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## A RATIO OF THE SHEAR VISCOSITY TO THE DENSITY OF ENTROPY FOR HELIUM

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We have studied the ratio  $(\eta/s)$  of the shear viscosity  $\eta$  to the density of entropy  $s$  for helium as a function of the temperature and have established that the minimal value,  $(\eta/s)_{\min}$ , satisfies the Kovtun–Son–Starinets inequality,  $(\eta/s)_{\min} \geq (\hbar/4\pi k_B)$ .

*Key words:* Helium, ratio of the shear viscosity to the density of entropy, Kovtun–Son–Starinets inequality.

### 1. Introduction

In 1998, J.M. Maldacena established a gauge-string relation [1], which appears to be a very efficient method to study the strongly interacting quantum systems. Using this relation, P.K. Kovtun, D.T. Son, and A.O. Starinets showed in 2005 that, for such systems, the ratio of the shear viscosity  $\eta$  to the entropy density  $s$  satisfies the relation

$$(\eta/s) \geq (\hbar/4\pi k_B) \simeq 6.08 \times 10^{-13} \text{ K} \cdot \text{s},$$

and the equality is reached only for the so-called perfect liquid [2] (see, e.g., the recent review [3]). Here,  $\hbar$  is the Planck constant, and  $k_B$  is the Boltzmann constant. The authors provided evidence that this inequality could be valid for a wider class of systems. Till now, this statement was tested and proved for a number of substances (see, e.g., [4, 5]).

In this paper, we study the temperature dependence of the ratio  $(\eta/s)$  and consider the applicability of the above inequality for helium.

### 2. Liquid Helium

At temperatures less than  $T_\lambda = 2.18 \text{ K}$ , helium is liquid. According to L. Tisza and L. Landau, we can imagine ourselves the liquid helium as a mixture of two “liquids”: superfluid liquid and normal liquid. If we assume the liquid helium incompressible, then the

equation for it splits into two equations: an equation for superfluid liquid and an equation for normal liquid. Of two liquids, only the normal liquid is viscous, and its hydrodynamics for small velocities is described with the equation

$$\frac{\partial}{\partial t} \rho v_i + \frac{\partial \Pi_{ik}}{\partial x_k} = 0,$$

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right),$$

where  $\eta$  is the shear viscosity.

At temperatures  $T \ll T_\lambda$  (more exactly, at  $T < 0.8\text{--}1.0 \text{ K}$ ), we can consider the normal liquid as a Bose gas of phonons with the energy spectrum  $\epsilon(p) = pc$  where  $c$  is a velocity of sound. Therefore, the mass density  $\rho_{nph}$  and the entropy  $S_{ph}$  of the normal liquid are described with the expressions

$$\rho_{nph} = \frac{2\pi^2}{45\hbar^3 c^5} (k_B T)^4, \quad S_{ph} = \frac{2\pi^2}{45\hbar^3 c^3} (k_B T)^3 k.$$

At such temperatures, the viscosity of the normal liquid is defined by phonon scattering processes [6]:

$$\eta_{ph} = (1/5)c^2 \rho_{nph} \tau_{ph},$$

where the inverse mean free time

$$\frac{1}{\tau_{ph}} = \frac{9 \cdot 13!}{2^{13}} (u+1)^4 \left( \frac{k_B T}{2\pi\hbar} \right)^7 \left( \frac{k_B T}{\rho c^5} \right)^2, \quad u = \frac{\rho}{c} \frac{\partial c}{\partial \rho}.$$

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Therefore, the viscosity of the normal helium looks as follows:

$$\eta_{ph} = \frac{2^{21} \pi^9}{5^2 \cdot 9^2 \cdot 13!} \frac{1}{(u+1)^4} \left(\frac{\hbar}{\rho c}\right)^3 \left(\frac{\rho c^2}{k_B T}\right)^5 \hbar.$$

