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UNIFIED $(p,q;\alpha,\gamma,l)$-DEFORMATIONS OF OSCILLATOR AND HYBRID OSCILLATOR ALGEBRAS AND TWO-DIMENSIONAL CONFORMAL FIELD THEORY

The unified multiparametric generalizations of the well-known two-parameter deformed oscillator and hybrid oscillator algebras are introduced. The basic versions of these deformations are obtained by imputing the new free parameters in the structure functions and by a generalization of defining relations of these algebras. The generalized Jordan–Schwinger and Holstein–Primakoff realizations of the $U_{pq}^{\alpha\gamma l}(su(2))$ algebra by the creations and annihilations operators of the basic versions of these deformations are found. The $(p,q;\alpha,\gamma,l)$-deformation of the two-dimensional conformal field theory is considered. The pole structure of the $(p,q;\alpha,\gamma,l)$-deformed operator product expansion (OPE) of the holomorphic component of the energy-momentum tensor with primary fields is found. The two-point correlation function of the $(p,q;\alpha,\gamma,l)$-deformed two-dimensional conformal field theory is calculated.

Keywords: generalized deformed oscillator algebra, structure function, generalized Jordan–Schwinger and Holstein–Primakoff transformations, deformed two-dimensional conformal field theory.

1. Introduction

The important tool in the study of universal enveloping algebras of the simple Lie algebras is their realization by the creation and annihilation operators (Jordan–Schwinger, Holdstein–Primakoff, and other constructions) of the oscillator algebra. In order to generalize these constructions to the quantum algebras (the deformations of the universal enveloping algebras of simple Lie algebras), Biedenharn [1] and Macfarlane [2] introduced independently *the q-deformed creation and annihilation operators.*

Long before, another q-deformation of the canonical commutation relations has been used by Arik and Coon [3] for the operator description of the generalized Veneziano amplitude obtained by the replacement of the Γ-function by the Γ_q-function. The oscillator algebras generated by these operators belong to the general class of *generalized oscillator algebras* [4].

A generalized oscillator algebra is an associative algebra generated by the generators $\{1,a,a^+,N\}$, where a, a^+ are Hermitian conjugate operators, and N is the self-adjoint operator, whose defining relations are as follows:

$$aa^+ = f(N+1), \quad a^+a = f(N),\\ [N,a] = -a, \quad [N,a^+] = a^+. \tag{1}$$

A positive analytic function $f(x)=[x]$ with $f(0)=0$ is called *a structure function* [4]. This function defines a deformation scheme and, along with defining relations, the deformed oscillator algebra. The well-known example of a structure function is the (at $p=q$, one-) two-parameter function [5]

$$f(x) = [x]_{pq} = \frac{p^{-x}-q^{x}}{p^{-1}-q}, \tag{2}$$

where $p,q\in\mathbb{R}$, of the (one-)two-parameter deformed oscillator algebra. The structure function and the defining relations

$$aa^+ - q^{-1}a^+a = p^N, \quad [N,a] = -a,\\ [N,a^+] = a^+, \tag{3}$$

or

$$aa^+ - qa^+a = p^{-N}, \quad [N,a] = -a,\\ [N,a^+] = a^+ \tag{4}$$

define the (p,q)-deformed oscillator algebra [5–7]. This algebra describes the Arik–Coon ($p=1,q$) [3], Biedenharn–Macfarlane ($p=q^\alpha,q^\gamma$), [1,2], Kwek–Oh ($p=q^\alpha,q^\gamma$) [8] deformations of oscillator algebras in the unified framework.

In [9, 10], a modification of the oscillator algebra with the defining relations

$$aa^+ - a^+a = 1 + 2\nu(1-K),$$
$$K = (-1)^N, \quad [N,a] = -a, \quad [N,a^+] = a^+, \quad \nu \in \mathbb{R} \tag{5}$$

underlying the two-particle Calogero model has been introduced. This oscillator algebra belongs to the generalized oscillator algebras with the structure function

$$f(x) = [x]_\nu = n + \nu(1 + (-1)^n) \tag{6}$$

and, in the case where 2ν integer ≥ 1, is equivalent to the one-mode oscillator algebra of a para-Bose system with the commutation relations

$$[\mathcal{N}, a] = -a, [\mathcal{N}, a^+] = a^+, \tag{7}$$

where $\mathcal{N} = \frac{1}{2}(a^+a + aa^+) - \frac{p}{2}$ and $2\nu = p-1$, where $p = 1, 2, ...$ is the para-Bose oscillator order.

In [11], the q-deformed "hybrid" version of the oscillator algebra (4) with deformation parameter q and the modified oscillator algebra (5) with parameter ν have been constructed.

The structure function of this deformation,

$$f(n) = [n]_{q\nu} = \frac{q^{-n} - q^n}{q^{-1} - q} + 2\nu \frac{(-1)^n q^{-n} - q^n}{q^{-1} + q}, \tag{8}$$

where $q, \nu \in \mathbb{R}$, and the commutation relations

$$aa^+ - qa^+a = q^{-N}(1 + 2\nu K),$$
$$[N, a] = -a, \quad [N, a^+] = a^+, \tag{9}$$
$$Ka = -aK, \quad Ka^+ = a^+K, \quad [N, K] = 0$$

define the new deformation of the oscillator algebra.

