doi: 10.15407/ujpe61.11.0943

V.I. ZHABA

Uzhgorod National University
(54, Voloshyn Str., Uzhgorod 88000, Ukraine; e-mail: viktorzh@meta.ua)

APPROXIMATION OF THE DEUTERON WAVE FUNCTION WITH THE USE OF NIJMEGEN POTENTIALS AND DEUTERON POLARIZATION CHARACTERISTICS

PACS 21.45.Bc, 13.75.Cs

The coefficients of analytic forms for the deuteron wave function in the configuration representation for Nijmegen potentials have been calculated numerically. The obtained wave functions do not contain redundant nodes. The calculated parameters of a deuteron agree well with experimental and theoretical data. The polarization characteristics T_{20} and A_{yy} calculated with the use of the obtained wave functions are comparable with earlier results.

K e y w o r d s: wave function, analytic form, deuteron, polarization, node.

1. Introduction

A deuteron is the simplest nucleus consisting of two strongly interacting particles: a proton and a neutron. The simplicity of the deuteron structure makes it a convenient laboratory for studying the nucleon-nucleon forces.

At present, the deuteron is well studied both experimentally and theoretically. In particular, the calculations of its static parameters – the binding energy, magnetic moment, electric quadrupole moment, and so on – agree well with experimental data [1]. Nevertheless, there are a number of theoretical discrepancies. For instance, in some theoretical works, either one (Bonn potential) [2] or both (Moscow potential) [3] components of the wave function have nodes near the coordinate origin. This behavior of the wave function contradicts the general mathematical theorem about the number of nodes of the characteristic functions of boundary-value problems [4].

The ground state of any system is described by a function that has no nodes in the middle part of the interval of a boundary-value problem. The presence of nodes in the wave function for the ground (single) state of a deuteron may testify to certain mismatches in the realization of numerical algorithms applied to the solution of such problems. In work [5], it was shown that the asymptotics of components for the solution of a system of equations is not determined by only the orbital moment L, as it takes place for the solutions of a single Schrödinger equation. Instead, it is associated with the total orbital momentum J and the asymptotic behavior of the tensor potential at short distances, which gives rise to that the corresponding Schrödinger equations form a system of equations. The issue how the choice of numerical algorithms affects the solutions of a problem was analyzed in works [5–7].

Such potentials of nucleon-nucleon interaction as Bonn [2], Moscow [3], Nijmegen (NijmI, NijmII, Nijm93) [8], Argonne v18 [9], and Paris [10] ones have rather difficult structures and cumbersome expressions. The original Reid potential, Reid68, was parametrized on the basis of the phase analysis carried out by the Nijmegen group and called Reid93. Fifty parameters of the potential were parametrized with $\chi^2/N_{\rm data} = 1.03$ [8].

In addition, the wave function of a deuteron can be presented in a tabular form by means of the cor-

responding arrays of values for the radial wave functions. Sometimes, it is rather difficult to operate with such arrays of numbers in numerical calculations. Moreover, the corresponding computational code becomes too large. Therefore, it is expedient to obtain simpler analytic forms describing the wave functions of a deuteron.

Table 1. **Coefficients of the analytic form (1) for the potential NijmI**

i	A_i	a_i	B_i	b_i
1	−0.5985755	1.2486288	0.1341184	1.9583386
2	0.0719881	0.0906381	0.0178965	0.7636754
3	0.0638354	0.0909362	0.0472457	0.8111266
4	0.4978059	5.8301436	−0.0139045	1.4014708
5	−0.4845549	7.2874431	−0.0139327	0.7102277
6	0.0644466	0.0908163	−0.0144313	1.4089862
7	0.1226875	0.0288025	0.0528106	0.8135834
8	0.3600914	8.5601832	−0.0137620	1.3980507
9	−0.4845549	7.2874431	0.0239859	0.1663082
10	0.0742084	0.0101564	0.0002893	0.0170395
11	0.1095527	0.0273786	0.0435291	0.7991969
12	0.3600914	8.5601832	−0.3830299	3.1380815
13	−0.4845549	7.2874431	0.0043372	0.0603978
14	0.0735584	0.0101562	0.0141490	0.7841558
15	0.0291958	0.0039399	0.0664880	0.3772602
16	0.3600914	8.5601832	0.1219109	1.9261182
17	−0.1282627	20.4355213	−0.0144508	1.3816406

