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RELATIVISTIC PSEUDOSPIN AND SPIN SYMMETRIES OF THE ENERGY-DEPENDENT YUKAWA POTENTIAL INCLUDING A COULOMB-LIKE TENSOR INTERACTION

We solve the Dirac equation for the energy-dependent Yukawa potential including a tensor interaction term within the framework of the pseudospin and spin symmetry limits with arbitrary spin-orbit quantum number κ. We obtained explicitly the energy eigenvalues and the corresponding wave function using the Nikiforov–Uvarov method. The limiting cases of this model are reduced to the energy-dependent Yukawa and Coulomb potentials, respectively.

Keywords: Dirac equation, energy-dependent Yukawa potential, Nikiforov–Uvarov method.

1. Introduction

The concept of relativistic symmetries of the Dirac Hamiltonian discovered many years ago were recognized empirically in nuclear and hadronic spectroscopies [1]. The relativistic Dirac equation, which describes the motion of a spin-$\frac{1}{2}$ particle, has been used successfully in solving many physical problems of nuclear and high-energy physics [2–4]. Within the framework of the Dirac equation, the pseudospin symmetry was used to feature deformed nuclei, superdeformation, and to establish an effective shell model [5]. The pseudospin concept was firstly introduced in nuclear physics and has been associated with the Dirac equation, as suggested by J.N. Ginocchio some decade ago [6]. It was shown that the exact pseudospin symmetry occurs in the Dirac equation when $\frac{d\Sigma(r)}{dr} = 0$, where $\Sigma(r) = V(r) + S(r) = c_{ps} = \text{const}$ [7] and $V(r)$, $S(r)$ are repulsive and attractive scalar potentials, respectively. On the other hand, the exact spin symmetry occurs in the Dirac equation when $\frac{d\Delta(r)}{dr} = 0$, where $\Delta(r) = V(r) - S(r) = c_s = \text{const}$ [8]. The detailed recent review of spin and pseudospin symmetries is given in Ref. [9]. Consequently, the concepts of spin and pseudospin symmetries of solutions of the Dirac equation for some potential models have been investigated by many authors using various methods such as the asymptotic iteration method (AIM) [10], Nikiforov–Uvarov (NU) method [11], supersymmetric quantum mechanics (SUSSYQM) [12], shape invariance (SI) [13], and exact quantization rule method [14]. The pseudospin symmetry is usually referred to as a quasidegeneracy of single nucleon doublets with non-relativistic quantum number $\left(n, l, j = l + \frac{1}{2}\right)$ and $\left(n-1, l+2, j = l + \frac{3}{2}\right)$, where n, l and j are single-nucleon radial, orbital, and total angular quantum numbers, respectively. The total angular momentum is $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ is a pseudoangular momentum, and $\tilde{s}$ is a pseudospin angular momentum [15]. Similarly, the tensor interaction term was introduced into the Dirac equation with the replacement $\mathbf{p} \to \mathbf{p} - iM\omega\beta \cdot \hat{r}U(r)$, and a spin-orbit coupling is added to the Dirac Hamiltonian [16]. Wave equations with energy-dependent potentials occur in relativistic quantum mechanics, firstly in the Pauli– Schrödinger equation [17] and recently in the Hamiltonian formulation of the relativistic many-body system [18]. The energy-dependent potentials have been used as a source in a nonlinear Hamiltonian equation [19] and in the problem of soliton propagation [20]. The energy-dependent potentials were also involved in the relativistic treatment of a point charge in an external Coulomb field described by the Klein–Gordon equation [21]. In recent times, the bound state solutions with energy-dependent potentials have been investigated in [22, 24, 35]. The main aim of the present paper is to obtain approximate solutions of the Dirac equation with energy-dependent Yukawa (EDY) potential including a Coulomb-like potential under the pseudospin symmetric limit. The paper is organized as follows. In Section 2, we give a brief introduction to the Nikiforov–Uvarov (NU) method. In Section 3, the Dirac equation for spin and pseudospin including the Coulomb interaction term is briefly introduced. We solve the Dirac equation under the pseudospin and spin symmetries in Section 4. Remarks and the discussion are given in Section 5. Finally, the conclusion is presented in Section 6.

2. The NU Method

The method is used to obtain a solution of the second-order differential equations having the form [11]

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \quad (1)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most of the second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without any need to check the validity of a solution, we present a shortcut for the method. Using the transformation

$$\psi(s) = W(s)\Phi(s), \quad (2)$$

Eq. $(A.1)$ is reduced to the well-known hypergeometric-type equation

$$\sigma(s)\Phi''(s) + \tau(s)\Phi'(s) + \lambda\Phi(s) = 0, \quad (3)$$

and

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad \tilde{\tau}(s) < 0. \quad (4)$$

Here, Eq. $(A.1)$ has a particular solution with degree n when the following relation is satisfied:

$$\lambda = \lambda_n = -n\,\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, \dots . \quad (5)$$

In order to obtain the equation for energy eigenvalues, the following definitions given below are required in the NU method:

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \pm\sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \quad (6)$$

$$\lambda = k + \pi'(s), \quad (7)$$

where λ is a constant. Here, the expression under the square root in the polynomial in $\pi(s)$ must be the square of a polynomial of the first degree, since $\pi(s)$ is the first-degree polynomial. Then, one can obtain the k values by considering that discriminant of the square root has to be zero in Eq. (6). Consequently, the equation for energy eigenvalues is obtained by comparing Eq. (7) with Eq. (5). The function $\Phi(s)$ given in Eq. (3) is a hypergeometric-type function, and its solution can be written in terms of polynomials, which are given by the Rodrigues relation

$$\Phi_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \quad (8)$$

where C_n is the normalization constant, and the weight function $\rho(s)$ should satisfy the condition

$$[\sigma(s)\rho(s)]' = \tau(s)\rho(s). \quad (9)$$

On the other hand, the other factor $W(s)$ satisfies the logarithmic equation

$$\frac{d}{ds}\ln W(s) = \frac{\sigma(s)}{\pi(s)}. \quad (10)$$

2.1. Parametric generalization of the NU method

We consider the second-order differential equation, whose form represents a general Schrödinger-type equation, to obtain a parametric generalization of the NU method [25]:

$$\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1 - c_3 s)}\right)\psi_n'(s) +$$

$$+ \left(\frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1 - c_3 s)^2}\right)\psi_n(s) = 0. \quad (11)$$