This deformation breaks down $q \leftrightarrow q^{-1}$ symmetry of the Biedenharn–Macfarlane oscillator algebra obtained from (4) at $p = q$. This algebra has been generalized to the deformed C_λ-extended oscillator algebra [12].

Naturally, it would be desirable to generalize a particular mathematical structure as much as possible. In particular, this concerns the generalization of the q-deformed universal enveloped of the Lie and hybrid oscillator algebras.

In [13], this oscillator algebra has been generalize to the (p,q)-hybrid two-parameter algebra.

A generalization of the traditional scheme for the deformed algebras to the multiparametric case has been done in [14–16]. In this approach, only the structure functions are changed, but a part of the defining relations was kept invariable.

We generalize the defining relations (3) (or (4)) to obtain the various generalized deformed versions of the oscillator algebras at a fixed structure function (10). We consider only the generalized Daskaloyannis (GD), generalized Chakrabarti–Jagannathan (GCh-J), and generalized Hong Yang (GHY) versions of these algebras.

In this article, we include the short review of our publication [19] on the construction of the unified $(p,q;\alpha,\gamma,l)$-deformation of the oscillator algebras which envelop, as particular cases, the well-known deformations [1–3, 5, 17, 18]. The structure functions ("generalized $(p,q;\alpha,\gamma,l)$-numbers") of these deformations [19] generalize the $(p,q;\alpha,\gamma,l)$-deformation of the oscillator algebra:

$$f(x) = [x]^{\alpha\gamma l}_{pq} = \frac{p^{-\alpha x} - q^{\gamma x}}{p^{-l/\gamma} - q^{l/\alpha}}. \tag{10}$$

Here, $\alpha, \gamma, l \in \mathbb{R}$, contain the additional deformation parameters as compared with (2).

We also study the generalization of this construction to the $(p,q;\alpha,\gamma,l)$-deformed hybrid oscillator algebras with the structure functions

$$\bar{f}(n) = [n]^{\alpha\gamma l}_{pq\nu} = \frac{p^{-\alpha n} - q^{\gamma n}}{p^{-l/\gamma} - q^{l/\alpha}} + 2\nu \frac{(-1)^n p^{-\alpha n} - q^{\gamma n}}{p^{-l/\gamma} - q^{l/\alpha}}, \tag{11}$$

where $p, q, \nu, \alpha, \gamma, l \in \mathbb{R}$, in the case of the broken $(p \leftrightarrow q, \alpha \leftrightarrow -\gamma)$ symmetry and

$$\tilde{f}(n) = [n]^{\alpha\gamma l}_{pq\nu} = \frac{p^{-\alpha(n+\nu(1-(-1)^n)} - q^{\gamma(n+\nu(1-(-1)^n)}}{p^{-l/\gamma} - q^{l/\alpha}}, \tag{12}$$

where $p, q, \nu, \alpha, \gamma, l \in \mathbb{R}$, in the case of the conserved $(p \leftrightarrow q, \alpha \leftrightarrow -\gamma)$ symmetry of the deformed oscillator algebra.

2. Oscillator Algebra and Its Unified Deformations

At the fixed structure function (10), we will consider the basic versions of the generalized deformed oscillator algebras.

The generalized $(p,q;\alpha,\gamma,l)$-deformed Daskaloyannis version of the oscillator algebra. In this case, the defining relations (1) take the form

$$aa^+ = \frac{p^{-\alpha N - l/\gamma} - q^{\gamma N + l/\alpha}}{p^{-l/\gamma} - q^{l/\alpha}}, \quad a^+a = \frac{p^{-\alpha N} - q^{\gamma N}}{p^{-l/\gamma} - q^{l/\alpha}},$$
$$[N, a] = -\frac{l}{\alpha\gamma} a, \quad [N, a^+] = \frac{l}{\alpha\gamma} a^+. \tag{13}$$

The generalized $(p, q; \alpha, \gamma, l)$-deformed Chakrabarti–Jagannathan version of the oscillator algebra. In [19], we have generalized the two-parameter deformed oscillator algebra [5] by introducing the additional parameters α, γ, l in the structure function. This is a generalization of the well-known deformations [1–3, 5, 8, 17, 18] of these algebras. The generators $\{I, a, a^+, N\}$, the structure function (10), and the defining relations

$$aa^+ - p^{-l/\gamma} a^+ a = q^{\gamma N},$$
$$[N, a] = -\frac{l}{\alpha\gamma} a, \quad [N, a^+] = \frac{l}{\alpha\gamma} a^+, \tag{14}$$

or

$$aa^+ - q^{l/\alpha} a^+ a = p^{-\alpha N},$$
$$[N, a] = -\frac{l}{\alpha\gamma} a, \quad [N, a^+] = \frac{l}{\alpha\gamma} a^+, \tag{15}$$

define the generalized $(p, q; \alpha, \gamma, l)$-deformed Chakrabarti–Jagannathan version of the oscillator algebra. It is easy to see that this algebra is represented in the Hilbert space $\mathbb{H}$ with the basis $\{|n\rangle\}_{n=0}^{\infty}$ by

$$a|n\rangle = \left(\frac{p^{-\alpha n} - q^{\gamma n}}{p^{-l/\gamma} - q^{l/\alpha}}\right)^{1/2} \left| n - \frac{l}{\alpha\gamma} \right\rangle,$$
$$a^+|n\rangle = \left(\frac{p^{-\alpha n - l/\gamma} - q^{\gamma n + l/\alpha}}{p^{-l/\gamma} - q^{l/\alpha}}\right)^{1/2} \left| n + \frac{l}{\alpha\gamma} \right\rangle, \tag{16}$$
$$N|n\rangle = n|n\rangle,$$

where $p, q, \alpha, \gamma \in \mathbb{R}$ and $\frac{l}{\alpha\gamma} \in \mathbb{Z}$. The other version of this generalized deformed oscillator algebra is

