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BEHAVIOR OF THE GRAVITATIONAL SYSTEM CLOSE TO THE PLANCK EPOCH

The evolution of a quantum gravitational system (QGS) with the maximally symmetric geometry in the epoch close to the Planck one is investigated. The state vector of the QGS satisfies the set of wave equations which describes the time evolution of the quantum system in the space of quantum fields. It is shown that, for the time arrow from past to future, the state vector describes the QGS contracting for the negative values of the cosmic scale factor and expanding for its positive values. The intensity distributions of matter for two exactly solvable models of spatially closed and flat QGSs formed by dust and radiation are calculated. The analogies with known phenomena in quantum mechanics and optics are drawn.

K e y w o r d s: quantum gravity, quantum geometrodynamics, cosmology.

1. Introduction

The standard cosmological model claims that our universe has nucleated from the initial cosmological singularity point as a result of the Big-Bang. The question about the Big-Bang prehistory and its mechanism requires passing to the theory which considers matter and gravitation as quantum fields.

The method of constraint system quantization [1–12] can be taken as a basis of the quantum theory of gravity suitable for the investigation of cosmological and other quantum gravitational systems similar to our universe near the Planck epoch [13–20]. In this theory, the state vector of a quantum gravitational system (QGS) satisfies the set of wave equations which describes the time evolution of a quantum system in a generalized space of quantum fields. The probabilistic interpretation of the state vector of the QGS implies its normalizability.

In the simplest case of the maximally symmetric geometry with the Robertson–Walker metric, the geometric properties of the system are determined by a single variable, namely the cosmic scale factor a. We will consider the homogeneous isotropic QGS formed by matter in the form of a uniform scalar field ϕ. This field can be interpreted as a surrogate of all possible real physical fields of matter averaged with respect to spin, space, and other degrees of freedom. In addition, it will be accepted that the QGS is filled with a perfect fluid in the form of a relativistic matter (further referred as radiation), which defines a material reference frame enabling us to introduce the time variable (recognize the instants of time) [5, 9, 11].

It is convenient to formulate quantum theory in terms of dimensionless variables and parameters, in which length, proper time, mass-energy, energy density, and pressure are measured in modified Planck units: $l_{\rm P} = \sqrt{2G\hbar/(3\pi c^3)}$, $t_{\rm P} = l_{\rm P}/c$, $m_{\rm P} = \hbar/(l_{\rm P}c)$, $\rho_{\rm P} = 3c^4/(8\pi G l_{\rm P}^2)$, where G is Newton's gravitational constant. The scalar field is taken in $\phi_{\rm P} = \sqrt{3c^4/(8\pi G)}$. In these units, the basic equations of the QGS model have the form of the following set of two partial differential equations for the state vector Ψ [11, 12, 18–20]

$$\left(-i\partial_T - \frac{2}{3}E\right)\Psi = 0, \tag{1}$$

$$\left(-\partial_a^2 + \mathrm{k}a^2 - 2aH_\phi - E\right)\Psi = 0, \tag{2}$$

where T is a conformal time expressed in radians. In general relativity, the cosmic scale factor a describes the overall expansion or contraction of the cosmological system, being a function of the proper time τ connected with the conformal time T by the equation

$$d\tau = a\,dT. \tag{3}$$

In Eqs. (1) and (2), the quantities a, ϕ, and T are independent variables of the state vector $\Psi = \Psi(a, \phi, T)$. The parameter E is a real constant, which is determined by the energy density of a perfect fluid ρ_γ taken in the form

$$\rho_\gamma = \frac{E}{a^4}. \tag{4}$$

In natural physical units, E has the dimension of [Energy × Length], $[E] = [\hbar c]$. The coefficient 2/3 in Eq. (1) is caused by the choice of the parameter T as the conformal time variable.