Here, we give only the basic ingredients of the generalized NU method [11]. By comparing Eq. (11) with Eq. (1), one can obtain

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad (12a)$$

$$\sigma(s) = s(1 - c_3 s), \quad (12b)$$

$$\tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \quad (12c)$$

Inserting the above equations into Eq. (6) leads to

$$\pi(s) = c_4 + c_5 s \pm \sqrt{(c_6 - k c_3)s^2 + (c_7 + k)s + c_8}, \quad (13)$$

where

$$c_4 = \frac{1}{2}(1 - c_1), \quad (14)$$

$$c_5 = \frac{1}{2}(c_2 - 2c_3), \quad (15)$$

$$c_6 = c_5^2 + \xi_1, \quad (16)$$

$$c_7 = 2c_4 c_5 - \xi_2, \quad (17)$$

$$c_8 = c_4^2 + \xi_3. \quad (18)$$

Considering that the discriminant of the square root has to be zero in Eq. (6), we obtain

$$k_{1,2} = -(c_7 + 2c_3 c_8) \pm 2\sqrt{c_8 c_9} \quad (19)$$

with

$$c_9 = c_3 c_7 + c_3^2 c_8 + c_6. \quad (20)$$

From Eq. (13), one can easily see that different k values lead to different $\pi(s)$. If we take

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9}, \quad (21)$$

$\pi(s)$ becomes

$$\pi(s) = c_4 + c_5 s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}), \quad (22)$$

and then we find

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}]. \quad (23)$$

The energy eigenvalue equation can be readily obtained by using Eqs. (5) and (6) with the above results as follows:

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) +$$

$$+ n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0. \quad (24)$$

In order to obtain the wave functions, one can use the relations

$$\rho(s) = s^{\alpha_{10}-1}(1 - c_3 s)^{(c_{11}/c_3) - c_{10} - 1}, \quad (25a)$$

$$\Phi_n(s) = P_n^{(c_{10}-1, (c_{11}/c_3) - c_{10} - 1)}(1 - 2c_3 s), \quad (25b)$$

$$W(s) = s^{c_{12}}(1 - c_3 s)^{-c_{12} - (c_{13}/c_3)}, \quad (25c)$$

$$\Psi_n(s) = s^{c_{12}}(1 - c_3 s)^{-c_{12} - (c_{13}/c_3)} \times$$

$$\times P_n^{(c_{10}-1, (c_{11}/c_3) - c_{10} - 1)}(1 - 2c_3 s), \quad (25d)$$

where $P_n^{(c_{10}-1, (c_{11}/c_3) - c_{10} - 1)}(1 - 2c_3 s)$ is a Jacobi polynomial and

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \quad (26)$$

$$c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}), \quad (27a)$$

$$c_{12} = c_4 + \sqrt{c_8}, \quad (27b)$$

$$c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}), \quad (27c)$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$. This method has been used extensively to solve various second-order differential equations in quantum mechanics such as the Schrödinger equation, Klein–Gordon equation, Duffin–Kemmer–Petiau equation, spinless Salpeter equation, and the Dirac equation [26].

3. Dirac Equation with a Tensor Coupling

The Dirac equation for spin-$\frac{1}{2}$ particles moving in an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$, and a tensor potential $U(r)$ in the relativistic units ($\hbar = c = 1$) is [27]

$$[\boldsymbol{\alpha}\cdot\mathbf{p} + \beta(M + S(r) - i\beta\boldsymbol{\alpha}\cdot\hat{r}U(r)]\,\psi(r) =$$

$$= [E - V(r)]\,\psi(r), \tag{28}$$

where E is the relativistic energy of the system, $\mathbf{p} = -i\boldsymbol{\nabla}$ is the three-dimensional momentum operator, and M is the mass of a fermionic particle, $\boldsymbol{\alpha}$ and β are the 4×4 Dirac matrices given as

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{29}$$

where I is a 2×2 unitary matrix, and $\boldsymbol{\sigma}_i$ are the Pauli three-vector matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{30}$$

The eigenvalues of the spin-orbit coupling operator are $\kappa = \left(j + \frac{1}{2}\right) > 0, \kappa = -\left(j + \frac{1}{2}\right) < 0$ for the unaligned $j = l - \frac{1}{2}$ and aligned $j = l + \frac{1}{2}$ spins, respectively. The set $\left(H, K, J^2, J_z\right)$ forms a complete set of conserved quantities. Thus, we can write the spinors as [28]

$$\psi_{n\kappa}(r) = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)Y^l_{jm}(\theta,\phi) \\ iG_{n\kappa}(r)Y^{\tilde{l}}_{jm}(\theta,\phi) \end{pmatrix}, \tag{31}$$

where $F_{n\kappa}(r), G_{n\kappa}(r)$ represent the upper and lower components of the Dirac spinors, $Y^l_{jm}(\theta,\varphi), Y^{\tilde{l}}_{jm}(\theta,\varphi)$ are the spin and pseudospin spherical harmonics, and m is the projection on the z-axis. With other known identities [29],