The generalized $(p, q; \alpha, \gamma, l)$-deformed Hong Yan version of the oscillator algebra. The Hong Yan version of the oscillator algebra [20] is generalized to a deformed algebra as follows.

The generators $\{I, a, a^+, N\}$, structure function (10), and the defining relations

$$aa^+ - \left(p^{-\alpha} q^{\gamma}\right)^{\frac{l}{2\alpha\gamma}} a^+ a = \frac{p^{-\alpha N - \frac{l}{2\gamma}} + q^{\gamma N + \frac{l}{2\alpha}}}{p^{-\frac{l}{2\gamma}} + q^{\frac{l}{2\alpha}}},$$
$$Na - aN = -\frac{l}{\alpha\gamma} a, \quad Na^+ - a^+ N = \frac{l}{\alpha\gamma} a^+ \tag{17}$$

define the generalized $(p, q; \alpha, \gamma, l)$-deformed Hong Yan (GHY) version of the oscillator algebra. The first relation in (17) can be rewritten in the form

$$aa^+ - \left(p^{-\alpha} q^{\gamma}\right)^{\frac{l}{\alpha\gamma}} a^+ a =$$
$$= \left[N + \frac{l}{\alpha\gamma}\right]_{pq}^{\alpha\gamma l} - \left(p^{-\alpha} q^{\gamma}\right)^{\frac{l}{\alpha\gamma}} [N]_{pq}^{\alpha\gamma l}. \tag{18}$$

Relations (13)–(15), (17) define three versions of the generalized $(p, q; \alpha, \gamma, l)$-deformed oscillator algebra. It is easy to see that (13)–(15), (17) at the corresponding values of the deformation parameters are reduced to (1), (3)–(5), respectively.

3. The Generalized $(p, q; \alpha, \gamma, l)$-Deformed Analog of the Harmonic Oscillator

The generalized $(p, q; \alpha, \gamma, l)$-deformed oscillator is described by the Hamiltonian

$$H = \frac{\hbar\omega}{2m}(aa^+ + a^+a), \tag{19}$$

where a, a^+ are the corresponding deformed creation and annihilation operators,

$$H = \frac{\hbar\omega}{2m} \begin{cases} [N]_{pq}^{\alpha\gamma l} + [N + \frac{l}{\alpha\gamma}]_{pq}^{\alpha\gamma l} & \text{(GD) oscill.,} \\ [N]_{pq}^{\alpha\gamma l} + [N + \frac{l}{\alpha\gamma}]_{pq}^{\alpha\gamma l} - \\ -(p^{-\alpha} q^{\gamma})^N C_2 (1 + p^{l/\gamma} q^{l/\alpha}) & \text{(GHY) oscill.,} \\ [N]_{pq}^{\alpha\gamma l} + [N + \frac{l}{\alpha\gamma}]_{pq}^{\alpha\gamma l} - \\ -q^{-\alpha N} C_1 (1 + p^{-l/\alpha}) & \text{(GChJ) oscill.,} \end{cases} \tag{20}$$

where C_1, C_2 are the Casimir operators of the corresponding oscillator algebras.

The spectrum of the Hamiltonian H in the Fock space is

$$E_n = q^{\gamma n} + \left(1 + p^{-l/\gamma}\right) [n]_{pq}^{\alpha\gamma l} \tag{21}$$

or

$$E_n = p^{-\alpha n} + \left(1 + q^{l/\alpha}\right) [n]_{pq}^{\alpha\gamma l}. \tag{22}$$

Apart from this, we have the relation

$$p^{-\alpha m} E_n - p^{-\alpha n} E_m =$$
$$= \frac{1 + l/\alpha}{p^{-l/\alpha} - q^{l/\gamma}} (p^{-\alpha n} q^{\gamma m} - p^{-\alpha m} q^{\gamma n}). \tag{23}$$

The generalized $(p,q;\alpha,\gamma,l)$-deformed oscillator algebras with the structure function (10) conserve the $(p\leftrightarrow q,\alpha\leftrightarrow -\gamma)$ symmetry. Under the transition to the $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebras with the structure function (11), this symmetry breaks down.

4. Some Aspects of $(p,q;\alpha,\gamma,l)$-Hybrid Oscillator Algebras with Broken $(p\leftrightarrow q,\alpha\leftrightarrow -\gamma)$ Symmetry

We consider the basic versions of these $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebras.

The replacement of the structure function (10) by the structure function (11) leads to the $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebras with broken symmetry. We consider the basic versions of these $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebras.

