes of the underlying geometry can be described in different ways, for instance, by evolution equations, by averaging over surface moduli or parameters, *etc.* Then it is naturally to require the surface deformation to be represented by a continuous smooth trajectory in a some space with properties, which should be carefully studied.

Although the case of genus two gives us access to quite explicit calculations, most of the problems cannot be solved in general. This fact forces us to concentrate on a family of surfaces with a reduced number of geometric degrees of freedom. Using the more convenient geometric approach, we consider the surfaces represented by two-parametric hyperbolic octagons embedded into a unit disk.

Assuming that an octagon form remains the same under the rotation by $\pi/2$, we firstly construct the fundamental domain with opposite sides identified and the associated Fuchsian group, by using the two real parameters as the “input”: length and angle determining the position of vertices, i.e., the octagon geometry. Although the general formalism linking the geometric data and the Fuchsian group is known [9], we pay a great attention to manifest the dependence of the octagon boundary segments and the isometry group generators on these parameters in order to make the functions straightforwardly applicable to the forthcoming calculations.

We aim to investigate a real-analytic structure of the parameter space, that is dictated by the isometry group of a Teichmüller space, which is usually called as the mapping class group and essentially determines an initial octagon evolution in various physical problems. To realize this, we introduce a Te-

[1] Here, we refer to few works but directly related to a given topic.

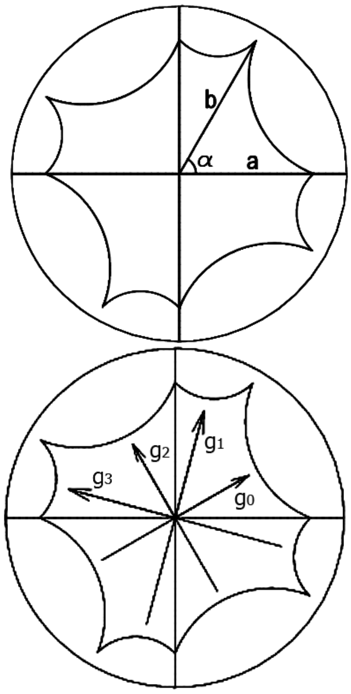

Fig. 1. Symmetric octagon with $a = 0.8$, $\alpha = \pi/3$, and generators g_k of the Fuchsian group

ichmüller space for the surfaces under consideration as a subset of the total Teichmüller space for all surfaces in genus two and compute the Fenchel–Nielsen variables regarding as the global coordinates on it.

We perform the main analysis (in Section 3) within the Weil–Petersson geometry allowing us to endow the parameter space with a symplectic two-form, which is invariant, by definition, under the action of the mapping class group. The key tool is Wolpert's formula [10] allowing us to express this form in terms of the Fenchel–Nielsen (FN) variables. As a result, we shall see that the accessibility domain of geometric data used is a symplectic orbifold. Furthermore, the symmetry group of the reduced Teichmüller space is expected to be wider than the mapping class group because of geometric constraints imposed. Note that the involution of the surfaces with an order four automorphism and the associated generators are discussed in [11] in detail.

We supplement our results by the description of isoperimetric orbits in the parameter space (Section 4), which gives us an additional information about the structure of this space and reflects a particular diffeomorphism of the octagon. On the other hand, the dense set of isoperimetric orbits serves as a tool for the further geometric quantization independent of the octagon automorphisms and the pants decompositions.

To apply the geometric approach to physics, we firstly introduce (in Section 5.1) a pair of canonical action–angle variables for the isoperimetric orbits by associating the action variable with the WP area in the parameter space. This is a main point of our formalism using the global characteristics. Other authors usually operate by the local Fenchel–Nielsen parameters (e.g., see [12]). Then, in our terms, it looks natural to quantize the WP area (the action variable). Such a problem is similar to the one within the loop quantum gravity/cosmology, where the methods and the results of area quantization are intensively studied. However, in order to exploit the gravitational approach, it is necessary to extend the algebra of observables, namely, we should replace the pair of action–angle variables by the generators of the $\mathfrak{su}(1,1)$ algebra associated with the Lorentz algebra reflecting the symmetry of a (2+1)-dimensional space-time. Such an approach opens also the perspective of describing the classical geometrodynamics of the Riemann surfaces as the canonical transformation generated by a boost. Its investigation is performed in Section 5.2 and leads to a "big bounce" (see [13]) in the parameter space.

2. Model Surfaces

We concentrate on the properties of a Riemann surface S in genus $g = 2$, which is understood here as a compact two-dimensional orientable manifold with the Riemannian metric of a constant negative curvature. Such a surface is obtained from a *hyperbolic* simply connected octagon $\mathcal{F}$ embedded into the unit disk $\mathbb{D} = \{z = x + \mathrm{i}y \in \mathbb{C} | |z| < 1\}$, via gluing the opposite sides formed by eight geodesic arcs, whose intersections serve as vertices.