$$\begin{aligned} &(\boldsymbol{\sigma}\cdot\mathbf{A})\left((\boldsymbol{\sigma}\cdot\mathbf{B})\right) = \mathbf{A}\cdot\mathbf{B} + i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B}), \\ &\boldsymbol{\sigma}\cdot\mathbf{p} = \boldsymbol{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\mathbf{p} + i\frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r}\right), \end{aligned} \tag{32}$$

as well as

$$\begin{aligned} &(\boldsymbol{\sigma}\cdot\mathbf{L})\,Y^{\tilde{l}}_{jm}(\theta,\varphi) = (\kappa-1)Y^{\tilde{l}}_{jm}(\theta,\varphi), \\ &(\boldsymbol{\sigma}\cdot\mathbf{L})\,Y^{l}_{jm}(\theta,\varphi) = -(\kappa-1)Y^{l}_{jm}(\theta,\varphi), \\ &(\boldsymbol{\sigma}\cdot\hat{r})\,Y^{l}_{jm}(\theta,\varphi) = -Y^{\tilde{l}}_{jm}(\theta,\varphi), \\ &(\boldsymbol{\sigma}\cdot\hat{r})\,Y^{\tilde{l}}_{jm}(\theta,\varphi) = -Y^{l}_{jm}(\theta,\varphi), \end{aligned} \tag{33}$$

we have two coupled first-order Dirac equations [29]

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))\,G_{n\kappa}(r), \tag{34}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))\,F_{n\kappa}(r), \tag{35}$$

where

$$\Delta(r) = V(r) - S(r), \tag{36}$$

$$\Sigma(r) = V(r) + S(r). \tag{37}$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ in Eqs. (23) and (24), we obtain the second-order Schrödinger-like equation as

$$\begin{aligned} &\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) - \right. \\ &- (M + E_{n\kappa} - \Delta(r))\,(M - E_{n\kappa} + \Sigma(r)) + \\ &\left. + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)}{(M + E_{n\kappa} - \Delta(r))}\right\}F_{n\kappa}(r) = 0, \end{aligned} \tag{38}$$

$$\begin{aligned} &\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa U(r)}{r} + \frac{dU(r)}{dr} - U^2(r) - \right. \\ &- (M + E_{n\kappa} - \Delta(r))\,(M - E_{n\kappa} + \Sigma(r)) + \\ &\left. + \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)}{(M + E_{n\kappa} - \Sigma(r))}\right\}G_{n\kappa}(r) = 0, \end{aligned} \tag{39}$$

where $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1), \kappa(\kappa+1) = l(l+1)$. The radial wave functions are required to satisfy the necessary conditions, i.e., $F_{n\kappa}(r) = G_{n\kappa}(r) = 0$ and $F_{n\kappa}(r) = G_{n\kappa}(r) \to 0$ at infinity. At this stage, $\Delta(r)$ or $\Sigma(r)$ takes the form of an energy-dependent Yukawa (EDY) potential. However, Eqs. (38) and (39) can be solved exactly for $\kappa = 0, -1$ and $\kappa = 0, 1$, respectively.

4. Solution of the Dirac equation with Energy-Dependent Yukawa and Tensor Potentials

In this section, we are going to solve the Dirac equation with the EDY potential and a tensor potential by using the Nikiforov–Uvarov method.

4.1. Pseudospin symmetry

The exact pseudospin symmetry was proved in [30]. It occurs in the Dirac equation when $\frac{d\Sigma(r)}{dr}=0$ or $\Sigma(r)=C_{ps}=\text{const}$ [7]. In this limit, we take $\Delta(r)$ as the EDY potential and a Coulomb-like potential [31] for the tensor potential added,

$$\Delta(r,E)=c_v(1+\eta E_{n\kappa}^{ps})\frac{e^{-\alpha r}}{r}, \tag{40}$$

$$U(r)=-\frac{H}{r},\quad H=\frac{z_a z_b e^2}{4\pi\varepsilon_0},\quad r\geq R_e, \tag{41}$$

where c_v and η are constant coefficients, $R_e=7.78$ fm is the Coulomb radius, z_a and z_b denote the charges of the projectile a and target b nuclei, respectively [31]. Substituting Eqs. (40)–(41) into Eq. (39), we obtain

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-\frac{2\kappa H}{r^2}+\frac{H}{r^2}-\frac{H^2}{r^2}-\right.$$
$$-\left(M+E_{n\kappa}^{ps}-c_v(1+\eta E_{n\kappa}^{ps})\frac{e^{-\alpha r}}{r}\right)\times$$
$$\left.\times\left(M-E_{n\kappa}^{ps}+C_{ps}\right)\right\}G_{n\kappa}^{ps}(r)=0. \tag{42}$$

Simplifying Eq.(42) yields

$$\left\{\frac{d^2}{dr^2}-\frac{\Lambda_\kappa(\Lambda_\kappa-1)}{r^2}+\tilde{\gamma}\left(c_v(1+\eta E_{n\kappa}^{ps})\frac{e^{-\alpha r}}{r}\right)-\varepsilon^2\right\}\times$$
$$\times G_{n\kappa}^{ps}(r)=0, \tag{43}$$

where

$$\tilde{\gamma}=\left(M-E_{n\kappa}^{ps}+C_{ps}\right),$$
$$\varepsilon^2=\left(M+E_{n\kappa}^{ps}\right)\left(M-E_{n\kappa}^{ps}+C_{ps}\right),\quad \Lambda_\kappa=\kappa+A. \tag{44}$$

Since the Dirac equation with the EDY potential has no exact solution, we use an approximation for the centrifugal term as [30]

$$\frac{1}{r^2}=\lim_{\alpha\to0}\left[4\alpha^2\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right], \tag{45}$$

$$\frac{1}{r}=\lim_{\alpha\to0}\left[2\alpha\frac{e^{-\alpha r}}{(1-e^{-2\alpha r})}\right]. \tag{46}$$