The Daskaloyannis version of the $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebra. In this case, the defining relations (1) can be written as

$$\begin{aligned} &aa^+=\frac{p^{-\alpha N-l/\gamma}-q^{\gamma N+l/\alpha}}{p^{-l/\gamma}-q^{l/\alpha}}+\\ &+2\nu\frac{(-1)^N(-1)^{\frac{l}{\alpha\gamma}}p^{-\alpha N-l/\gamma}-q^{\gamma N+l/\alpha}}{p^{-l/\gamma}+q^{l/\alpha}},\\ &a^+a=\frac{p^{-\alpha N}-q^{\gamma N}}{p^{-l/\gamma}-q^{l/\alpha}}+2\nu\frac{(-1)^Np^{-\alpha N}-q^{\gamma N}}{p^{-l/\gamma}+q^{l/\alpha}},\\ &[N,a]=-\frac{l}{\alpha\gamma}a,\quad \left[N,a^+\right]=\frac{l}{\alpha\gamma}a^+. \end{aligned}\tag{24}$$

The Chakrabarti–Jagannathan version of the $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebra. The relations

$$aa^+-q^{l/\alpha}a^+a=p^{-\alpha N}\begin{cases}(1+2\nu K), & \text{for } \frac{l}{\alpha\gamma} \text{ odd;}\\ (1-2\nu KX), & \text{for } \frac{l}{\alpha\gamma} \text{ even.}\end{cases}$$

$$[N,a]=-\frac{l}{\alpha\gamma}a,\quad \left[N,a^+\right]=\frac{l}{\alpha\gamma}a^+ \tag{25}$$

define the generalized $(p,q;\alpha,\gamma,l)$-hybrid Chakrabarti–Jagannathan version of the oscillator algebra. The Casimir operator of this algebra is

$$C_3=q^{-\gamma N}\left(\frac{p^{-\alpha N}-q^{\gamma N}}{p^{-l/\gamma}-q^{l/\alpha}}+\right.$$
$$\left.+2\nu\frac{(-1)^Np^{-\alpha N}-q^{\gamma N}}{p^{-l/\gamma}-q^{l/\alpha}}-a^+a\right). \tag{26}$$

The generators of algebra (25) in the Hilbert space $\mathbb{H}$ with the basis $\{|n\rangle\}_{n=0}^{\infty}$ are represented by the relations

$$\begin{aligned} &N|n\rangle=n|n\rangle,\\ &a|n\rangle=\left(\frac{p^{-\alpha n}-q^{\gamma n}}{p^{-l/\gamma}-q^{l/\alpha}}+2\nu\frac{(-1)^np^{-\alpha n}-q^{\gamma n}}{p^{-l/\gamma}+q^{l/\alpha}}\right)^{1/2}\times\\ &\times\left|n-\frac{l}{\alpha\gamma}\right\rangle,\\ &a^+|n\rangle=\left(\frac{p^{-\alpha n-l/\gamma}-q^{\gamma n+l/\alpha}}{p^{-l/\gamma}-q^{l/\alpha}}+\right.\\ &\left.+2\nu\frac{(-1)^n(-1)^{\frac{l}{\alpha\gamma}}p^{-\alpha n-l/\gamma}-q^{\gamma n+l/\alpha}}{p^{-l/\gamma}+q^{l/\alpha}}\right)^{1/2}\left|n+\frac{l}{\alpha\gamma}\right\rangle, \end{aligned}\tag{27}$$

where $p,q,\alpha,\gamma\in\mathbb{R}$ and $\frac{l}{\alpha\gamma}\in\mathbb{Z}$.

The Hong Yan version of the $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebra. The generators $\{I,a,a^+,N\}$ satisfy the relations

$$aa^+-\left(p^{-\alpha}q^{\gamma}\right)^{\frac{l}{2\alpha\gamma}}a^+a=$$
$$=\begin{cases}\frac{p^{-\alpha N-l/(2\gamma)}+q^{\gamma N+l/(2\alpha)}}{p^{-l/(2\gamma)}+q^{l/(2\alpha)}}+\\ +2\nu K(p^{-\alpha N-\frac{l}{2\gamma}}+q^{\gamma N+\frac{l}{2\alpha}}), \quad \text{if } \dfrac{l}{\alpha\gamma} \text{ odd;}\\ \frac{p^{-\alpha N-l/(2\gamma)}+q^{\gamma N+l/(2\alpha)}}{p^{-l/(2\gamma)}+q^{l/(2\alpha)}}-\\ -2\nu K(p^{-\alpha N-\frac{l}{2\gamma}}+q^{\gamma N+\frac{l}{2\alpha}}), \quad \text{if } \dfrac{l}{\alpha\gamma} \text{ even.}\end{cases}\tag{28}$$

$$[N,a]=-\frac{l}{\alpha\gamma}a,\quad \left[N,a^+\right]=\frac{l}{\alpha\gamma}a^+,$$

and define the generalized $(p,q;\alpha,\gamma,l)$-deformed Hong Yan (GHY) hybrid version of the oscillator algebra.

5. Some Aspects of $(p,q;\alpha,\gamma,l)$-Hybrid Oscillator Algebras with Preserving $(p\leftrightarrow q,\alpha\leftrightarrow -\gamma)$ Symmetry

At the fixed structure function (12), we will consider the basic versions of the generalized deformed oscillator algebras.