The operator H_ϕ in Eq. (2) is the Hamiltonian of the field ϕ. This Hamiltonian is defined in a curved space-time and, therefore, depends on a scale factor a as a parameter, $H_\phi = H_\phi(a)$. If the potential term of the uniform scalar field ϕ is described by the scalar function $V(\phi)$, then

$$H_\phi = \frac{1}{2}a^3\rho_\phi, \quad \rho_\phi = \frac{2}{a^6}\partial_\phi^2 + V(\phi), \tag{5}$$

and

$$L_\phi = \frac{1}{2}a^3 p_\phi, \quad p_\phi = \frac{2}{a^6}\partial_\phi^2 - V(\phi), \tag{6}$$

where ρ_ϕ and p_ϕ are the operators of energy density and pressure. The L_ϕ can be interpreted as the Lagrangian of the scalar field. The variable ϕ is defined on the interval: $\phi \in (-\infty, +\infty)$.

In Eq. (2), we single out the curvature constant k, $\mathrm{k} = +1, 0, -1$ for spatially closed, flat, and open QGSs, respectively. The derivation of Eqs. (1) and (2) [11, 18] does not depend on the numerical value of k.

The variables a and ϕ satisfy the commutation relations

$$[a, -i\partial_a] = i, \quad [\phi, -i\partial_\phi] = i.$$

All other commutators vanish.

Equations (1) and (2) can be rewritten as one time equation of the Schrödinger type in the space of two variables a and ϕ with the time-independent operator with the dimension of [Energy × Length] instead of [Energy].

2. Proper Mass-Energy in a Definite Quantum State

It is convenient to pass from the (a, ϕ)-representation of the state vector Ψ to a representation, in which the continuous variable ϕ is replaced by a discrete or continuous set of values of the quantum number k, which characterizes the states of the matter field in a comoving volume $\frac{1}{2}a^3$. With that end in view, we introduce the complete set of orthonormalized state vectors of the scalar field $\langle\chi|u_k\rangle$ in the representation of a rescaled variable $\chi = \chi(a, \phi)$, in which the Hamiltonian H_ϕ is diagonalized,

$$\langle u_k|H_\phi|u_{k'}\rangle = M_k(a)\,\delta_{kk'}. \tag{7}$$

After the averaging of H_ϕ with respect to the field χ, we transform from the scalar field to the new effective matter in the discrete and/or continuous kth state with the proper mass-energy $M_k(a) = \frac{1}{2}a^3\rho_m$. The energy density ρ_m and pressure p_m of such an averaged matter are

$$\rho_m = \langle u_k|\rho_\phi|u_k\rangle, \quad p_m = \langle u_k|p_\phi|u_k\rangle. \tag{8}$$

Introducing the equation of state parameter

$$w_m(a) = -\frac{1}{3}\frac{d\ln M_k(a)}{d\ln a}, \tag{9}$$

we find that the averaged matter is a barotropic fluid with the equation of state

$$p_m = w_m(a)\rho_m. \tag{10}$$

The explicit forms of the field χ and the mass-energy $M_k(a)$ are different for the different functions $V(\phi)$. If, for example, $V(\phi) = \lambda_\alpha\phi^\alpha$, where λ_α is the coupling constant, α takes arbitrary non-negative values, and the summation with respect to α is not assumed, then [18]

$$\chi = \left(\sqrt{2\lambda_\alpha}\frac{a^3}{2}\right)^{\frac{2}{2+\alpha}}\phi,$$
$$M_k(a) = \epsilon_k\left(\frac{\lambda_\alpha}{2}\right)^{\frac{2}{2+\alpha}} a^{\frac{3(2-\alpha)}{2+\alpha}}, \tag{11}$$
$$w_m(a) = \frac{\alpha - 2}{\alpha + 2},$$