Table 2. **Coefficients of the analytic form (1) for the potential NijmII**

i	A_i	a_i	B_i	b_i
1	−0.0278778	1.6021971	0.0004746	0.0792755
2	0.0589271	0.1092446	0.1662781	0.3393720
3	0.0426398	0.0045215	0.1148787	0.3375295
4	0.6597067	4.9769248	0.1129263	0.0202313
5	−0.4068244	1.1886875	−0.1309857	0.5165437
6	0.0716066	0.0389344	0.1866132	0.3393763
7	0.1108224	0.0106422	−0.1191002	0.3391629
8	0.3002277	1.3102922	−0.1408905	0.4176138
9	−0.4068244	1.1886875	−0.1433174	0.5206460
10	0.1093643	0.0180780	0.1866614	0.4180024
11	0.1245369	0.0389761	−0.1656813	0.4172884
12	0.3002277	1.3102922	−0.1245236	0.4172083
13	−0.4068244	1.1886875	−0.1694870	0.5206702
14	0.0516403	0.1092001	0.2488424	0.4164459
15	0.0473036	0.1082425	0.0004746	0.4178253
16	0.3002277	1.3102922	0.1662781	0.4163094
17	−0.9274669	3.3417195	0.1148787	0.4852219

2. Analytic Form for the Deuteron Wave Function

The known numerical values of the radial wave function of a deuteron in the coordinate representation can be approximated analytically with the help of either convenient series expansions [11]

$$\begin{cases} u_a(r) = \sum\limits_{i=1}^{N_a} A_i \exp(-a_i r^2), \\ w_a(r) = r^2 \sum\limits_{i=1}^{N_a} B_i \exp(-b_i r^2) \end{cases} \tag{1}$$

or as the asymmetric double-sigmoid function [7]

$$R_l = C_0 + C_1 \frac{1}{1+\exp\left(-\frac{r-C_2+C_3/2}{C_4}\right)} \times \\ \times \left(1 - \frac{1}{1+\exp\left(-\frac{r-C_2-C_3/2}{C_5}\right)}\right). \tag{2}$$

The parameter C_0 in Eq. (2) is not always positive. In this case, there arises a redundant node.

Despite the cumbersome and long calculations when minimizing χ^2 to values less than 10^{-4}, we had to approximate 839×4-arrays of numerical values for the deuteron wave functions for the Nijmegen-group potentials (NijmI, NijmII, and Nijm93 [12,13]) within the interval $r = 0 \div 25$ fm. The values obtained in the case $N_a = 17$ for the parameters A_i, a_i, B_i, and b in formulas (1) are quoted in Tables 1 to 3. The corresponding values for the case $N_a = 13$ can be found in work [11].

Besides Eq. (1), there is another analytic form for the deuteron wave function [2, 10, 14],

$$\begin{cases} u_b(r) = \sum\limits_{j=1}^{N_b} C_j \exp(-m_j r), \\ w_b(r) = \sum\limits_{j=1}^{N_b} D_j \exp(-m_j r)\left[1 + \frac{3}{m_j r} + \frac{3}{(m_j r)^2}\right], \end{cases} \tag{3}$$

where $m_j = \beta + (j-1)m_0$, $\beta = \sqrt{ME_d}$, $m_0 = 0.9$ fm^{-1}, M is the nucleon mass, and E_d the deuteron binding energy.