In our model declared and geometrically described in [3], we assume that the vertices are at the points $a\exp(\mathrm{i}k\pi/2)$, $b\exp[\mathrm{i}(\alpha + k\pi/2)]$, where $0 < \alpha < \pi/2$, $0 < a, b < 1$, $k = \overline{0,3}$ (see Fig. 1, top panel). We also require the sum of the inner angles of $\mathcal{F}$ to be equal to 2π. This is the same as requiring $\mathrm{area}(\mathcal{F}) = 2\pi(2g-2) = 4\pi$ in accordance with the Gauss–Bonnet theorem [14].

The octagon we have obtained is stable under the rotation by $\pi/2$. This means that the surface has an order four automorphism. Note that the connection between this geometric model and algebraic curves was intensively explored in the works of P. Buser and R. Silhol (e.g., see [11, 15] and references therein).

Actually, the model octagon is two-parametric, and we choose a pair (a, α) as independent real variables, while the parameter b together with the parameters of geodesics (sides) are functions of those. It can be shown that $b = (\sqrt{2}a \cos\tilde{\alpha})^{-1}$, where $\tilde{\alpha} = \alpha - \pi/4$. Therefore, the model octagon $\mathcal{F}$ can be viewed as a "minimal deformation" of the regular hyperbolic octagon with $b = a = 2^{-1/4}$, $\alpha = \pi/4$, well studied in the context of the chaology (see, e.g., [5] and references therein).

Note that to manifest the dependence of the octagon parameters on the pair (a, α) is necessary in the different problems, where geometry is not fixed. For instance, (a, α) would be dynamical variables in topological field theory and gravity; it is able to average over (a, α) in statistical physics, *etc.*

Due to the Gauss–Bonnet theorem, one can determine the domain of variety of the parameters (a, α):

$$-\pi/4 < \tilde{\alpha} < \pi/4, \quad (\sqrt{2}\cos\tilde{\alpha})^{-1} < a < 1, \tag{1}$$

which is sketched in Fig. 3. We denote this parameter space by $\mathcal{A}$, whose points completely determine the geometry of the octagon $\mathcal{F}$. Our aim is to investigate the structure of $\mathcal{A}$ and to quantize it.

Since the opposite sides of $\mathcal{F}$ have the same lengths by construction, we have, therefore, a uniquely defined orientation-preserving isometry g_k mapping the geodesic boundary segment s_{k+4} onto s_k for all $k = \overline{0,3}$ (see Fig. 1, bottom panel). For these isometries, we get $g_k[\mathcal{F}] \cap \mathcal{F} = s_k$, where $g_k[\mathcal{F}]$ means the set $\{g_k[z] | z \in \mathcal{F}\}$. Pasting the sides s_{k+4} and s_k together by identifying any $z \in s_{k+4}$ with $g_k[z] \in s_k$, we obtain a closed surface in genus two that carries the hyperbolic metric inherited from $\mathcal{F}$.

Four isometries g_k and their inverses g_k^{-1} generate the Fuchsian group Γ (isomorphic to the fundamental group π_1) with a single relation,

$$g_0 g_1^{-1} g_2 g_3^{-1} g_0^{-1} g_1 g_2^{-1} g_3 = \mathrm{id}. \tag{2}$$

Then the surface S is purely defined as a quotient $\mathbb{D}/\Gamma$, and $\pi : \mathbb{D} \to S$ is the natural covering map. This is a Fuchsian model Γ of the Riemann surface S under consideration.

The isometries g_k in the unit disk model are naturally presented by matrices belonging to the $SU(1,1)$ group and acting by the following rule:

$$z \mapsto \gamma[z] = \frac{uz + v}{\overline{v}z + \overline{u}}, \quad \gamma = \begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix}, \tag{3}$$

where $|u|^2 - |v|^2 = 1$, and the bar symbol means complex conjugation.

For $g_k^{\pm 1} \in \Gamma$ defined via the so-called half turns, let p_k be the mid-point of the k-th side, $k = \overline{0,3}$. The generators are then written as $g_k = H(p_k)$ (see [9]), where

$$H(p) = \frac{-1}{1 - |p|^2} \begin{pmatrix} 1 + |p|^2 & 2p \\ 2\overline{p} & 1 + |p|^2 \end{pmatrix}. \tag{4}$$

The operation of the matrices $H(p)$ consists of the composition of the half turn (rotation with angle π) of a geodesic segment around the origin $z = 0$ and the half turn around the point p.

Due to the symmetry of our model, $p_0 = p_+$, $p_1 = p_-$, $p_2 = \mathrm{i}p_+$, and $p_3 = \mathrm{i}p_-$, where

$$p_\pm = \frac{\omega_\pm}{1 + \sqrt{1 - |\omega_\pm|^2}}, \tag{5}$$

$$\omega_\pm = \frac{b e^{\mathrm{i}\alpha}(1 - a^2) + a e^{\mathrm{i}\pi(1 \mp 1)/4}(1 - b^2)}{1 - a^2 b^2}. \tag{6}$$

Note that the explicit form of the generators g_k in terms of (a, α) is shown in [21].