We have plotted the centrifugation term and its approximation in Fig. 1. Substituting Eqs. (45)–(46)

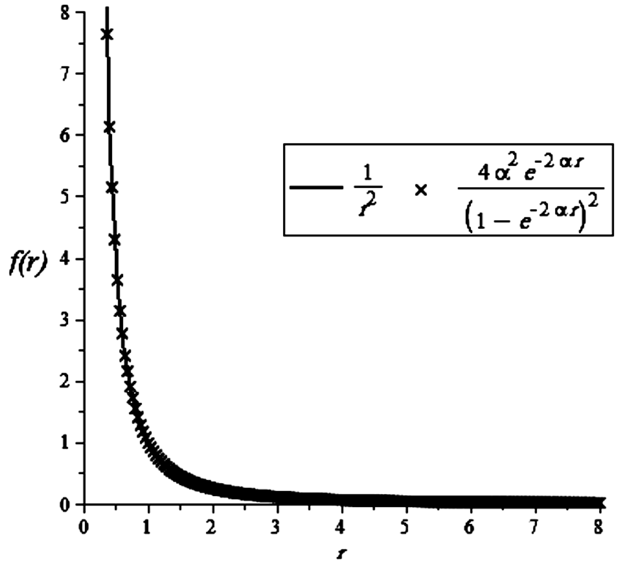

Fig. 1. $\frac{1}{r^2}$ and its approximation for $\alpha=0.05$

into Eq. (43) and using the change of the variable $s=e^{-2\alpha r}$, we obtain

$$\frac{d^2G_{n\kappa}^{ps}}{dr^2}+\frac{(1-s)}{s\,(1-s)}\frac{dG_{n\kappa}^{ps}}{dr}+\frac{1}{s^2\,(1-s)^2}\times$$
$$\times\left[-A^{ps}s^2+B^{ps}s-C^{ps}\right]G_{n\kappa}^{ps}=0, \tag{47}$$

where

$$A^{ps}=\frac{\varepsilon}{4\alpha^2}+\frac{\tilde{\gamma}c_v(1+\eta E_{n\kappa}^{ps})}{2\alpha},$$
$$B^{ps}=\frac{2\varepsilon}{4\alpha^2}+\frac{\tilde{\gamma}c_v(1+\eta E_{n\kappa}^{ps})}{2\alpha}-\Lambda_\kappa(\Lambda_\kappa-1), \tag{48}$$
$$C^{ps}=\frac{\varepsilon}{4\alpha^2}.$$

By comparing Eq. (47) with Eq. (11), we obtain the parameters

$$c_1=1,\quad \xi_1=A^{ps},\quad c_2=1,\quad \xi_2=B^{ps},$$
$$c_3=1,\quad \xi_3=C^{ps}. \tag{49}$$

Equation (11) determines the other coefficients as

$$c_4=0,\ c_5=-\frac{1}{2},\ c_6=\frac{1}{4}+\frac{\varepsilon}{4\alpha^2}+\frac{\tilde{\gamma}c_v(1+\eta E_{n\kappa}^{ps})}{2\alpha},$$
$$c_7=-\frac{2\varepsilon}{4\alpha^2}-\frac{\tilde{\gamma}c_v(1+\eta E_{n\kappa}^{ps})}{2\alpha}+\Lambda_\kappa(\Lambda_\kappa-1),$$
$$c_8=\frac{\varepsilon}{4\alpha^2},\ c_9=\Lambda_\kappa(\Lambda_\kappa-1)+\frac{1}{4},\ c_{10}=1+2\sqrt{\frac{\varepsilon}{4\alpha^2}},$$

$$c_{11}=2+2\left(\sqrt{\Lambda_\kappa(\Lambda_\kappa-1)+\frac{1}{4}}+\sqrt{\frac{\varepsilon}{4\alpha^2}}\right),\ c_{12}=\sqrt{\frac{\varepsilon}{4\alpha^2}},$$
$$c_{13}=-\frac{1}{2}-\left(\sqrt{\Lambda_\kappa(\Lambda_\kappa-1)+\frac{1}{4}}+\sqrt{\frac{\varepsilon}{4\alpha^2}}\right). \tag{50}$$

With the aid of Eqs. (10) and (50), we obtain the energy eigenvalues for the EDY potential model with the pseudospin symmetry concept for any spin-orbit quantum number $\kappa = \pm1, \pm2...$ in the Dirac theory as

$$n+\frac{(2n+1)}{2}+(2n+1)\left(\sqrt{\Lambda_\kappa(\Lambda_\kappa-1)+\frac{1}{4}}+\sqrt{\frac{\varepsilon}{4\alpha^2}}\right)+$$
$$+n\,(n-1)-\frac{\tilde{\gamma}c_v(1+\eta E^{ps}_{n\kappa})}{2\alpha}+\Lambda_\kappa(\Lambda_\kappa-1)+$$
$$+2\sqrt{\frac{\varepsilon}{4\alpha^2}}\left(\Lambda_\kappa(\Lambda_\kappa-1)+\frac{1}{4}\right)=0. \tag{51}$$

In what follows, we find the lower component of the wave function as

$$G^{ps}_{n\kappa}=N_{n\kappa}\left(e^{-2\alpha r}\right)^{\sqrt{\frac{\varepsilon}{4\alpha^2}}}\left(1-e^{-2\alpha r}\right)^{\frac{1}{2}+\sqrt{\Lambda_\kappa(\Lambda_\kappa-1)+\frac{1}{4}}}\times$$
$$\times\,{}_2F_1\left(-n, n+2\sqrt{\frac{\varepsilon}{4\alpha^2}}+\right.$$
$$\left.+2\sqrt{\Lambda_\kappa(\Lambda_\kappa-1)+\frac{1}{4}}+1; 1+2\sqrt{\frac{\varepsilon}{4\alpha^2}}; e^{-2\alpha r}\right), \tag{52}$$

where $N_{n\kappa}$ is the normalization constant. The upper spinor component of the Dirac equation can be calculated as

$$F^{ps}_{n\kappa}(r)=\frac{1}{M-E^{ps}_{n\kappa}+C_{ps}}\left(\frac{d}{dr}-\frac{\kappa}{r}-\frac{H}{r}\right)G^{ps}_{n\kappa}(r), \tag{53}$$

where $E_{n\kappa}\neq M+C_{ps}$. When $C_{ps}=0$ (exact psudospin symmetry), this means that only negative energy solutions are possible.