The asymptotics of the deuteron wave function at $r \to \infty$ are

$u(r) \sim A_S \exp(-\beta r)$,

$$w(r) \sim A_D \exp(-\beta r) \left[1 + \frac{3}{\beta r} + \frac{3}{(\beta r)^2}\right],$$

where A_S and A_D are the normalizing coefficients for the S- and D-state, respectively.

At $N_b > 11$, the coefficients in the analytic form (3) are determined from the formulas [14]

$$\sum_{j=1}^{N_b} C_j = 0, \quad \sum_{j=1}^{N_b} D_j = \sum_{j=1}^{N_b} D_j m_j^2 = \sum_{j=1}^{N_b} \frac{D_j}{m_j^2} = 0. \quad (4)$$

The accuracy of parametrization (1) is characterized by the quantities

$$I_S = \left(\int_0^\infty [u(r) - u_a(r)]^2 \, dr \right)^{1/2} = 2.5 \cdot 10^{-4},$$

$$I_D = \left(\int_0^\infty [w(r) - w_a(r)]^2 \, dr \right)^{1/2} = 1.7 \cdot 10^{-4}.$$

The coefficients of the analytic form (3) were determined for the Bonn [2] and Paris [10] potentials in the case $N_b = 11$.

Table 3. **Coefficients of the analytic form (1) for the potential Nijm93**

i	A_i	a_i	B_i	b_i
1	−0.8525983	1.8425206	0.0007130	0.0281536
2	0.0348329	0.0042124	0.0410172	0.4153763
3	0.0910268	0.0769507	0.0410173	0.4153765
4	0.8599717	2.6359631	0.0087985	1.6922442
5	−0.3003584	0.7363712	0.0493539	0.9800333
6	0.0048787	0.1298680	0.0099698	0.8098339
7	0.1605685	0.3899728	−0.0028394	0.8482049
8	0.4708801	1.1261553	0.0177657	1.7152607
9	−0.3003584	0.7363712	0.0493539	0.9800333
10	0.1142999	0.0341238	0.0053267	0.0738306
11	0.0871801	0.0906278	0.0099695	0.8098346
12	0.4708801	1.1261553	0.1264523	2.4340326
13	−0.3003584	0.7363712	0.0493539	0.9800333
14	0.1210117	0.0234265	0.0254155	0.1787281
15	0.1390945	0.0104395	0.0000441	0.0095141
16	0.4708801	1.1261553	0.0177657	1.7152607
17	−1.2640547	1.8246238	−0.3900541	4.0909729