Since the different octagons may lead to the same surface, we mark a surface by generators of Γ. Two marked surfaces (S, Γ) and (S', Γ') are called marking equivalent if there exists an isometry $\gamma : S \to S'$ satisfying $g'_k = \gamma g_k \gamma^{-1}$ $(k = \overline{0,3})$. Then all marking equivalent surfaces form a marking equivalence class $[S, \Gamma]$ representing the Riemann surface S together with a structure defined on it.

It is useful sometimes to mark a surface by selecting a curve system Σ of simple closed geodesics on it. Then the marking equivalence also means the existence of the isometry $\gamma : S \to S'$ sending $\Sigma \to \Sigma'$. In this case, the equivalence class is formed by a pair $[S, \Sigma]$.

The set of all marking equivalence classes of the closed compact Riemann surfaces in genus g forms the Teichmüller space denoted by $\mathcal{T}_g$. The definition of $\mathcal{T}_g$ depends in general on the choice of a marking of Riemann surfaces. In any case, the real dimension

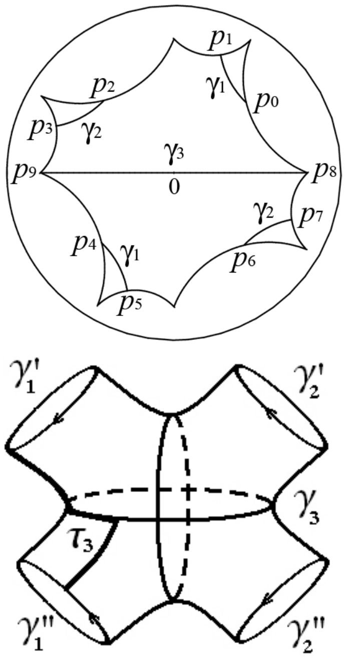

Fig. 2. Top panel: Pants decomposition of the octagon with $a = 0.8$, $\alpha = \pi/3$. Bottom panel: Two pants glued along γ_3

of $\mathcal{T}_g$ like a vector space equals $6g - 6$ in accordance with the Riemann–Roch theorem. We immediately note that the Riemann surfaces constructed with geometrical constraints imposed above result only in the subset of the total $\mathcal{T}_2$ of dimension six. In this sense, we call such a space as the *reduced* Teichmüller one.

3. Weil–Petersson Symplectic Two-Form

In this section, we firstly introduce the Fenchel–Nielsen variables by means of the pants decomposition of the surface. However, we omit the most of computations, which have been already performed in [21]. Using these variables, the main aim of this section is to write down the Weil–Petersson symplectic two-form in the parameter space $\mathcal{A}$ that is needed for the further calculations.

A starting point of the following constructions is the fact that a hyperbolic Riemann surface in genus g without boundary always contains a system of $3g-3$ simple closed geodesics that are neither homotopic to one another nor homotopically trivial. Regardless of which curve system we choose, the cut along these geodesics always decomposes the surface into $2g - 2$ pairs of pants (three-holed spheres), playing a role of natural building blocks for Riemann surfaces (e.g., see [16]).

In the case at hand, the surface S constructed is a two-holed torus, which can be decomposed into two pairs of pants by a system of three closed geodesics. Such a surgery allows us to calculate the global Fenchel–Nielsen (FN) parameters: lengths of these geodesics and twists, needed for the further investigation and defined as follows.

Let us consider the geodesic arcs from p_0 to p_1 and from p_5 to p_4 on the octagon $\mathcal{F}$ (see Fig. 2, top panel). On the surface S obtained by gluing the sides of the octagon, these two arcs together form a smooth closed geodesic γ_1. Similarly, a closed geodesic γ_2 is obtained from the arcs running from p_2 to p_3 and from p_7 to p_6, respectively. The line p_8p_9 results in a closed geodesic γ_3.

The triple γ_1, γ_2, γ_3 dissects S into two pairs of pants determined up to the isometry by the hyperbolic lengths ℓ_k, $k = \overline{1,3}$.

Note that hyperbolic distance between the complex coordinates z and w in the unit disk model is denoted by $\mathrm{dist}_{\mathbb{D}}(z, w)$ and determined from the relation

$$\cosh \mathrm{dist}_{\mathbb{D}}(z, w) = 1 + \frac{2|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}, \tag{7}$$

where $|z - w|$ is the Euclidean distance.

Then, the immediate calculations yield

$$\ell_{1,2} \equiv 2\ \mathrm{dist}_{\mathbb{D}}(p_+, p_-) = 2\ \mathrm{arccosh}\ \frac{a^2}{1 - a^2}, \tag{8}$$

$$\ell_3 \equiv 2\ \mathrm{dist}_{\mathbb{D}}(0, a) = 2 \ln \frac{1 + a}{1 - a}, \tag{9}$$

where $\mathrm{dist}_{\mathbb{D}}(p_{n-1}, p_n) = \mathrm{dist}_{\mathbb{D}}(p_+, p_-)$ for $n = 1, 3, 5, 7$.