At last, the ratio of the shear viscosity  $\eta$  to the volume density of entropy  $s$  is

$$\frac{\eta_{ph}}{s_{ph}} = \frac{2^{20} \pi^7}{5 \cdot 9 \cdot 13!} \frac{1}{(u+1)^4} \left[\left(\frac{\hbar}{\rho c}\right)^3 \left(\frac{\rho c^2}{k_B T}\right)^4\right]^2 \frac{\hbar}{k_B}.$$

We can present this expression as

$$\frac{\eta_{ph}}{s_{ph}} = \left(\frac{\eta_{ph}}{s_{ph}}\right)_{T=T_1} \left(\frac{T_1}{T}\right)^8, \quad (1)$$

where

$$\left(\frac{\eta_{ph}}{s_{ph}}\right)_{T=T_1} = \frac{2^{22} \pi^8}{5 \cdot 9 \cdot 13!} \frac{1}{(u+1)^4} \left[\left(\frac{\hbar}{\rho c}\right)^3 \left(\frac{\rho c^2}{k_B T_1}\right)^4\right]^2.$$

Thus, at temperatures  $T \ll T_\lambda$ , the ratio  $(\eta_{ph}/s_{ph})$  decreases with increase in the temperature.

### 3. Gaseous Helium

At temperatures  $T_\lambda \ll T$ , we can consider helium as a classical Boltzmann gas. In order to simplify notations in this section, we assume the Boltzmann constant  $k_B = 1$ .

The entropy of this gas [7]

$$s(T, P) = -N[\ln P + \chi'(T)],$$

where  $\chi'(T)$  is some function of the temperature. If we assume that the heat capacities  $c_v$  and  $c_p$  do not depend on the temperature,

$$c_p, c_v = \text{const},$$

then

$$\chi'(T) = -c_p \ln T - c_p - \zeta,$$

where  $\zeta$  is the chemical potential. In this case, the density of entropy  $s$  takes the form

$$s(T, P) = N[\ln(T^{c_p}/P) + c_p + \zeta].$$

A viscosity of the classical Boltzmann gas is equal to [8]

$$\eta \sim mvlN,$$

where  $v$  is the thermal velocity and  $l$  is the mean free path,

$$v \sim \left(\frac{T}{m}\right)^{1/2}, \quad l \sim \frac{1}{\sigma N}.$$

Therefore, the viscosity of the classical Boltzmann gas has the following dependence on the temperature:

$$\eta \sim \frac{(mT)^{1/2}}{\sigma}.$$

Thus, the ratio of the shear viscosity  $\eta$  to the volume density of entropy  $s$  reads

$$\frac{\eta}{s} \sim \frac{(mT)^{1/2}}{\sigma N} \frac{1}{\ln(T^{c_p}/P) + \zeta + c_p}.$$

It is convenient to represent this expression as follows:

$$\frac{\eta}{s} = \left(\frac{\eta}{s}\right)_{T=T_2} \frac{(T/T_2)^{1/2}}{\ln(eT/T_2)}, \quad (2)$$

where

$$T_2 = [P \exp(-\zeta)]^{1/c_p} = [P \exp(-\zeta)]^{(\gamma-1)/\gamma},$$

$$\gamma = c_p/c_v.$$

We can see that, at temperatures  $T_\lambda \ll T$ , the ratio  $(\eta_{ph}/s_{ph})$  increases with the temperature.

### 4. Helium at the temperature $T_\lambda$

At the temperature  $T_\lambda$ , the Bose–Einstein condensation takes place in helium. If we neglect the interaction of helium atoms, then the entropy of helium and its first derivative with respect to the temperature (i.e., specific heat) are continuous functions of the temperature at  $T_\lambda$ , but the second derivative of the entropy is discontinuous.

If we take the interaction of atoms in account, then helium near the  $\lambda$ -transition is described with a complex order parameter which is nothing else as the condensate wave function,

$$\psi = \sqrt{\rho_s} e^{i\phi}.$$

To describe helium at  $|T - T_\lambda| \ll T_\lambda$ , we should consider fluctuations of the order parameter.