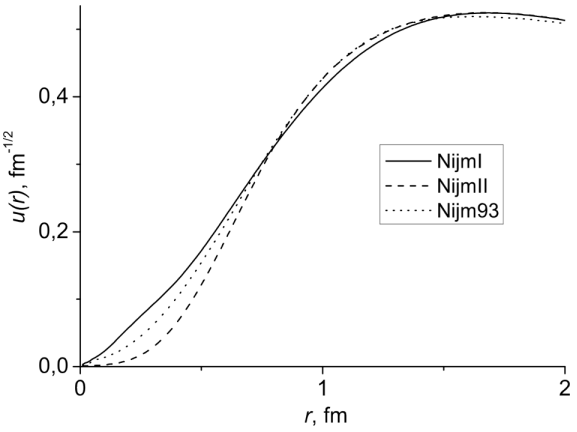

Fig. 1. Deuteron wave function $u(r)$

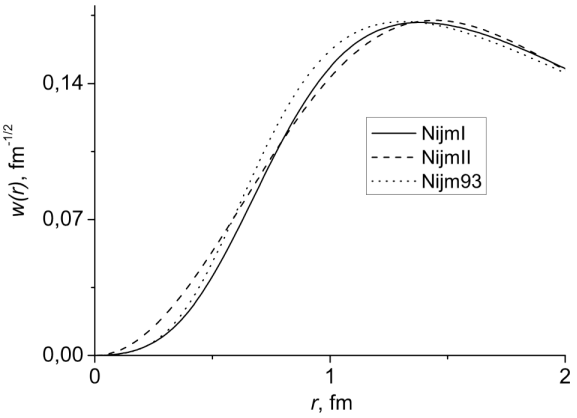

Fig. 2. Deuteron wave function $w(r)$

The radial wave functions of a deuteron, $u(r)$ and $w(r)$, calculated with the use of the analytic forms (1) in the configuration representation do not contain redundant nodes. Figures 1 and 2 illustrate the behavior of the deuteron wave functions near the coordinate origin. The obtained wave functions of a deuteron correlate well with the results of work [13].

3. Deuteron Parameters

On the basis of the wave functions (1) for a deuteron in the coordinate representation, the following deuteron parameters can be calculated [2, 6]:

- the deuteron radius

$$r_d = \frac{1}{2} \left\{ \int_0^\infty r^2 \left[u^2(r) + w^2(r)\right] dr \right\}^{1/2};$$

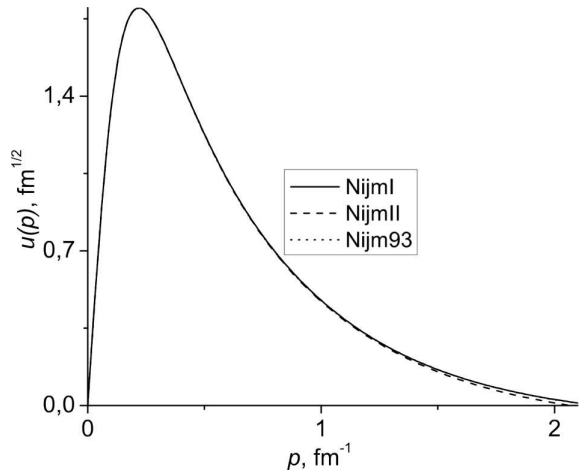

Fig. 3. Deuteron wave function $u(p)$

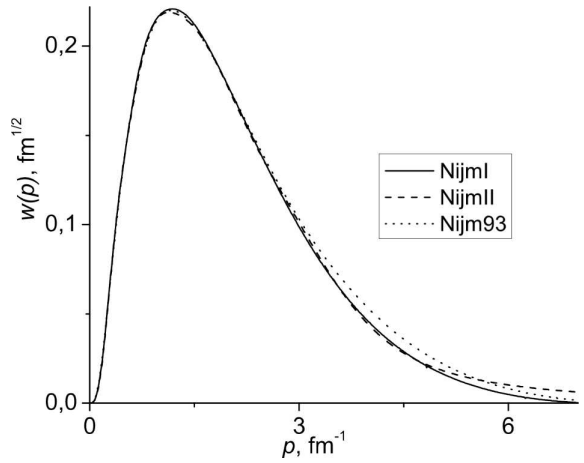

Fig. 4. Deuteron wave function $w(p)$

- the electric quadrupole moment

$$Q_d = \frac{1}{20}\int\limits_0^\infty r^2 w(r)\left[\sqrt{8}u(r) - w(r)\right]dr;$$

- the magnetic moment

$$\mu_d = \mu_s - \frac{3}{2}\left(\mu_s - \frac{1}{2}\right)P_D;$$

- the contribution of D-state

$$P_D = \int\limits_0^\infty w^2(r)dr;$$

- and the D/S-asymptotics of the state

$$\eta = A_D/A_S.$$

The calculated deuteron parameters are quoted in Table 4 (they are marked as (1)). They agree well with experimental results [1] and with theoretical results obtained in original work [8] for the same potentials.

4. Polarization Characteristics of a Deuteron

It is expedient to use the obtained wave functions (1) for the calculation of the polarization characteristics of a deuteron [15].