When the pairs of pants are pasted together again to recover S, there arise additional degrees of freedom at each γ_k, named the twist parameters τ_k and defined as follows. On each pair of pants, one takes disjoint orthogonal geodesic arcs between each pair of boundary geodesics. It is known that the feet of two perpendiculars on each geodesic are diametrically opposite. Let us paste together two tubular neighborhoods of pair(s) of pants with the boundaries of closed geodesics γ_k' and γ_k'' of the same orientation and hyperbolic length, and let us denote the weld by γ_k (see Fig. 2, bottom panel). In principle, the feet of perpendiculars, arriving at the previously separated

γ'_k and γ''_k, do not coincide on γ_k. The twist parameter τ_k is then the hyperbolic distance (shift) along γ_k between the feet of perpendiculars on opposite sides of the weld. Globally, the surfaces arising from different τ_k are not in general isometric. This fact is often used for the investigation of Riemann surface deformations [16–18].

Let us now concentrate on computational aspects. One of the convenient methods of computation of geodesic lengths is the matrix formalism. Here, we have used the algorithms from [9] based on it and realized in [21].

The twists in terms of model parameters are

$$\tau_{1,2} = \operatorname{arccosh}\left[\frac{2a^2-1}{a^2(1-b^2)} - 1\right], \quad \tau_3 = \ln\frac{1+a}{1-a}. \quad (10)$$

It is known that the Teichmüller space of marked Riemann surfaces in genus two forms a manifold homeomorphic to $\mathbb{R}^6$. This fact allows one to identify the FN variables with global coordinates on it. However, the Teichmüller space carries an additional structure, namely, the Weil–Petersson (WP) symplectic two-form. Actually, it is the imaginary part of a natural Kählerian metric. Due to the Wolpert theorem [10, 19] (see also Thm. 3 in [20]), the WP two-form for compact closed Riemann surfaces in genus g takes on a particularly simple form in terms of the FN variables,

$$\omega_{\mathrm{WP}} = \frac{1}{2}\sum_{k=1}^{3g-3} \mathrm{d}\ell_k \wedge \mathrm{d}\tau_k, \quad (11)$$

with respect to any pants decomposition. This says in the sense of theoretical mechanics that ℓ_k play the role of the action variables, whereas $\theta_k = 2\pi\tau_k/\ell_k$ are the angle variables. Indeed, the simple Dehn twist $\theta_k \to \theta_k + 2\pi$ gives us isometrically the same surface.

Using the pants decomposition presented in Fig. 2 (top panel) and substituting the functions ℓ_k and τ_k of (a, α) into (11), the WP symplectic form becomes

$$\omega_{\mathrm{WP}} = \frac{8a}{(1-a^2)(2a^2\cos^2\tilde{\alpha} - 1)}\mathrm{d}a \wedge \mathrm{d}\tilde{\alpha}. \quad (12)$$

To verify the uniqueness of the last formula, let us consider another pants decomposition by changing the connections between the arc mid-points and the main diagonal of the octagon, which gives us new $\gamma'_{1,2}$ and γ'_3, respectively. It is easily seen that a performed decomposition simply leads to the replacements,

$$a \leftrightarrow b, \qquad \tilde{\alpha} \leftrightarrow -\tilde{\alpha}, \quad (13)$$

in the length and twist functions of the previous decomposition.

Although we have obtained the set of new functions, the resulting two-form remains the same, that is, $\omega'_{\mathrm{WP}} = \omega_{\mathrm{WP}}$ due to the fact that $\mathrm{sgn}\tau_k = -\mathrm{sgn}\tau'_k$.

Thus, we can conclude that i) the permission domain $\mathcal{A}$ of the parameters (a, α) is a non-trivial symplectic manifold $(\mathcal{A}, \omega_{\mathrm{WP}})$; ii) the Weil–Petersson symplectic two-form (12) is closed and invariant under the action of the $\mathbb{Z}_2$ group represented by transformation (13). Formally, we can treat form (12) as an area element of the manifold $\mathcal{A}$ associated with the moduli space of Riemann surfaces under consideration.

Furthermore, introducing the quantities

$$T_k^{(\prime)} \equiv \cosh\frac{\tau_k^{(\prime)}}{2}, \quad L_k^{(\prime)} \equiv \cosh\frac{\ell_k^{(\prime)}}{2}, \quad (14)$$

we can establish the following relations among them:

$$L_3^{(\prime)} \equiv \cosh\frac{\ell_3^{(\prime)}}{2} = 2L_{1,2}^{(\prime)} + 1, \quad \tau_3^{(\prime)} = \ell_3^{(\prime)}/2, \quad (15)$$

$$T'_1 = \sqrt{\frac{L_1^2T_1^2 + L_1T_1^2 - L_1^2 + 1}{2L_1T_1^2 - L_1 + 1}}, \qquad L'_1 = T_1^2\frac{2L_1}{L_1 - 1} - 1. \quad (16)$$

These formulas reflect the symmetry of the model in terms of geometric constraints and correspond to a special case of the surface with an order four automorphism previously studied in ([11], Lm. 3.5).

4. Isoperimetric Curves in $\mathcal{A}$

We can also obtain additional information about the structure of $\mathcal{A}$ by means of the analysis of principal geometric characteristics. One of those is an area fixed by the Gauss–Bonnet theorem and equal to 4π for genus two. Therefore, the area cannot obviously be the measure of an octagon deformation (evolution) preserving genus.