where ϵ_k is an eigenvalue of the equation

$$(-\partial_\chi^2 + \chi^\alpha - \epsilon_k)|u_k\rangle = 0. \tag{12}$$

In a particular case of the ϕ^2-model, the parameter $w_m(a) = 0$ and the barotropic fluid becomes an aggregate of separate macroscopic bodies (dust) with the mass-energy $M_k = \sqrt{2\lambda_2}(k + \frac{1}{2})$, which does not depend on a, and $\epsilon_k = 2k + 1$, where $k = 0, 1, 2, \ldots$. Equation (12) for $\alpha = 2$ is the equation for a quantum oscillator. The mass M_k can be interpreted as a sum of masses of separate excitation quanta of the spatially coherent oscillations of the field χ about the equilibrium state $\chi = 0$. The quantum number k is the number of these excitation quanta with the mass $\sqrt{2\lambda_2}$. Taking the mass of proton $\approx$1 GeV ($\sim$$10^{-19}$ in Planck mass units) as such a mass, one obtains $M_k \sim 10^{80}$ GeV ($\sim$$10^{-61}$), when the number of protons $k \sim 10^{80}$. Such dust mass leads to the actual density of matter in the observed part of our universe [21]. The review of the properties of the barotropic fluid in the ϕ^α-models with $\alpha \neq 2$ is given in Ref. [18].

3. Separation of Time and Matter Degrees of Freedom

Using the completeness and orthonormality of the state vectors $\langle\chi|u_k\rangle$, the state vector Ψ of a QGS in the (a, χ)-representation can be represented in the form of a superposition of all possible kth states of the barotropic fluid

$$\Psi = \sum_k |u_k\rangle\langle u_k|\Psi\rangle, \tag{13}$$

where $\langle u_k|\Psi\rangle \equiv \psi_k(a, T)$ satisfies the differential equations

$$\left(-i\partial_T - \frac{2}{3}E\right)\psi_k(a, T) = 0, \tag{14}$$

$$\left(-\partial_a^2 + \mathrm{k}a^2 - 2aM_k(a) - E\right)\psi_k(a, T) = 0. \tag{15}$$

The general solution of this set has the form

$$\psi_k(a, T) = \sum_n c_{nk}(T) f_{nk}(a), \tag{16}$$

with

$$c_{nk}(T) = c_{nk}(T_0)\exp\left\{i\frac{2}{3}E_n(T - T_0)\right\}, \tag{17}$$

where the summation with respect to discrete values of n and the integration with respect to continuous ones is assumed.

The wave functions $f(a)$ in Eq. (16) satisfy the equation

$$\left(-\partial_a^2 + \mathrm{k}a^2 - 2aM_k(a) - E\right) f(a) = 0, \tag{18}$$

where indices of the function $f(a)$ and the eigenvalue E describing the discrete and continuous states of radiation are omitted. Thus, the QGS is considered as a material point moving in the potential

$$U_k(a) = \mathrm{k}a^2 - 2aM_k(a) \tag{19}$$

formed by the barotropic fluid in the kth state.

The parameter T_0 in Eq. (17) is an arbitrary constant taken as a time reference point. Equation (18) determines the stationary quantum state of the QGS at some fixed instant of time T_0, the choice of which is arbitrary, $f(a) \equiv f(a, T_0)$.

In the probabilistic interpretation of quantum theory, the coefficient $c_{nk}(T_0)$ gives the probability $|c_{nk}(T_0)|^2$ to find the QGS in the nth state of radiation and the kth state of the barotropic fluid at the time instant T_0.

The conditions of normalization and orthogonality can be imposed on the wave functions $f(a)$,

$$\langle f_{nk}|f_{n',k}\rangle = \delta_{nn'}. \tag{20}$$

Then the state vector Ψ appears be normalized to unity,

$$\langle\Psi|\Psi\rangle = \sum_k\sum_n |c_{nk}(T_0)|^2 = 1, \tag{21}$$

under the condition that the probability summed over all possible quantum states of radiation and a barotropic fluid is equal to unity, i.e. the QGS with the state vector Ψ exists.