The Daskaloyannis version of the $(p,q;\alpha,\gamma,l)$-hybrid oscillator algebra. In this case the defining relations (1) are as follows:

$$aa^+ = \frac{p^{-\alpha(N+\nu(1+(-1)^N)} - q^{\gamma(N+\nu(1+(-1)^N)}}{p^{-l/\gamma} - q^{l/\alpha}},$$
$$a^+a = \frac{p^{-\alpha(N+\nu(1-(-1)^N)} - q^{\gamma(N+\nu(1-(-1)^N)}}{p^{-l/\gamma} - q^{l/\alpha}}, \quad (29)$$
$$Na - aN = -\frac{l}{\alpha\gamma}a, \quad Na^+ - a^+N = \frac{l}{\alpha\gamma}a^+.$$

The Chakrabarti–Jagannathan version of the $(p,q;\alpha,\gamma,l)$ -hybrid oscillator algebra. The generators of this algebra $(a, a+, N)$ satisfy the relations

$$aa^+ - q^{l/(\alpha\gamma)+\nu(1-(-1)^{l/(\alpha\gamma)}K)}a^+a =$$

$$= p^{-\alpha(N+\nu(1-K))}\begin{cases}[l/(\alpha\gamma)+2\nu K]^{\alpha\gamma}_{pq\nu}, & \text{if } l/(\alpha\gamma) \text{ odd;}\\ 1, & \text{if } l/(\alpha\gamma) \text{ even,}\end{cases}$$

$$Na - aN = -\frac{l}{\alpha\gamma}a, \quad Na^+ - a^+N = \frac{l}{\alpha\gamma}a^+. \quad (30)$$

6. Generalized $(p,q;\alpha,\gamma,l)$-Deformed $U^{\alpha\gamma l}_{pq}(su(2))$ Algebras

Generalized $(p,q;\alpha,\gamma,l)$-deformed Jordan–Schwinger realization of the $U^{\alpha\gamma l}_{pq}(su(2))$ algebra. The classical Lie algebra $su(2)$ is defined by the generators (J_0, J_+, J_-) and the commutation relations

$$[J_0, J_+] = J_+, \ [J_0, J_-] = -J_-, \ [J_+, J_-] = 2J_0. \quad (31)$$

In this section, we will consider the generalized Jordan–Schwinger and Holstein–Primakoff realizations of the $U^{\alpha\gamma l}_{pq}(su(2))$ algebras by the creation and annihilation operators of the generalized $(p,q;\alpha,\gamma,l)$-deformed oscillator algebras.

Two independent basis collections (I, a, a^+N_a) and (I, b, b^+, N_b) of the generalized $(p,q;\alpha,\gamma,l)$-deformed oscillator algebras define the generalized Jordan–Schwinger realization of the $U^{\alpha\gamma l}_{pq}(su(2)$ algebra:

$$J_0 = \frac{1}{2}(N_a - N_b), \quad \tilde{C} = \frac{1}{2}(N_a + N_b).$$

$$J_+ = \left(p^\alpha q^{-\gamma}\right)^{N_b/2} a^+b, J_- = b^+a\left(p^\alpha q^{-\gamma}\right)^{N_b/2}. \quad (32)$$

In the case of collections of the generalized $(p,q;\alpha,\gamma,l)$-deformed Chakrabarti–Jagannathan type oscillator algebra (14) (or (15)), we obtain the $U^{\alpha\gamma l}_{pq}\Big(su(2)\Big)$ generalized algebra with the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad J_+J_- - (p^\alpha q^{-\gamma})^{\frac{l}{\alpha\gamma}}J_-J_+ =$$
$$= \left(1 - C_1\left(p^{-l/\gamma} - q^{l/\alpha}\right)\right)[2J_0]^{\alpha\gamma l}_{pq}, \quad (33)$$

where the Casimir operator C_1 of the oscillator algebra (14) is defined by

$$C_1 = p^{\alpha N}\left(\frac{p^{-\alpha N} - q^{\gamma N}}{p^{-l/\gamma} - q^{l/\alpha}} - a^+a\right). \quad (34)$$

In the case of collections of the $(p,q;\alpha,\gamma,l)$-deformed Hong Yan type oscillator algebra (17), the $U^{\alpha\gamma l}_{pq}\Big(su(2)\Big)$ generalized deformed algebra is defined by the commutation relations

$$[J_0, J_\pm] = \pm J_\pm,$$

$$J_+J_- - \left(p^\alpha q^{-\gamma}\right)^{\frac{l}{\alpha\gamma}}J_-J_+ =$$

$$= [2J_0]^{\alpha\gamma l}_{pq} + \left(p^\alpha q^{-\gamma}\right)^{\tilde{C}-J_0}C_2\left(\left[\tilde{C} - J_0 + \frac{l}{\alpha\gamma}\right] -\right.$$

$$\left.-\left(p^{-\alpha}q^{\gamma}\right)^{2J_0}[\tilde{C} - J_0] - p^{-\alpha}q^{\gamma}\right)^{2\tilde{C}}\left[\tilde{C} + J_0 + \frac{l}{\alpha\gamma}\right] +$$

$$+\left(p^{-\alpha}q^{\gamma}\right)^{\frac{l}{\alpha\gamma}}[\tilde{C} + J_0], \quad (35)$$

where C_2 is the Casimir operator of the oscillator algebra (17),

$$C_2 = \left(p^\alpha q^{-\gamma}\right)^N([N]^{\alpha\gamma l}_{pq} - a^+a). \quad (36)$$