The simplest way to describe changes globally consists in the consideration of the perimeter of a hyper-

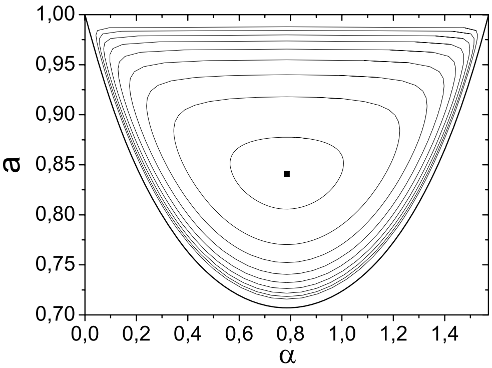

Fig. 3. Orbits of constant perimeter in the plane of octagon parameters

bolic octagon. Within the present model, the perimeter is given by the formula:

$$P = 8\ \mathrm{arccosh}\ \frac{1 - a^2b^2 + \sqrt{(1-a^2)^2 + (1-b^2)^2}}{(1-a^2)(1-b^2)}. \quad (17)$$

In a some sense, this characteristic is a good candidate due to the invariance of P under the octagon automorphisms and the pants decomposition. This means that P can take on the same value for various values of (a, α). In this section, we are aiming to describe the corresponding orbits.

For the further investigation, it is useful to introduce two auxiliary quantities,

$$T \equiv \tanh{(P/16)}, \quad \varepsilon = \pm 1, \quad (18)$$

where the latter one reflects the existence of two symmetric sheets in $\mathcal{A}$ labeled by sign $\tilde{\alpha}$.

For a given $T(P)$, the maximal and minimal values of parameter a are found at $\tilde{\alpha} = 0$, when $b = (\sqrt{2}a)^{-1}$. We get

$$a_\pm(T) = \frac{1}{2}\sqrt{2 + T^2 \pm \sqrt{(2+T^2)^2 - 8}}. \quad (19)$$

This means that one can parametrize a as follows:

$$a(T, \varphi) = \frac{1}{2}\sqrt{2 + T^2 + \cos\varphi\sqrt{(2+T^2)^2 - 8}}, \quad (20)$$

where the cyclic variable $\varphi \in [0, 2\pi)$ is used.

Let us now solve the algebraic equation $(2+T^2)^2 = 8$. We immediately obtain that $T_{\rm reg} = \sqrt{2\sqrt{2} - 2}$, $P_{\rm reg} = 8\ \mathrm{arccosh}\ \left(5 + 4\sqrt{2}\right)$, and $a_{\rm reg} = 2^{-1/4}$. At $\tilde{\alpha} = 0$, these quantities correspond to the regular hyperbolic octagon, as it must be. Thus, the trajectory in $\mathcal{A}$ for $P_{\rm reg}$ is contracted to a point. Moreover, $P_{\rm reg}$ is a minimal value of P among the possible ones. Therefore, the maximal symmetry of the regular octagon explains an extremum of information entropy observed previously in [3]. This fact could be important in the description of the physical systems, in which the geometry carrier (two-holed torus) changes.

Substituting (20) in (17) and resolving the equation obtained with respect to $\tilde{\alpha}$, we deduce that

$$\tilde{\alpha}(T, \varphi) = \arctan \frac{\sqrt{2}\sqrt{(2+T^2)^2 - 8}\sin\varphi}{2\sqrt{3T^2 - 2 - \cos\varphi\sqrt{(2+T^2)^2 - 8}}}. \quad (21)$$

Equations (20) and (21) allow us to reproduce the orbits $P = \text{const}$, presented in Fig. 3. The point corresponds to the parameters of the regular octagon ($P_{\rm reg} \approx 24.45713$); cyclic curves are orbits for P from $P = 25$ to $P = 41$ with step 2.

Since the set of orbits is dense in $\mathcal{A}$, there arises a possibility to geometrically quantize the symplectic orbifold $\mathcal{A}$ in a spirit of [2]. In order to realize it, it is necessary to consider a Weil- Petersson (WP) area $A_{\rm WP}(P)$ of the domain in $\mathcal{A}$ bounded by the isoperimetric orbit for some fixed P. Physically, $A_{\rm WP}(P)$ can be treated as an action variable, that is, the only integral of motion $\{a(T,t),\ \alpha(T,t) | t \in \mathbb{R}\}$. Canonical quantization in terms of $A_{\rm WP}(P)$ and the conjugate angle variable has to give us the number of quantum states inside a domain in $\mathcal{A}$. We develop the quantum geometry of $\mathcal{A}$ and the reduced Teichmüller space in the next section.