The measurement of the polarization characteristics for the deuteron fragmentation reaction $A(d,p)X$ at intermediate and high energies remains one of the basic instruments for studying the deuteron structure. In the momentum representation, the momentum dependence of the component of the sensitivity tensor to the deuteron polarization T_{20} is completely determined by the wave function, which depends only on the momentum p [16]:

$$T_{20} = \frac{1}{\sqrt{2}}\frac{2\sqrt{2}u(p)w(p) - w(p)^2}{u(p)^2 + w(p)^2}, \tag{5}$$

where $u(p)$ and $w(p)$ are the deuteron wave functions in the momentum representation. They can be obtained from the functions $u(r)$ and $w(r)$ by means of the Hankel transformations,

$$u(p) = \int\limits_0^\infty u(r)j_0(pr)dr;$$

$$w(p) = \int\limits_0^\infty w(r)j_2(pr)dr,$$

where $j_0(pr)$ and $j_2(pr)$ are the Bessel functions of the zero and second orders, respectively. Here, the functions $u(r)$ and $w(r)$ have the analytic form (1).

Table 4. **Deuteron parameters**

Potential	P_D, %	r_m, fm	Q_d, fm^2	η
Nijm I (1)	5.66388	1.9664	0.271371	0.0272395
Nijm I [8]	5.664	1.967	0.2719	0.0253
Nijm II (1)	5.62951	1.96747	0.269724	0.0274549
Nijm II [8]	5.635	1.968	0.2707	0.0252
Nijm 93 (1)	5.75307	1.96583	0.270526	0.0252098
Nijm 93 [8]	5.755	1.966	0.2706	0.0252

The deuteron wave functions calculated in the momentum representation are depicted in Figs. 3 and 4. In the small-momentum interval, the shape of the component $u(p)$ is somewhat different from the values quoted in work [12]. Evidently, this discrepancy may be associated with the method of calculation of $u(p)$ or with the choice of a numerical procedure.

The formula for T_{20} is the same as in the one-nucleon exchange model [16] and in work [17]. Experimentally, the component T_{20} is determined in terms of the cross-sections corresponding to the spin projections of initial deuteron $(+1, 0, -1)$ onto the axis [16],

$$T_{20} = \frac{1}{\sqrt{2}} \frac{\left(\frac{d\sigma}{d\Omega}\right)_+ + \left(\frac{d\sigma}{d\Omega}\right)_- - 2\left(\frac{d\sigma}{d\Omega}\right)_0}{\left(\frac{d\sigma}{d\Omega}\right)_+ + \left(\frac{d\sigma}{d\Omega}\right)_- + \left(\frac{d\sigma}{d\Omega}\right)_0}.$$

The quantity T_{20} calculated by formula (5) for the Nijmegen potentials (Fig. 5) correlates well with the results of work [18].

Besides the component T_{20} of the deuteron polarization sensitivity tensor, the deuteron wave function in the momentum representation can be used to calculate the polarization transfer K_0 (see Fig. 6):

$$K_0 = \frac{u(p)^2 - w(p)^2 - u(p)w(p)/\sqrt{2}}{u(p)^2 + w(p)^2}. \tag{6}$$

The value of the tensor analyzing power A_{yy} is obtained experimentally from the numbers of protons registered for different beam polarization modes (n^+, n^-, and n^0) after making a correction for the dead time of the installation. The corresponding expression looks like [19]

$$A_{yy} = 2\frac{p_z^-(n^+/n^0 - 1) - p_z^+(n^-/n^0 - 1)}{p_z^- p_{zz}^+ - p_z^+ p_{zz}^-}. \tag{7}$$