4. Contracting and Expanding QGSs

According to Eq. (5), Eq. (2) is invariant under the inversion $a \to -a$. Since the Robertson–Walker line element contains only even powers of a, and the sign of a has not been fixed, while deriving Eqs. (1) and (2), Eq. (18) can be generalized by extending to the domain of negative values of a, so that $a \in (-\infty, +\infty)$ (cf. Refs. [22, 23]).

In order to clarify the physical meaning of the solutions of Eq. (18) in the domain $a < 0$, let us integrate Eq. (3),

$$T(\tau) = T_0 + \int_0^\tau \frac{d\tau'}{a(\tau')}, \quad \text{at } T(0) = T_0. \tag{22}$$

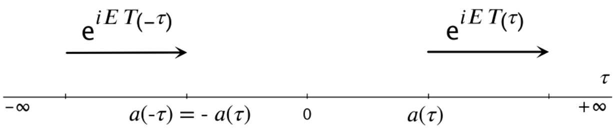

Fig. 1. Wave expanding in the proper time τ

We get

$$T(\tau) = T(-\tau), \quad \text{at } a(-\tau) = -a(\tau). \tag{23}$$

From Eqs. (13)–(17), it follows that the dependence of the state vector Ψ on the time $T(\tau)$ is determined by the exponential multiplier $\exp(iET)$, where the inessential multiplier 2/3 is omitted and the natural condition $T_0 = 0$ is imposed. Let this exponential function describe the wave expanding from $\tau = -\infty$ in the direction $\tau = +\infty$ and passing through the point $\tau = 0$, where $a(0) = 0$ according to Eq. (23). The illustration is given in Fig. 1.

Then the scale factor $a \in (-\infty, 0]$ corresponds to the values $\tau \in (-\infty, 0]$, and the scale factor $a \in [0, +\infty)$ corresponds to the values $\tau \in [0, +\infty)$. As a result for the time arrow from $\tau = -\infty$ to $\tau = +\infty$, the state vector Ψ describes the QGS contracting on the semiaxis $a < 0$, since $|a|$ decreases, and the QGS expanding on the semiaxis $a > 0$, because $|a|$ increases.

The instant of time $\tau = 0$ can be interpreted as the instant of the nucleation of a quantum system expanding in time from the point $a = 0$, although any nucleation "from nothing" does not occur physically. What happens at the instant $\tau = 0$ is that the regime of the preceding contraction of the system changes into the subsequent expansion. Equation (18) describes the stationary states of the QGS for a given constant E. The state vector (13) contains all information about the system as a whole: the cross-section $|a| = \text{const}$ determines the quantum state of the QGS at the time instant τ, when such a value of the scale factor holds.

If one applies the above-described scenario to our universe at the Planck epoch, interpreting the passage through the point $a = 0$ at $\tau = 0$ as the nucleation of an expanding universe with $a > 0$ at $\tau > 0$, then the answer to the question "What was with the quantum system before the instant of the nucleation of the universe of our (expanding) type?" can be given: there has existed another universe with the same mass-energy $M_k(a)$ and wave function $f(a)$ characterized by the same quantum numbers for matter and radiation as the nucleated universe; however, that universe has been contracting up to the state with $a = 0$, which not necessarily will be singular (see Sect. 5 below).

5. Intensity Distribution

Let us consider a QGS, in which the barotropic fluid (matter) and radiation are in some definite quantum states. In such a quantum system, the intensity distribution of matter-energy as a function of a is given by the expression

$$I(a) = M(a)|f(a)|^2, \tag{24}$$

where indices of the states of matter (k) and radiation (n) are omitted. The wave function $f(a)$ is the solution of Eq. (18) complemented with the appropriate boundary conditions, which determine, for example, the behavior of $f(a)$ in the asymptotic domain of large values of $|a|$.