Here, using the WP symplectic form (12) and Eqs. (20) and (21), we limit ourselves by the introduction and the analysis of the WP area

$$A_{\rm WP}(P) = \int\limits_{P_{\rm reg}}^{P} \frac{8a}{(1-a^2)(2a^2\cos^2\tilde{\alpha} - 1)} \mathrm{d}a\mathrm{d}\tilde{\alpha}. \quad (22)$$

This double integral is reduced to a single one,

$$A_{\rm WP}(P) = \int\limits_{x_-(T)}^{x_+(T)} \frac{8\mathrm{d}x}{(1-x)\sqrt{2x - 1}}\ \mathrm{arccosh}\ f(x, T), \quad (23)$$

where $x \equiv a^2$; the functions $x_\pm(T) \equiv a_\pm^2(T)$ are determined by (19);

$$f(x,T) = \frac{1}{\sqrt{1-T^2}}\sqrt{\frac{2x-1}{x}}\sqrt{\frac{T^2-x}{T^2-2x+1}}, \tag{24}$$

and convention (18) is applied.

Further calculations are performed numerically, and the result is demonstrated in Fig. 4. The semi-analytical analysis shows that the curve $A_{\rm WP}(P)$ at relatively small $P - P_{\rm reg}$ can be approximated by a parabola, $c_1(P-P_{\rm reg})^2 + c_2(P-P_{\rm reg})$, with accuracy of the order $\mathrm{O}(\exp(-P/8))$. The best fit in the presented range of P gives $c_1 = 0.05622$, $c_2 = 2.62132$. In principle, the parameters $c_{1,2}$ slowly depend on P, so that $c_1 \to 1/16 = 0.0625$ as $P \to \infty$.

5. Quantization of $A_{\rm WP}$

In this section, we are aiming to quantize a domain area in the parameter space $\mathcal{A}$ on the basis of the symplectic form (12) and the idea to cover $\mathcal{A}$ by isoperimetric orbits. Such an approach requires one to define the action and the angle variables. The former, as mentioned above, is the WP area (up to a constant multiplier) as a function of P. At the present stage, the angle variable is unknown and should be found. Actually, we need to find a canonical transformation from the local Fenchel–Nielsen parameters to the global ones like the WP area, ware invariant under automorphisms and the pants decomposition, that is, the mapping class group.

Thus, our formalism allows us to describe the diffeomorphism preserving the perimeter of a hyperbolic octagon and to evaluate the number of quantum "cells" in $A_{\rm WP}$. It seems to be simple at first sight. We would like to apply our formalism to quantize a physically reasonable system. Here, we stop at the model with the $SU(1,1)$ symmetry corresponding to the (2+1)-dimensional gravity, inspiring us by the known results of the area quantization.

Thus, we start from finding the angle variable. To realize this, let us re-write $\omega_{\rm WP}$ in terms of $x = a^2$ and P. We get

$$\omega_{\rm WP} = \frac{\sqrt{2}\varepsilon(1-T^2)}{8\sqrt{(x-x_-)(x_+-x)(T^2-x)}}\times$$
$$\times\left(\frac{1}{1-T^2} - \frac{1}{1-x} + \frac{1}{T^2-2x+1}\right)\mathrm{d}x \wedge \mathrm{d}P, \tag{25}$$

where the notation from the previous section is used.

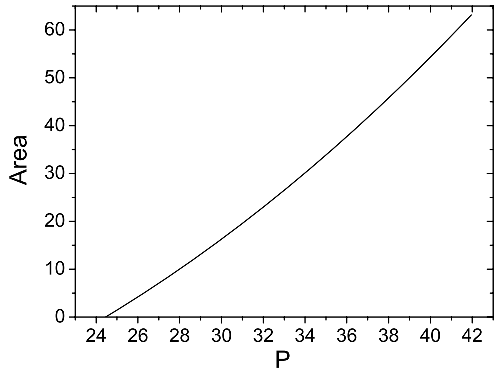

Fig. 4. WP area of a domain bounded by the isoperimetric curve $P = \mathrm{const}$

Now, let us define the function

$$Q(x,T) = \frac{\sqrt{2}}{4}\frac{1-T^2}{\sqrt{T^2-x_-}}\times$$
$$\times\left[\frac{F(u,k)}{1-T^2} - \frac{\Pi(u,\nu_1,k)}{1-x_-} + \frac{\Pi(u,\nu_2,k)}{T^2-2x_-+1}\right], \tag{26}$$

where F and Π are elliptic integrals of the first and third kinds, respectively.

The other quantities are

$$u = \sqrt{\frac{x-x_-}{x_+-x_-}}, \quad k = \sqrt{\frac{x_+-x_-}{T^2-x_-}},$$
$$\nu_1 = \frac{x_+-x_-}{1-x_-}, \quad \nu_2 = 2\frac{x_+-x_-}{T^2-2x_-+1}, \tag{27}$$

where u is the amplitude; k is the module; ν_1 and ν_2 are parameters.

Then, it is easy to verify that the symplectic form becomes

$$\omega_{\rm WP} = \varepsilon \mathrm{d}Q(x,T) \wedge \mathrm{d}P. \tag{28}$$

On the other hand, the integration over x yields

$$\mathrm{d}A_{\rm WP} = \left(\oint\limits_{P=\mathrm{const}} \varepsilon \mathrm{d}Q(x,T)\right)\mathrm{d}P =$$
$$= \left(\int\limits_{x_-}^{x_+}\mathrm{d}Q(x,T) - \int\limits_{x_+}^{x_-}\mathrm{d}Q(x,T)\right)\mathrm{d}P =$$
$$= 2Q(x_+,T)\mathrm{d}P. \tag{29}$$

This means that

$$\frac{\mathrm{d}A_{\mathrm{WP}}}{\mathrm{d}P} = \frac{\sqrt{2}}{2}\frac{1-T^2}{\sqrt{T^2-x_-}}\times$$
$$\times\left[\frac{K(k)}{1-T^2} - \frac{\Pi(\nu_1,k)}{1-x_-} + \frac{\Pi(\nu_2,k)}{T^2-2x_-+1}\right], \quad (30)$$

where the complete elliptic integrals of the first and third kinds are used.