Theoretically, the tensor analyzing power A_{yy} in the momentum representation is calculated with the use of the formula [15]

$$A_{yy} = \frac{T_{00}^2 - T_{11}^2 + 4P^2T_{10}^2}{T_{00}^2 + 2T_{11}^2 + 4P^2T_{10}^2}, \tag{8}$$

where $T_{ij}(p/2)$ are amplitudes determined with the help of the wave functions $u(r)$ and $w(r)$ as follows:

$$T_{00} = S_0(p/2) + \sqrt{2}S_2(p/2),$$

$$T_{11} = S_0(p/2) - \frac{1}{\sqrt{2}}S_2(p/2),$$

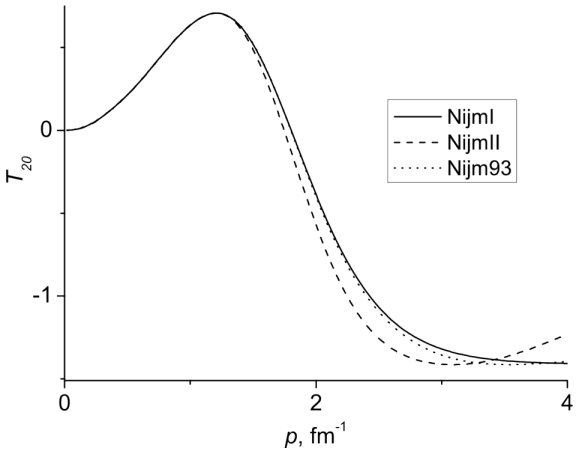

Fig. 5. Deuteron tensor polarization component T_{20}

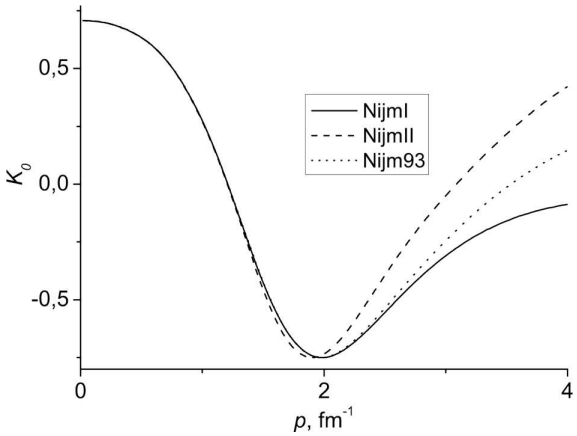

Fig. 6. Polarization transfer K_0

$$T_{10} = \frac{i}{\sqrt{2}} \int_0^\infty \left(u^2(r) - \frac{w^2(r)}{2}\right) j_0(rp/2)dr +$$

$$+ \frac{i}{2} \int_0^\infty w(r) \left(u(r) + \frac{w(r)}{\sqrt{2}}\right) j_2(rp/2)dr. \tag{9}$$

Here, S_0 and S_2 are the spherical and quadrupole, respectively, form factors of a deuteron:

$$S_0(p/2) = \int_0^\infty (u^2(r) + w^2(r)) j_0(rp/2)dr,$$

$$S_2(p/2) = \int_0^\infty 2w(r) \left(u(r) - \frac{1}{2\sqrt{2}}w(r)\right) j_2(rp/2)dr. \tag{10}$$

In formula (8), the parameter $P = 0.4p$ was introduced. The values of the quantity A_{yy} calculated for