The intensity summed over all possible values of a gives the mean mass-energy of matter in the QGS in the state $f(a)$ normalized to unity,

$$\langle M(a)\rangle = \int da\, I(a). \tag{25}$$

We note that $\langle M(a)\rangle = M = \text{const}$ for the ϕ^2-model. As an example, we will calculate intensity (24) in the exactly solvable models of a spatially closed and flat QGS filled with dust and radiation.

5.1. Spatially closed QGS

In the model of spatially closed QGS formed by dust, whose mass does not depend on a, $M(a) = M = \text{const}$, Eq. (18) is reduced to the equation for an oscillator by the substitution of the variable $a = \xi + M$,

$$\left(-\partial_\xi^2 + \xi^2 - \epsilon\right) f(\xi) = 0, \tag{26}$$

where an eigenvalue $\epsilon = E + M^2$. Changing from the variable a to ξ restores the inversion invariance of Eq. (18) violated in the case where matter is represented by dust. The variable ξ describes a deviation of a from its equilibrium value at the point $a = M$. This variable lies in the interval $[-M, +\infty)$, if a takes only positive values and zero. For $M \gg 1$, the interval of change of ξ in the normalization integral for the function $f(\xi)$ can be extended to the

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whole semiaxis of the negative values of ξ. The error arising here is $\sim O((2M)^{2n-1}\exp(-M^2))$, where $n = 0, 1, 2, ...$ [11, 12]. Then the solution of Eq. (26), decreasing at $|\xi| = +\infty$ and normalized to unity, gives the intensity distribution of dust matter in the form

$$I_n(a) = \frac{M}{2^n n! \sqrt{\pi}}\, e^{-(a-M)^2} H_n^2(a-M), \tag{27}$$

where $H_n(\xi)$ is the Hermite polynomial and the variable a takes any values in the interval $(-\infty, +\infty)$. The constant E is quantized in accordance with the condition

$$E = 2n + 1 - M^2. \tag{28}$$

It takes a sequence of discrete positive values for $2n + 1 > M^2$ and discrete negative values for $2n + 1 < M^2$. In the latter case, radiation as a perfect fluid is characterized by the negative energy density (4) and pressure $p_\gamma = -\frac{1}{3}|\rho_\gamma|$, i.e. a perfect fluid acquires the properties of the antigravitating matter at small quantum numbers n and large masses M.

Equation (27) determines the intensity distribution of dust matter in a QGS both in the regime of its contraction, when $a < 0$, and in the regime of expansion, when $a > 0$. The quantities n and M in Eqs. (27) and (28) are free parameters.

In Fig. 2, it is shown the intensity distribution for the parameters $n = 16$ and $M = 5$ (bold line), when $\rho_\gamma > 0$, and $M = 6$ (thin line), when $\rho_\gamma < 0$. In the case of the positive energy density of radiation, the intensity evolves so that the first maximum of $I_n(a)$ is reached in the domain of contraction, straight before the boundary point $a = 0$, where the regime of contraction changes into the expansion. It is as if the QGS accumulates the energy just before the beginning of the expansion. The intensity at the point $a = 0$ is found to be finite ($I_n(0) = 0.7$ in Fig. 2). In the case $\rho_\gamma < 0$, the negative pressure of radiation pushes out the first maximum into the domain of expansion near the point $a = 0$. In both cases, the intensity oscillates between maximum values and zero in the domain of expansion. The value $a = M$ corresponds to the smallest maximum and $I_n(0) = I_n(2M)$. For $a \gg 2M$, the intensity decreases exponentially.