Note that it is the exact formula, while the value of $A_{\mathrm{WP}}(P)$ is obtained numerically or by using an approximation, if we want it in analytic form.

Now, it is easily seen from (28) that the function $\Omega(x,T) = \varepsilon Q(x,T)$ satisfies the relation

$$\{P,\Omega\}_{\mathrm{WP}} = 2, \quad (31)$$

which defines the Poisson bracket.

Defining the action variable or "angular momentum" as

$$J(P) = \frac{1}{4\pi}A_{\mathrm{WP}}(P), \quad (32)$$

we can find the angle variable Φ from the equation $\omega_{\mathrm{WP}} = \varepsilon\mathrm{d}Q(x,T)\wedge\mathrm{d}P = 2\mathrm{d}\Phi\wedge\mathrm{d}J$. In other words, we have

$$\mathrm{d}\Phi = 2\pi\left(\frac{\mathrm{d}A_{\mathrm{WP}}(P)}{\mathrm{d}P}\right)^{-1}\varepsilon\mathrm{d}Q(x,T)\Bigg|_{P=\mathrm{const}} =$$
$$= \frac{\pi\varepsilon}{Q(x_+,T)}\mathrm{d}Q(x,T)\Bigg|_{P=\mathrm{const}}. \quad (33)$$

Using the rule of calculation of the integrals containing ε, which guarantees the counterclockwise integration (see (29)), one sees immediately that

$$\oint\limits_{P=\mathrm{const}} \mathrm{d}\Phi = 2\pi, \quad (34)$$

as it must be.

At $J(P) = \mathrm{const}$, we arrive at the exact expression for the angle variable:

$$\Phi = \pi\varepsilon\frac{Q(x,T)}{Q(x_+,T)}, \quad \Phi\in[-\pi,\pi]. \quad (35)$$

Note that $\{J,\Phi\}_{\mathrm{WP}} = 1$.

It is also worth noting that, for any Hamiltonian function $H = H(P)$, the angular frequency of "rotation" along the isoperimetric orbit $P = \mathrm{const}$ is given by

$$\dot{\Phi} = \{H,\Phi\}_{\mathrm{WP}} = \frac{2\pi}{Q(x_+,T)}\frac{\mathrm{d}H(P)}{\mathrm{d}P}. \quad (36)$$

This means that the frequency does not depend on the evolution parameter in accordance with the theorems of mechanics.

Now, it seems trivial to quantize the system with one degree of freedom in terms of the J and Φ variables, which leads immediately to the estimation (in the Planck units)

$$A_{\mathrm{WP}} \sim 4\pi n, \quad n\in\mathbb{N} \quad (37)$$

for positive and relatively large n.

However, we specify this formula due to the consideration of a physical system closely related to the low-dimensional gravity.

The geometrodynamics of Riemann surfaces within the considered model is undefined because of its purely gauge nature. This means that there were no time and Hamiltonian having the physical meaning and generating the evolution of J and Φ simultaneously. To resolve this problem, we appeal here to the ideas from the (2+1)-dimensional quantum gravity, where $SO(2,1)\sim SU(1,1)$ plays the role of the Lorentz group. We construct a model with the same symmetry as follows.

First, combining the basic variables J and Φ, we expand the set of observables up to

$$J_0 = J, \quad J_\pm = \sqrt{J^2 - C}\exp(\mp\mathrm{i}\Phi), \quad (38)$$

where $C \geqslant 0$ is an arbitrary constant for a moment, requiring to be $J^2 \geqslant C$.

One immediately sees that the Poisson algebra of these observables is the $\mathfrak{su}(1,1)$ Lie algebra:

$$\{J_+,J_-\}_{\mathrm{WP}} = 2\mathrm{i}J_0, \quad \{J_\pm,J_0\}_{\mathrm{WP}} = \pm\mathrm{i}J_\pm. \quad (39)$$

More generally, the generators $J_{0,\pm}$ may be replaced by the infinite number of quantities $L_n = J\exp(\mathrm{i}n\Phi)$, $n\in\mathbb{Z}$ generating the Witt algebra.

Since the evolution is usually described by canonical transformations, let us check that, indeed, $SU(1,1)$ transformations are canonical transformations of a given system. Let us introduce the 2×2 matrix

$$\mathcal{M} = \begin{pmatrix} J_0 & J_+ \\ J_- & J_0 \end{pmatrix}, \quad (40)$$

whose determinant is the Casimir operator of the $\mathfrak{su}(1,1)$ algebra

$$C \equiv J_0^2 - J_+J_-. \quad (41)$$

The action of a generic $SU(1,1)$ element,

$$U = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad |u|^2 - |v|^2 = 1, \tag{42}$$

reads

$$\mathcal{M} \to \tilde{\mathcal{M}} = U\mathcal{M}U^\dagger, \tag{43}$$

which preserves the Casimir operator, $C = \det\tilde{\mathcal{M}}$.