Generalized $(p,q;\alpha,\gamma,l)$-deformed Holstein–Primakoff realization of $U^{\alpha\gamma l}_{pq}(su(2))$ algebra. The algebra $su(2)$ is realized by the one basis collection (a, a^+, N_a) of the operators of the ordinary harmonic oscillator. It is defined by the Holstein–Primakoff transformations

$$J_+ = a^+(2j-N)^{1/2}, \ J_- = (2j-N)^{1/2}a, \ J_0 = N-j, \quad (37)$$

where j is a c-number. The q-deformed Holstein–Primakoff analog of the algebra $su(2)$ has been studied in [21].

The generalized Holstein–Primakoff realization of the $U_{pq}^{\alpha\gamma l}(su(2))$ algebra defined by the one collection of the $(p,q;\alpha,\gamma,l)$-deformed oscillator algebra is given by

$$J_+ = (p^\alpha q^{-\gamma})^{N/2} a^+ \sqrt{[2j-N]_{pq}^{\alpha\gamma l}},$$
$$J_- = \sqrt{[2j-N]_{pq}^{\alpha\gamma l}}\, a\left(p^\alpha q^{-\gamma}\right)^{N/2}, \qquad (38)$$
$$J_0 = N - j,$$

where j is some c-number. For the collection of the generalized Chakrabart–Jagannathan oscillator algebra (14) we obtain the $U_{pq}^{\alpha\gamma l}\Big(su(2)\Big)$ algebra with the defining relations

$$[J_0, J_\pm] = \pm J_\pm,$$
$$J_+J_- - (p^\alpha q^{-\gamma})^{\frac{l}{\alpha\gamma}} J_-J_+ = [-2J_0] + C_1 q^{-2\gamma J_0}. \quad (39)$$

7. The $(p,q;\alpha,\gamma,l)$-Deformation of the Two-Dimensional Conformal Field Theory

The different deformation schemes of oscillator algebras have been considered, in parallel, to the construction of the two-dimensional conformal field theories [22–25]. We study some properties of the primary fields of the two-dimensional deformed conformal field theory with the employment of the $(p,q;\alpha,\gamma,l)$-deformation scheme. Let $\phi(z,\bar z)$, be the primary field of conformal weights $(h,\bar h)$ of the $(p,q;\alpha,\gamma,l)$-deformed conformal field theory. Its generalized infinitesimal $(p,q;\alpha,\gamma,l)$-transformations are defined by the relation

$$\delta_{\varepsilon,pq}^{\alpha\gamma l}\phi(z) = \varepsilon(z)^{1-h} D_{pq}^{\alpha\gamma l}(\varepsilon(z))^h\phi(z)), \qquad (40)$$

where

$$D_{pq}^{\alpha\gamma l}\phi(z) = \frac{\phi(p^{-\alpha}z) - \phi(q^\gamma z)}{z(p^{-l/\gamma} - q^{l/\alpha})} =$$
$$= \frac{1}{z}\cdot\frac{p^{-\alpha z\partial} - q^{\gamma z\partial}}{p{-}l/\alpha - q^{l/\gamma}}\phi(z) = \frac{1}{z}[z\partial]_{pq}^{\alpha\gamma l}\phi(z). \qquad (41)$$

Henceforth, we will restrict ourselves by the consideration of the holomorphic terms. We have

$$\delta_{n,pq}^{\alpha\gamma l}\phi(z) = z^n[z\partial + h(n+1)]_{pq}^{\alpha\gamma l}\phi(z) \qquad (42)$$

for the $(p,q;\alpha,\gamma,l)$-deformed primary field $\phi(z)$ with the conformal weight h. As in the classical case, the variation of a conformal field $\phi(z)$ is given by the "equal-time" commutator

$$\delta_{n,pq}^{\alpha\gamma l}\phi(z) = \left[\oint_{C_0} \frac{dz}{2\pi i} z^n T(z), \phi(w)\right] =$$
$$= \frac{1}{2\pi i}\oint_{C_P} dz z^{n+1} R\Big(T(z)\phi(w)\Big)_{p,q}^{\alpha\gamma l}, \qquad (43)$$

where $T(z)$ is the holomorphic component of the energy-momentum tensor, and C_0 and C_P are contours encircling the origin and the all poles in the OPE of $\big(T(z)\phi(w)\big)_{pq}^{\alpha\gamma l}$, respectively. The notation of the time ordering on the z plane is replaced by that of radial ordering

$$R(A(z)B(w)) = \begin{cases} A(z)B(w), & \text{if } |z| > |w|, \\ B(w)A(z), & \text{if } |z| < |w|. \end{cases}$$

Defining the product of two field operators by the formula

$$\big(A(z)B(w)\big)_{pq}^{\alpha\gamma l} = A(zq^\gamma)B(wp^{-\alpha})$$

we see that

$$\big(T(z)\phi(w)\big)_{pq}^{\alpha\gamma l} =$$
$$= \frac{1}{w(p^{-l/\gamma} - q^{l/\alpha})}\left\{\frac{\phi(wp^{-\alpha})}{z - wp^{-\alpha h}} - \frac{\phi(wq^\gamma)}{z - wq^{\gamma h}}\right\} \qquad (44)$$

leads to the correct variation (42) in $\phi(z)$ with the help of the evaluation of the integral in (43) with C_P taken as a contour encircling the points $wp^{-\alpha h}, wq^{\gamma h}$.