4.2. Spin symmetry

In the spin symmetry limit, $\frac{d\Delta(r)}{dr}=0$ or $\Delta(r)=C_s=$ $=$ const. Like the previous section, we consider

$$\Sigma(r,E)=c_v(1+\eta E^s_{n\kappa})\frac{e^{-\alpha r}}{r}. \tag{54}$$

The substitution of this relation in Eq. (38) gives

$$\left\{\frac{d^2}{dr^2}-\frac{\lambda_\kappa(\lambda_\kappa-1)}{r^2}+\right.$$
$$\left.+\gamma\left(c_v(1+\eta E^s_{n\kappa})\frac{e^{-\alpha r}}{r}\right)-\mu^2\right\}F^s_{n\kappa}(r)=0, \tag{55}$$

where

$$\gamma=(-M-E^s_{n\kappa}+C_s),$$
$$\mu=(M-E^s_{n\kappa})(-M-E^s_{n\kappa}+C_s), \tag{56}$$
$$\lambda_\kappa=\kappa+H+1,$$

and $\kappa=\ell$ and $\kappa=-\ell-1$ for $\kappa<0$ and $\kappa>0$, respectively. We now introduce the NU method to proceed. In this limit, we have to deal with

$$\frac{d^2F^s_{n\kappa}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dF^s_{n\kappa}(s)}{ds}+$$
$$+\frac{1}{(s(1-s))^2}(-A^s s^2+B^s s-C^s)F^s_{n\kappa}(z)=0, \tag{57}$$

where

$$A^s=-\frac{\mu}{4\alpha^2}+\frac{\gamma c_v(1+\eta E^s_{n\kappa})}{2\alpha},$$
$$B^s=-\frac{2\mu}{4\alpha^2}+\frac{\gamma c_v(1+\eta E^s_{n\kappa})}{2\alpha}-\lambda_\kappa(\lambda_\kappa-1),$$
$$C^s=-\frac{\mu}{4\alpha^2}. \tag{58}$$

By comparing Eq. (14)–(27c) with Eq. (57), we find the correspondence

$$c_1=1,\ c_2=1,\ c_3=1,\ \xi_1=A^s,\ \xi_2=B^s,\ \xi_3=C^s,$$
$$c_4=0,\ c_5=-\frac{1}{2},\ c_6=\frac{1}{4}-\frac{\mu}{4\alpha^2}+\frac{\gamma c_v(1+\eta E^s_{n\kappa})}{2\alpha},$$
$$c_7=\frac{2\mu}{4\alpha^2}-\frac{\gamma c_v(1+\eta E^s_{n\kappa})}{2\alpha}+\lambda_\kappa(\lambda_\kappa-1),\ c_8=-\frac{\mu}{4\alpha^2},$$
$$c_9=\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4},\ c_{10}=1+2\sqrt{-\frac{\mu}{4\alpha^2}}, \tag{59}$$
$$c_{11}=2+2\left(\sqrt{\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4}}+\sqrt{-\frac{\mu}{4\alpha^2}}\right),$$
$$c_{12}=\sqrt{-\frac{\mu}{4\alpha^2}},$$
$$c_{13}=-\frac{1}{2}-\left(\sqrt{\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4}}+\sqrt{-\frac{\mu}{4\alpha^2}}\right).$$

Table 1. **Energies in the Pseudospin Symmetry Limit for $\alpha = 0.05$, $M = 5$ fm^{-1}, $C_{ps} = -5$, $C_v = 0.5$, $\eta = -0.4$**

$\tilde{\ell}$	$n, \kappa < 0$	(ℓ, j)	$E^{ps}_{n\kappa}$ (fm^{-1}) ($H = 0.5$)	$E^{ps}_{n\kappa}$ (fm^{-1}) ($H = 0$)	$n-1$, $\kappa > 0$	$(\ell+2, j+1)$	$E^{ps}_{n\kappa}$ (fm^{-1}) ($H = 0.5$)	$E^{ps}_{n\kappa}$ (fm^{-1}) ($H = 0$)
1	1, –1	$1S_{\frac{1}{2}}$	–4.785875896	–4.682066333	0.2	$0d_{\frac{3}{2}}$	–4.785875896	–4.853025137
2	1, –2	$1P_{\frac{3}{2}}$	–4.898252803	–4.853025137	0.3	$0f_{\frac{5}{2}}$	–4.898252803	–4.929649176
3	1, –3	$1d_{\frac{5}{2}}$	–4.951904203	–4.929649176	0.4	$0g_{\frac{7}{2}}$	–4.951904203	–4.967867988
4	1, –4	$1f_{\frac{7}{2}}$	–4.979344475	–4.967867988	0.5	$0h_{\frac{9}{2}}$	–4.979344475	–4.98751631
1	2, –1	$2S_{\frac{1}{2}}$	–4.898252803	–4.853025137	1.2	$1d_{\frac{3}{2}}$	–4.898252803	–4.929649176
2	2, –2	$2P_{\frac{3}{2}}$	–4.951904203	–4.929649176	1.3	$1f_{\frac{5}{2}}$	–4.951904203	–4.967867988
3	2, –3	$2d_{\frac{5}{2}}$	–4.979344475	–4.967867988	1.4	$1g_{\frac{7}{2}}$	–4.979344475	–4.98751631
4	2, –4	$2f_{\frac{7}{2}}$	–4.993182832	–4.98751631	1.5	$1h_{\frac{9}{2}}$	–4.993182832	–4.99689886