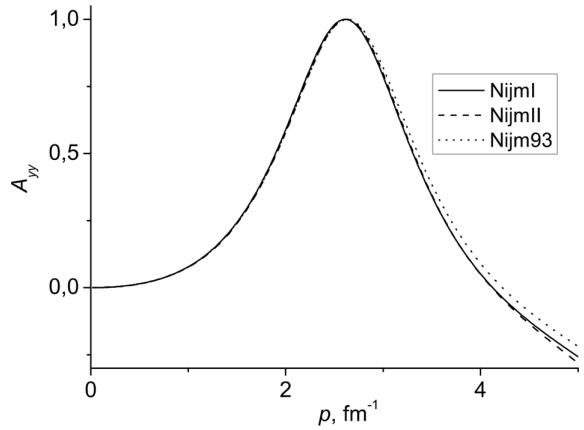

Fig. 7. Tensor analyzing power A_{yy}

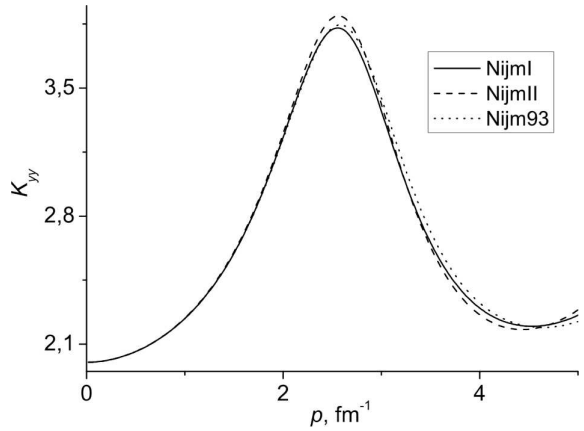

Fig. 8. Tensor-tensor polarization transfer K_{yy}

the Nijmegen-group potentials are comparable with the results obtained for the Bonn potential [20].

The tensor-tensor polarization transfer K_{yy} (Fig. 8) was calculated with the use of the formula [14]

$$K_{yy} = \frac{5T_{11}^2 + T_{00}^2 - 8P^2T_{10}^2}{T_{00}^2 + 2T_{11}^2 + 4P^2T_{10}^2}. \quad (11)$$

It should be noted that the values of the polarization characteristics T_{20} and A_{yy} obtained for the Nijmegen potentials in this work almost coincide with the corresponding values obtained for the potential Reid93 in work [6]. In the cited work, the wave functions of a deuteron also do not contain redundant nodes in the coordinate and momentum representations. The deviation between the relevant results amounts to 1–2% in the high-momentum interval.

5. Conclusions

To summarize, the coefficients in the analytic forms approximating the numerical values of deuteron wave function in the coordinate representation are numerically calculated for the realistic phenomenological Nijmegen-group potentials (NijmI, NijmII, and Nijm93). The obtained wave functions do not contain redundant nodes.

The calculated wave functions of a deuteron in the coordinate and momentum representations are used to determine the deuteron parameters and the deuteron tensor polarization component T_{20}, the polarization transfer K_0, the tensor analyzing power A_{yy}, and the tensor-tensor polarization transfer K_{yy}. The results obtained are compared with the available experimental and theoretical data.

1. R. Machleidt. The nuclear force in the third millennium. *Nucl. Phys. A* **689**, 11 (2001) [DOI: 10.1016/S0375-9474(01)00814-4].
2. R. Machleidt. High-precision, charge-dependent Bonn nucleon-nucleon potential. *Phys. Rev. C* **63**, 024001 (2001) [DOI: 10.1103/PhysRevC.63.024001].
3. V.I. Kukulin *et al.* Moscow-type NN potentials and three-nucleon bound states. *Phys. Rev. C* **57**, 535 (1998) [DOI: 10.1103/PhysRevC.57.535].
4. R. Courant, D. Hilbert. *Methods of Mathematical Physics.* (Interscience, 1953) [ISBN: 10:047017952X].
5. I. Gaisak, V. Zhaba. On nodes of the deuteron wave function. *Visn. Lviv. Univ. Ser. Fiz.* **44**, 8 (2009).
6. I.I. Gaisak, V.I. Zhaba. Deuteron: its wave function and parameters. *Nauk. Visnyk Uzhgorod. Univ. Ser. Fiz.* **36**, 100 (2014).
7. V.S. Bokhinyuk, V.I. Zhaba, O.M. Parlag. To the question of the energy dependence of the cross-section of (γ, γ') reaction. *Nauk. Visnyk Uzhgorod. Univ. Ser. Fiz.* **31**, 111 (2012).
8. V.G.J. Stoks, R.A.M. Klomp, C.P.F. Terheggen, J.J. de Swart. Construction of high-quality NN potential models. *Phys. Rev. C* **49**, 2950 (1994) [DOI: 10.1103/ PhysRevC.49.2950].
9. R.B. Wiringa, V.G.J. Stoks, R. Schiavilla. Accurate nucleon-nucleon potential with charge-independence breaking. *Phys. Rev. C* **51**, 38 (1995) [DOI: 10.1103/ PhysRevC.51.38].
10. M. Lacombe *et al.* Parametrization of the deuteron wave function of the Paris NN potential. *Phys. Lett. B* **101**, 139 (1981) [DOI: 10.1016/0370-2693(81)90659-6].
11. S.B. Dubovichenko. *Properties of Light Atomic Nuclei in the Potential Cluster Model* (Daneker, 2004) (in Russian).
12. NN-OnLine. Deuteron wavefunctions. http: //nn-online.org/NN/nn.php?page=deuteronwavefunctions-table.
13. J.J. de Swart, R.A.M.M. Klomp, M.C.M. Rentmeester, Th.A. Rijken. The Nijmegen potentials. *Few-Body Syst.*