Introducing the modes

$$L_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} T(z), \quad n \in \mathbb{Z} \qquad (45)$$

of a holomorphic component of the energy-momentum tensor $T(z)$ and the modes

$$\phi_n = \oint_{C_0} \frac{dw}{2\pi i} w^{n+h-1}\phi(w) \qquad (46)$$

of a primary field $\phi(w)$ of the conformal weight h, we obtain

$$p^{\alpha(m+h))} q^{-\gamma(n+2)} L_n\phi_m - p^{\alpha(n+2)} q^{-\gamma(m+h)}\phi_m L_n =$$
$$= [(h-1)n - m]_{pq}^{\alpha\gamma l}\phi_{m+n}. \qquad (47)$$

If we set $\phi_m = L_m$, $h = 2$ in this equation, we obtain the centerless $(p,q;\alpha,\gamma,l)$-deformed Virasoro algebra

$$p^{\alpha(m+2))}q^{-\gamma(n+2)}L_nL_m - p^{\alpha(n+2)}q^{-\gamma(m+2)}L_mL_n =$$
$$= [n-m]^{\alpha\gamma}_{pq}L_{m+n}, \quad (48)$$

which coincides at the $\alpha = \gamma = l$ and $p = q$ with the algebra obtained in [26].

The generalized $U^{\alpha\gamma l}_{pq}\Big(su(1,1)\Big)$-deformed subalgebra of the $(p,q;\alpha,\gamma,l)$-deformed Virasoro algebra $U^{\alpha\gamma l}_{pq}(\mathcal{V}ir)$ is defined by the commutation relations

$$\begin{aligned} &K_-K_+ - (p^\alpha q^{-\gamma})^{\frac{l}{\alpha\gamma}}K_+K_- == [2K_0]^{\alpha\gamma l}_{pq}, \\ &[K_0,K_+] = K_+, \quad [K_0,K_-] = -K_-, \end{aligned} \quad (49)$$

where

$$[2K_0]^{\alpha\gamma l}_{pq} = \frac{p^{-2\alpha K_0} - q^{2\gamma K_0}}{p^{-l/\gamma} - q^{l/\alpha}}.$$

It is convenient to define the generators $M = p^{-\alpha K_0}$, $N = q^{\gamma K_0}$.

There exists the following representation of this algebra on the conformal fields $\phi(z)$ with conformal weights h

$$\begin{aligned} &K_{-1}\phi(z) = \frac{1}{z}\frac{\phi(p^{-\alpha}z) - \phi(q^\gamma z)}{p^{-l/\gamma} - q^{l/\alpha}}, \\ &K_{+1}\phi(z) = z\frac{p^{-2\alpha h}\phi(p^{-\alpha}z) - q^{2\gamma h}\phi(q^\gamma z)}{p^{-l/\gamma} - q^{l/\alpha}}, \\ &M\phi(z) = p^{-\alpha h}\phi(p^{-\alpha}z), \quad N = q^{\gamma h}\phi(q^\gamma z). \end{aligned} \quad (50)$$

The co-algebra structure of this algebra is defined on the generators of this algebra by the expressions

$$\begin{aligned} &\Delta(K_\pm) = M\otimes K_\pm + K_\pm\otimes N, \quad \Delta(M) = M\otimes M, \\ &\Delta(N) = N\otimes N. \end{aligned} \quad (51)$$

By the analogy with [23, 24], the $(p,q;\alpha,\gamma,l)$-deformed Ward–Takahashi identities for the n-point correlation functions of the primary fields $\phi_1(z_1), \phi_2(z_2), \ldots, \phi_n(z_n)$ can be written as

$$\overbrace{(\Delta\otimes id\ldots\otimes id)}^{N-1}\overbrace{(\Delta\otimes id\ldots\otimes id)}^{N-2}\ldots(\Delta\otimes id)\Delta(K_\pm)$$
$$\langle(\phi_1(z_1)\ldots\phi_N(z_N)\rangle^{\alpha\gamma l}_{pq} = 0. \quad (52)$$

The two-point correlation function $\langle\phi_1(z_1)\phi_2(z_2)\rangle^{\alpha\gamma l}_{pq}$ of the primary fields $\phi_1(z_1), \phi_2(z_2)$ is defined from the equations

$$\begin{aligned} &\Delta(K_{-1})\langle\phi_1(z_1)\phi_2(z_2)\rangle^{\alpha\gamma l}_{pq} = 0, \\ &\Delta(K_{+1})\langle\phi_1(z_1)\phi_2(z_2)\rangle^{\alpha\gamma l}_{pq} = 0. \end{aligned} \quad (53)$$

If $h_1 = h_2 = h$, the solution of this system can be represented in the form [19]

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle^{\alpha\gamma l}_{pq} = z_1^{-2h}\frac{(az;r)_\infty}{(z;r)_\infty}, \quad (54)$$

where $a = (p^\alpha q^\gamma)^{2h}, z = (p^\alpha q^\gamma)^{-h}\frac{z_2}{z_1}, r = p^\alpha q^\gamma$, (see the notations in [31]).