Table 2. **Energies in the Spin Symmetry Limit for $\alpha = 0.05$, $M = 5$ fm^{-1}, $C_s = 5$, $C_v = -0.5$, $\eta = 0.4$**

ℓ	$n, \kappa < 0$	(ℓ, j)	$E^{s}_{n\kappa}$ (fm^{-1}) ($H = 0$)	$E^{s}_{n\kappa}$ (fm^{-1}) ($H = 0.5$)	n, $\kappa > 0$	(ℓ, j)	$E^{s}_{n\kappa}$ (fm^{-1}) ($H = 0$)	$E^{s}_{n\kappa}$ (fm^{-1}) ($H = 0.5$)
1	0, –2	$0P_{\frac{3}{2}}$	4.513012873	4.219234736	0.1	$0P_{\frac{1}{2}}$	4.513012873	4.682066333
2	0, –3	$0d_{\frac{5}{2}}$	4.785875896	4.682066333	0.2	$0d_{\frac{3}{2}}$	4.785875896	4.853025137
3	0, –4	$0f_{\frac{7}{2}}$	4.898252803	4.853025137	0.3	$0f_{\frac{5}{2}}$	4.898252803	4.929649176
4	0, –5	$0g_{\frac{9}{2}}$	4.951904203	4.929649176	0.4	$0g_{\frac{7}{2}}$	4.951904203	4.967867988
1	1, –2	$1P_{\frac{3}{2}}$	4.785875896	4.682066333	1.1	$1P_{\frac{1}{2}}$	4.785875896	4.853025137
2	1, –3	$1d_{\frac{5}{2}}$	4.898252803	4.853025137	1.2	$1d_{\frac{3}{2}}$	4.898252803	4.929649176
3	1, –4	$1f_{\frac{7}{2}}$	4.951904203	4.929649176	1.3	$1f_{\frac{5}{2}}$	4.951904203	4.967867988
4	1, –5	$1g_{\frac{9}{2}}$	4.979344475	4.967867988	1.4	$1g_{\frac{7}{2}}$	4.979344475	4.98751631

Substituting Eq. (59) in Eq. (24) immediately gives

$$n+\frac{(2n+1)}{2}+(2n+1)\left(\sqrt{\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4}}+\sqrt{-\frac{\mu}{4\alpha^2}}\right)+ +n(n-1)+\frac{2\mu}{4\alpha^2}-\frac{\gamma c_v(1+\eta E^s_{n\kappa})}{2\alpha}+\lambda_\kappa(\lambda_\kappa-1)- -\frac{2\mu}{4\alpha^2}+2\sqrt{-\frac{\mu}{4\alpha^2}\left(\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4}\right)}=0. \quad (60)$$

The upper and lower components of the wave function are

$$F^s_{n\kappa}(r)=(e^{-2\alpha r})^{\sqrt{-\frac{\mu}{4\alpha^2}}}(1-e^{-2\alpha r})^{\frac{1}{2}+\sqrt{\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4}}}\times \times P_n^{(2\sqrt{-\frac{\mu}{4\alpha^2}},2\sqrt{\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4}})}(1-2e^{-2\alpha r})\times \times {}_2F_1\left(-n,n+2\sqrt{-\frac{\mu}{4\alpha^2}}+2\sqrt{\lambda_\kappa(\lambda_\kappa-1)+\frac{1}{4}}+1;1+ +2\sqrt{-\frac{\mu}{4\alpha^2}};e^{-2\alpha r}\right), \quad (61)$$

and

$$G^s_{n\kappa}(r)=\frac{1}{M+E^s_{n\kappa}-C_s}\left(\frac{d}{dr}+\frac{k}{r}+\frac{H}{r}\right)F^s_{n\kappa}(r). \quad (62)$$

We have obtained the energy eigenvalues in the absence ($H = 0$) and presence ($H = 0.5$) of the Coulomb tensor potential for various values of the quantum numbers n and κ. The results are reported in Tables 1 and 2 under the condition of the pseudospin and spin symmetries. We can clearly see that there is the degeneracy between the bound states. In the presence of the tensor interaction, these degeneracies are changed. Our numerical data reveal that in the pseudospin and spin symmetry limits. We show