As a consequence, the entropy of helium  $s(T, P)$  at  $T \rightarrow T_\lambda$  can be presented in the form of a sum of two parts,

$$s(T, P) = s_0(T, P) + s_{\text{sing}}(T, P),$$

where  $s_0(T, P)$  is the continuous part of the entropy and  $s_{\text{sing}}(T, P)$  is its singular part, which appears due to fluctuations. The singular part of the entropy is

$$s_{\text{sing}}(T, P) \sim |T - T_\lambda|^{1-\alpha},$$

where the thermodynamical critical index  $\alpha$  for helium is negative and small,

$$\alpha \simeq -0.02.$$

Therefore, the entropy of helium  $s(T, P)$  at  $T_\lambda$  is finite.

The viscosity of helium  $\eta$  at  $|T - T_\lambda| \ll T_\lambda$  can be presented also as a sum of two terms,

$$\eta(T) = \eta_0(T) + \eta_{\text{sing}}(T).$$

The singular term  $\eta_{\text{sing}}(T)$  is characterized with the dynamical critical indices [8, 9, 10]. We can express these dynamical critical indices in terms of the thermodynamic critical index  $\alpha$  due to a special circumstance: the non-equilibrium free energy of helium depends only on the modulus of the order parameter [10, 11, 12]. Considerations similar to that for the entropy lead us to the conclusion that the viscosity of helium  $\eta(T, P)$  at  $T_\lambda$  is finite. Therefore, the ratio  $(\eta(T_\lambda, P)/s(T_\lambda, P))$  is also finite.

## 5. Conclusion

According to our consideration, when the temperature  $T$  grows, starting from the absolute zero, then the ratio  $(\eta/s)$  of the shear viscosity  $\eta$  to the density of entropy  $s$  for helium decreases at first, achieves a finite minimum  $(\eta/s)_{\text{min}}$  near the temperature of the phase transition  $T_\lambda$ , and then increases. The inspection of the experimental data for helium [13, 14] shows that this conclusion is qualitatively valid. However, we should remark that the experimental value  $(\eta/s)_{\text{min}}$  is approximately 10 times larger than  $(\hbar/4\pi k_B)$ . Therefore, helium is far from the perfect liquid in this sense. Although for the unitary limit, the value  $(\eta/s)$  can achieve 1.3 [15].

In this work, we assume that the liquid helium is not compressible. If we take the compressibility of helium into account, the above consideration becomes more complicated as a result of the interaction of normal and superfluid motions in helium and the necessity to consider, due to that, three more volume viscosities along with the shear viscosity.

In addition, we should remark that, besides helium, we have nowadays, due to special experimental techniques, the gases of alkali metals which can also undergo to the Bose–Einstein condensation and can have the same properties, but their viscosities are not studied accurately enough.

We believe also that since the superfluidity and the superconductivity are related phenomena, the latter will demonstrate similar properties.

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ВІДНОШЕННЯ В'ЯЗКОСТІ  
ЗСУВУ ДО ГУСТИНИ ЕНТРОПІЇ В ГЕЛІЇ

Резюме

Досліджено для гелію відношення  $(\eta/s)$  в'язкості зсуву  $\eta$  до густини ентропії  $s$  як функцію температури і встановлено, що мінімальна величина,  $(\eta/s)_{\min}$ , задовольняє нерівність Ковтуна–Сона–Старинця,  $(\eta/s)_{\min} \geq (\hbar/4\pi k_B)$ .

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ОТНОШЕНИЕ СДВИГОВОЙ  
ВЯЗКОСТИ К ПЛОТНОСТИ ЭНТРОПИИ В ГЕЛИИ

Резюме

Исследовано для гелия отношение  $(\eta/s)$  сдвиговой вязкости  $\eta$  к плотности энтропии  $s$  как функцию температуры и установлено, что минимальное значение,  $(\eta/s)_{\min}$ , удовлетворяет неравенство Ковтуна–Сона–Старинца,  $(\eta/s)_{\min} \geq (\hbar/4\pi k_B)$ .