8. Summary and Conclusions

In this article, we have presented the construction of the generalized $(p,q;\alpha,\gamma,l)$-deformed oscillator and $(p,q;\alpha,\gamma,l)$-hybrid algebras. This unification conserves the basic properties of the well-known deformed algebras. We have found the Jordan–Schwinger and Holstein–Primakoff realizations of the $U^{\alpha\gamma l}_{pq}(su(2))$ algebra by the generalized deformed creation and annihilation operators. This unification is an "attempt to introduce some order in the rich and variable choice of deformed commutation relations" [27]. We also hope for that it will find the applications to the solutions of specific physical problems [28–30].

1. L.C. Biedenharn, J. Phys. A **22**, L 837 (1989).
2. J. Macfarlane, J. Phys. A **22**, L 581 (1989).
3. M. Arik and D.D. Coon, J. Math. Phys. **17**, 524 (1976).
4. C. Daskaloyannis, J. Phys: Math. General. A **24**, 789 (1991).
5. R. Chakrabarti and R. Jagannathan, J. Phys A **24**, L 711 (1991).
6. S. Jing, IL Nuovo Cimento A **105**, 1267 (1997).
7. M. Arik, E. Demircan, T. Turgut, L. Ekinci, and M. Mungan, Z. Phys. C **55**, 89 (1992).
8. L.C. Kwek and C.H. Oh, Eur. J. Phys. C **5**, 189 (1998).
9. M.A. Vasiliev, J. Mod. Phys. A **6**, 1115 (1991).
10. L. Brink, T.H. Hanson, and M.A. Vasiliev, Phys. Lett. B **109**, 1115 (1992).
11. T. Brzezinski, I.L. Egusquiza, and A. Macfarlane, Phys. Lett. B **276**, 109 (1992).
12. C. Quesne and N. Vanstineekiste, J. Theor. Phys. **39**, 1175 (2000).
13. R. Chakrabarti and R. Jagannathan, J. Phys. A: Math. Gen. **25**, 2325 (1992).

14. I.M. Burban, Phys. Lett. A **366**, 308 (2007).
15. J.B. Geloun and M.N. Hounkonnou, J. Phys. A: Math. Gen. **40**, 817 (2007).
16. I.M. Burban, Ukr. J. Phys. **57**, 396 (2012).
17. W-Sn. Chung, K.S. Chung, S.T. Nam, and C.I. Um, Phys. Lett. A **183**, 363 (1993).
18. V.V. Borzov, E.V. Damaskinsky, and S.B. Yagorov, J. Phys. **22**, 83 (1989).
19. I.M. Burban, Phys. Lett. A **377**, 2863 (2013).
20. H. Yan, J. Phys:. Math. Gen. **23**, L 1150 (1990).
21. M. Chaichian, D. Ellinas, and P. Kulsh, Phys. Rev. Lett. **65**, 980 (1990).
22. C.H. Oh and K. Singh, J. Phys. A **25**, L 149 (1992).
23. D. Bernard and A. LeClair, Phys. Lett. B **227**, 41 (1989).
24. T. Matsuzaki, Suzuki, Phys. Lett. B **296**, 33 (1992).
25. W-Sn. Chung, IL Nuovo Cimento B **109**, 599 (1994).
26. T.L. Curtright and C.K. Zachos, Phys. Lett. B **243**, 237 (1990).
27. C. Quesne and N. Vanstineekiste, Czech. J. Phys. B **47**, 115 (1997).
28. A.N.F. Aleixo and A.B. Balatekin, J. Math. Phys. **50**, 112103 (2009).
29. D. Bonatos and C. Dascaloyannis, Mod. Phys. Lett. B **366**, 308 (1996).
30. S.S. Mizrahi, J.P. Camargo Lima, and V.V. Dodonov, J. Phys. A: Math. General **37**, 3707 (2004).
31. G. Gasper and R. Rahma, *Basic Hypergeometric Series* (Cambridge Univ. Press, Cambridge, 1990).

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ОБ’ЄДНАНІ $(p, q; \alpha, \gamma, l)$-ДЕФОРМАЦІЇ ОСЦИЛЯТОРНИХ ТА ГІБРИДНИХ ОСЦИЛЯТОРНИХ АЛГЕБР І ДВОВИМІРНОЇ КОНФОРМНОЇ ТЕОРІЇ ПОЛЯ

Р е з ю м е

Метою цієї статті є огляд і доповнення наших результатів щодо побудови узагальнених $(p, q; \alpha, \gamma, l)$-деформованих осциляторних і гібридних осциляторних алгебр. Основні версії цих деформацій отримані за допомогою введення нових вільних параметрів в структурні функції і узагальнення визначальних співвідношень цих алгебр. Побудовані узагальнені Йордан–Швінгера та Голстейн–Примакова реалізації цих алгебр. Побудована $(p, q; \alpha, \gamma, l)$-деформація двовимірної конформної теорії поля. Знайдена полюсна структура голоморфної компоненти тензора енергії-імпульсу. Обчислена двохточкова кореляційна функція у конформній теорії поля.