Suppl. **8**, 438 (1995) [DOI: 10.1007/978-3-7091-9427-0_65].
14. A.F. Krutov, V.E. Troitsky. Parametrization of the deuteron wave function obtained within a dispersion approach. *Phys. Rev. C* **76**, 017001 (2007) [DOI: 10.1103/ PhysRevC.76.017001].
15. V.P. Ladygin, N.B. Ladygina. Polarization effects in the inelastic scattering of deutrons (d,d')X in the region of excitation of baryon resonances. *Yad. Fiz.* **65**, 188 (2002).
16. V.A. Karmanov. Polarization phenomena in the elastic pd-backscattering at energies near 1 GeV. *Yad. Fiz.* **34**, 1020 (1981).
17. S.S. Vasan. Deuteron spin alignment in high-energy elastic proton-deuteron backscattering. *Phys. Rev. D* **8**, 4092 (1973) [DOI: 10.1103/PhysRevD.8.4092].
18. V.P. Ladygin *et al.* Measurement of the tensor analyzing power T_{20} in the dd$\to^3$Hen and dd$\to^3$Hp at intermediate energies and at zero degree. *Phys. Lett. B* **598**, 47 (2004) [DOI: 10.1016/j.physletb.2004.08.002].
19. L.S. Azhgirey *et al.* Measurement of the tensor Ayy and vector Ay analyzing powers of the deuteron inelastic scattering of beryllium at 5.0 GeV/c and 178 mrad. *Phys. Atom. Nucl.* **68**, 991 (2005) [DOI: 10.1134/1.1954825].
20. V.P. Ladygin, L.S. Azhgirey, S.V. Afanasiev *et al.* e-print arXiv:nucl-ex/0412020 (2004). V.P. Ladygin *et al.* Tensor A_yy and vector A_y analyzing powers of the (d,p) and (d,d) reactions at 5 GeV/c and 178 MR, arXiv:nucl-ex/0412020.

Received 19.08.15.
Translated from Ukrainian by O.I. Voitenko

В.І. Жаба

АПРОКСИМАЦІЯ ХВИЛЬОВОЇ ФУНКЦІЇ ТА ПОЛЯРИЗАЦІЙНІ ХАРАКТЕРИСТИКИ ДЕЙТРОНА ДЛЯ ПОТЕНЦІАЛІВ НЕЙМЕГЕНСЬКОЇ ГРУПИ

Р е з ю м е

Чисельно розраховані коефіцієнти аналітичних форм для хвильової функції дейтрона в координатному представленні для потенціалів Неймегенської групи. Отримані хвильові функції не містять надлишкових вузлів. Розраховані параметри дейтрона добре узгоджуються з експериментальними і теоретичними даними. Розраховані по хвильових функціях поляризаційні характеристики T_{20} і A_{yy}, співрозмірні з раніше опублікованими.