Accordingly to the loop quantum gravity concept, the evolution is simply generated by the boost [13]:

$$U_\tau = \begin{pmatrix} \cosh(\tau/2) & \sinh(\tau/2) \\ \sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix}. \tag{44}$$

Then we can derive the trajectories for $J_{0,\pm}$ or, equivalently, for J and Φ, by computing $\mathcal{M}(\tau) = U_\tau\mathcal{M}(0)U_\tau^\dagger$ in accordance with (43). The evolution of J and Φ has to demonstrate the "big bounce" [13] in the parameter space by construction. If the Casimir invariant $C = 0$, the evolution starts at $\tau = -\infty$ from $(a = 1, \alpha = \pi/4)$ and ends at $\tau = +\infty$ with the parameters $(a = 1/\sqrt{2}, \alpha = \pi/4)$. The initial configuration is realized at $\tau = 0$.

Since J_0 describes the WP area $A_{\rm WP}$, its eigenvalues should be discrete and positive. Accordingly to the loop quantum gravity, we choose the irreducible representation with the standard basis diagonalizing the Casimir operator and J_0 and with the minimal positive spin $j = 1/2$, when $C = 1/4$. This leads to the spectrum (in the Planck units)

$$A_{\rm WP} = 4\pi\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}. \tag{45}$$

We can conclude that, in the classical picture, $A_{\rm WP} \equiv 0$ for $P_{\rm reg}$. At the quantum level, $A_{\rm WP} \neq 0$ always, i.e., there is a gap of the Planckian scale.

6. Conclusions

We have considered a simple model of the Riemann surface in genus two, represented by the two-parametric hyperbolic octagon embedded into a unit disk. We have shown that the parameter space associated with the moduli of the surface is not trivial. This is actually a symplectic orbifold. The fundamental symplectic two-form is induced on the base of Wolpert's theorems within the Weil–Petersson geometry. Moreover, the parameter space can be densely covered by a set of isoperimetric orbits. Each orbit determines a closed domain, whose area plays the role of a new global variable.

Instead of the local Fenchel–Nielsen parameters, we have proposed to use the Weil–Petersson area, which is invariant under automorphisms and the mapping class group, and the canonically conjugate angle in order to describe the surface configuration.

We have applied our formalism to the surface geometrodynamics, using the symmetries of the low-dimensional gravity. Such an approach allows us to quantize the Weil–Petersson area of a domain in the parameter space. It turns out that the area spectrum is equidistant and reproduces the known results.

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КВАНТУВАННЯ ПЛОЩІ ПРОСТОРУ ПАРАМЕТРІВ РІМАНОВИХ ПОВЕРХОНЬ РОДУ ДВА

Р е з ю м е

Ми розглядаємо модель компактних ріманових поверхонь роду два, представлених геометрично двопараметричними октагонами з $\pi/2$ автоморфізмами. Ми обчислюємо генератори групи Фукса і даємо аналітичний опис простору Тейхмюллера, параметризований змінними Фенхеля–Нільсона, у термінах геометричних даних. Ми визначаємо структуру простору параметрів за допомогою обчислення симплектичної два-форми Вейля–Петерссона (ВП) та аналізу ізопериметричних орбіт. Комбінуючи ці результати, знаходимо ВП площу у просторі параметрів, а також канонічні змінні дія–кут. Використовуючи ідеї з петльової квантової гравітації, ми застосуємо наш формалізм для опису класичної геометродинаміки, а також для квантування ВП площі. Результати праці можуть бути застосовні до квантової геометрії, хаотичних систем і низько-розмірної гравітації.

А.В. Назаренко

КВАНТОВАНИЕ ПЛОЩАДИ ПРОСТРАНСТВА ПАРАМЕТРОВ РИМАНОВЫХ ПОВЕРХНОСТЕЙ РОДА ДВА

Р е з ю м е

Мы рассматриваем модель компактных римановых поверхностей рода два, представленных геометрически двапараметрическими октагонами с $\pi/2$ автоморфизмами. Мы рассчитываем генераторы группы Фукса и даём аналитическое описание пространства Тейхмюллера, параметризованного глобальными параметрами Фенхеля–Нильсена, в терминах геометрических данных. Мы определяем структуру пространства параметров, вычисляя симплектическую фундаментальную два-форму Вейля–Петерссона и изопериметрические орбиты. Находим, комбинируя полученные результаты, площадь Вейля–Петерссона в пространстве параметров и канонические переменные действие–угол. Используя идеи из петлевой квантовой гравитации, мы применяем наш формализм для описания классической геометродинамики и квантования площади Вейля–Петерссона. Результаты статьи могут быть интересны в квантовой геометрии, хаотических системах и низко-размерной гравитации.