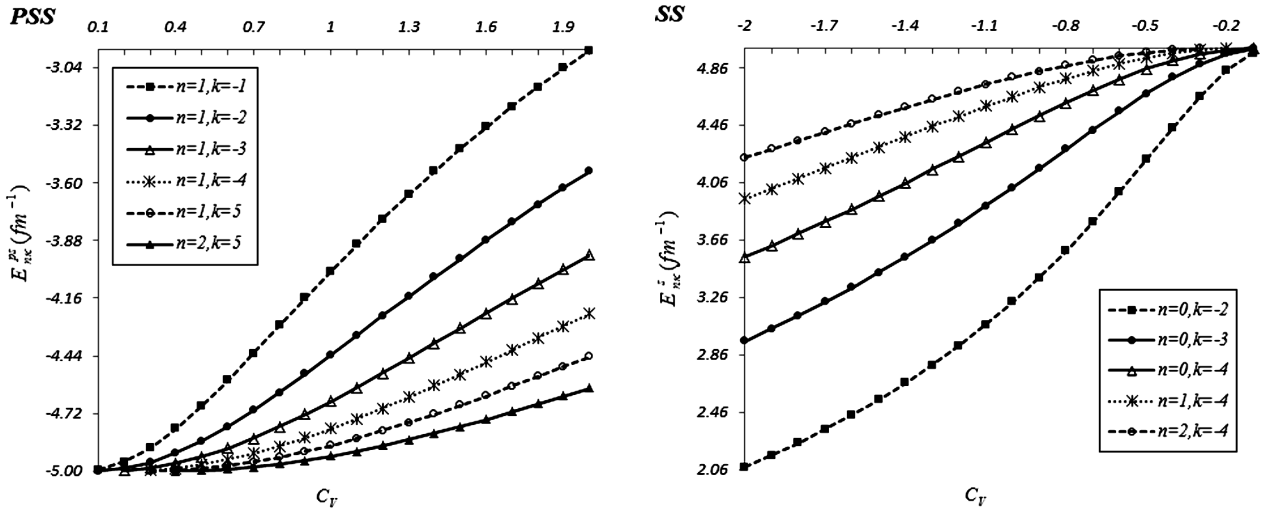

Fig. 2. ***PSS***: Energy vs. C_V for the pseudospin symmetry limit for $H = 0.5$, $\alpha = 0.05$, $M = 5$ fm^{-1}, $C_{ps} = -5$, $\eta = -0.4$. ***SS***: Energy vs. C_V for the spin symmetry limit for $H = 0.5$, $\alpha = 0.05$, $M = 5$ fm^{-1}, $C_s = 5$, $\eta = 0.4$

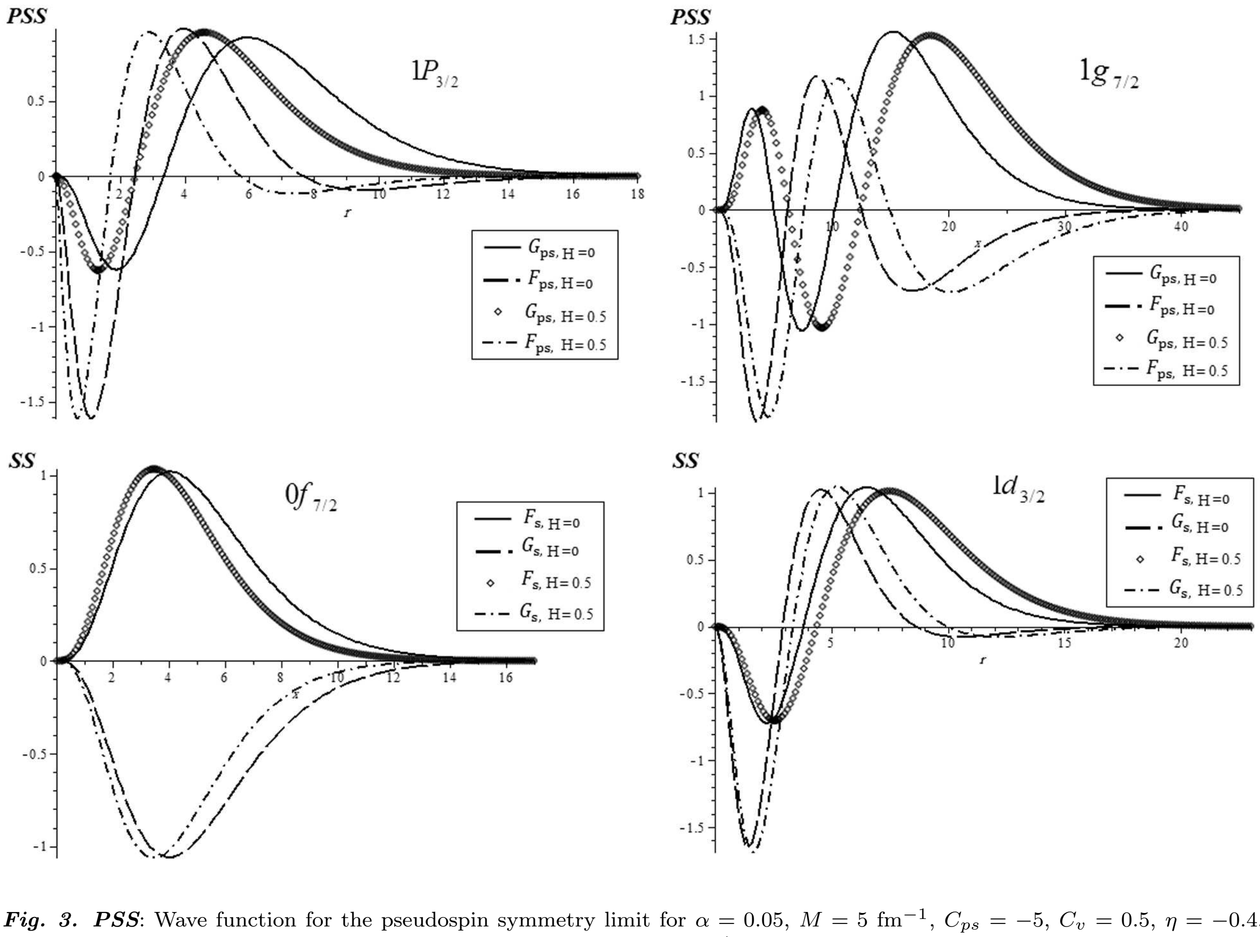

Fig. 3. ***PSS***: Wave function for the pseudospin symmetry limit for $\alpha = 0.05$, $M = 5$ fm^{-1}, $C_{ps} = -5$, $C_v = 0.5$, $\eta = -0.4$. ***SS***: Wave function for the spin symmetry limit for $\alpha = 0.05$, $M = 5$ fm^{-1}, $C_s = 5$, $C_v = -0.5$, $\eta = 0.4$

the effects of the C_V-parameter on the bound states under the condition of the pseudospin and spin symmetry limit for $H = 0.5$ in Fig. 2. In Fig. 3, the components of wave functions are plotted for the pseudospin and spin symmetry limits with and without a tensor interaction. It is seen in Fig. 3 that the tensor interaction affects only the shape of the wave functions and does not change the node structures of the radial upper and lower components of the Dirac spinors.

5. Few Special Cases

In this section, we will consider some special cases of interest of the energy-dependent Yukawa potential as follows.

5.1. Energy-dependent Coulomb Potential

When we set $\alpha \to 0$, the potential reduces into the energy-dependent Coulomb potential studied in [22] and [35], i.e,

$$V(r) = \frac{c_v(1+\eta E)}{r}. \tag{63}$$

Hamzavi and Ikhdair [23] studied the exact spin and pseudospin symmetry of the bound-state solutions of the Dirac equation with this potential for any spin-orbit κ, by using the asymptotic iteration method (AIM). Under this condition, we obtain the energy eigenvalues and the corresponding wave function for the energy-dependent Coulomb potential from Eqs. (43) and (44) as

$$(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}) =$$

$$= \frac{1}{4}\left[\frac{-(M - E_{n\kappa} + C_{ps})\, c_v(1+\eta E)}{n+\sigma}\right]^2, \tag{64}$$

$$G_{n\kappa}(r) = N_{n\kappa} r^{\beta+\frac{1}{2}} e^{-\varepsilon_{n\kappa} r} L_n^{2\beta}(2\varepsilon_{n\kappa} r), \tag{65}$$

where $\beta = \sqrt{\frac{1}{4} + \Lambda_\kappa(\Lambda_\kappa - 1)}$. In addition, when $C_{ps} = 0$, this result reduces to that in [22].

5.2. Yukawa Potential

Maghsoodi *et al.* [33] have obtained the approximate solutions of the Dirac equation in the presence of a Yukawa potential plus a tensor interaction term using the SUSSQM formalism. If we set $\eta = 0$ and $c_v = -V_0$, we obtain the Yukawa potential [34]

$$V(r) = -V_0\left(\frac{e^{-\alpha r}}{r}\right). \tag{66}$$

Substituting these parameters, we obtained the energy eigenvalues and wave functions for the Yukawa potential as

$$(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}) =$$

$$= \frac{\alpha^2}{4}\left[\frac{\frac{(M - E_{n\kappa} + C_{ps})V_0}{\alpha}}{n+\sigma} + (n+\sigma)\right]^2, \tag{67}$$

$$G_{n\kappa}(r) = N_{n\kappa}\left(e^{-2\alpha r}\right)^{\sqrt{\frac{\varepsilon^2}{4\alpha^2}}} \times$$
$$\times \left(1 - e^{-2\alpha r}\right)^{\left(\sqrt{\frac{1}{4}+\Lambda_\kappa(\Lambda_\kappa-1)}+\frac{1}{2}\right)} \times$$
$$\times\, {}_2F_1\left(-n, 2\left(1 + \sqrt{\frac{\varepsilon^2}{4\alpha^2}} + \sqrt{\frac{1}{4} + \Lambda_\kappa(\Lambda_\kappa - 1)}\right) +\right.$$
$$\left. + n; 2\left(1 + \sqrt{\frac{\varepsilon^2}{4\alpha^2}}\right); e^{-2\alpha r}\right). \tag{68}$$

This result is consistent with the one obtained by Maghsoodi et al. [31] and that of Hamzavi and Ikhdair [35] $S_0 = 0$. In addition, if we set $\eta = 0$, $C_{ps} = 0$, $E_{n\kappa} + M \to \frac{2\mu}{\hbar^2}$, $\kappa = l + 1$, and $E_{n\kappa} - M \to \to E_{n\kappa}$ [33], we obtain the energy spectrum in the non-relativistic limit of the Yukawa problem as [23]

$$E_{nl} = \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu V_0}{\hbar^2}}{\left(n + A + l - \frac{1}{2}\right)} + \alpha\left(n + A + l - \frac{1}{2}\right)\right]^2. \tag{69}$$

5.3. Coulomb Potential

Finally, when $\eta = 0$, $C_{ps} = 0$ and $\alpha \to 0$, Eq. (40) yields the energy formula for the Coulomb-like potential as [36]

$$E_{n\kappa} = -M\frac{4(n+\kappa)^2 - c_v^2}{4(n+\kappa)^2 + c_v^2}. \tag{70}$$

Furthermore, as $n \to \infty$, one easily obtains $E = -M$ (continuum states). This shows that, as n goes to infinity, the energy spectrum of Eq. (40) becomes infinite for the exact pseudospin symmetry case, as reported in Ref. [36].

6. Conclusions

In this paper, we have obtained the approximate solutions of the Dirac equation for the EDY potential including the tensor interaction term within the framework of psudospin and spin symmetry limits, by using the NU method. We have obtained the energy eigenvalues and the corresponding lower and upper wave functions in terms of the Jacobi polynomials. Moreover, the results obtained in this work have been compared with the previous works of other authors given in the literature. Finally, this work can be extended to other models [37], which will have many applications to physics and related fields [37].

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РЕЛЯТИВІСТСЬКІ ПСЕВДОСПІНОВА
І СПІНОВА СИМЕТРІЇ ПОТЕНЦІАЛУ ЮКАВИ,
ЗАЛЕЖНОГО ВІД ЕНЕРГІЇ, З КУЛОНОВОПОДІБНОЮ
ТЕНЗОРНОЮ ВЗАЄМОДІЄЮ

Р е з ю м е

Вирішено рівняння Дірака для потенціалу Юкави, залежного від енергії, з тензорною взаємодією для граничних псевдоспінової і спінової симетрій з довільним спін-орбітальним квантовим числом κ. Методом Никифорова–Уварова отримано точно власні значення енергії і відповідна хвильова функція. У граничних випадках ця модель зводиться до моделей із залежними від енергії потенціалами Кулона і Юкави.