For $n \gg 1$, the intensity distribution oscillates according to the law

$$I_n(a) = \frac{M}{\pi}\sqrt{\frac{2}{n}} \cos^2\left(\sqrt{2n}(a-M) - \frac{n\pi}{2}\right). \tag{29}$$

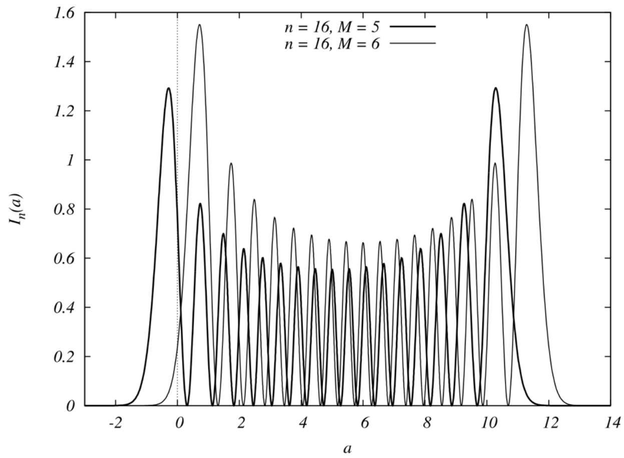

Fig. 2. Intensity distribution (27) for the cases $\rho_\gamma > 0$ (bold line) and $\rho_\gamma < 0$ (thin line)

The averaging with respect to oscillations gives the intensity, which does not depend on a,

$$\bar{I}_n(a) = \frac{M}{\pi}\sqrt{\frac{1}{2n}} \quad \text{at } n \to \infty. \tag{30}$$

The general behavior of intensity (27) reproduces a position probability density of a harmonic oscillator with respect to the variable a renormalized by the mass M. For the QGS under consideration, Eq. (26) is exact. Its application to the cosmological problem leads to nontrivial conclusions about the evolution of the intensity distribution of matter in the QGS with all possible parameters n and M close to the Planck epoch.

5.2. Spatially flat QGS

The prediction about the exponential decreasing of the intensity distribution of matter for $a \gg 2M$ is made in the model of spatially closed QGS. Now, let us consider another exactly solvable model, namely that of flat space. In the model of spatially flat QGS formed by dust with the constant mass M, Eq. (18) is reduced to

$$\left(\partial_\xi^2 + \xi\right) f(\xi) = 0, \tag{31}$$

by introducing the variable

$$\xi = (2M)^{1/3}\left(a + \frac{E}{2M}\right). \tag{32}$$

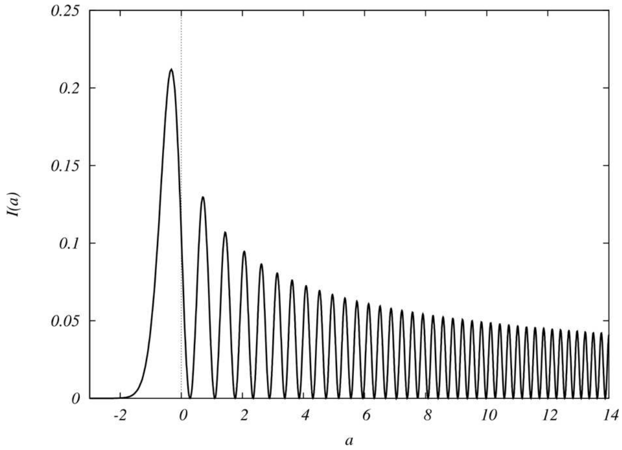

Fig. 3. Intensity distribution (33) for $M = 5$, $E = 8$

Its general solution is a linear superposition of the Airy functions Ai and Bi. We will look for the solution, which has a form of the outgoing wave at $\xi > 0$ and satisfies the boundary condition $f(-\infty) = 0$. By normalizing this solution to the delta-function as in Eq. (20), we get the following expression for the intensity distribution of matter (24):

$$I(a) = \frac{(2M)^{2/3}}{2\pi} Ai^2\left(-(2M)^{1/3}\left(a + \frac{E}{2M}\right)\right). \quad (33)$$

In Fig. 3, the intensity $I(a)$ (33) is depicted as a function of a for the same parameters as in Fig. 2 for $\rho_\gamma > 0$.

As in the case of spatially closed QGS, intensity (33) increases exponentially with the contraction of the system, reaching a maximum, passing through the point $a = 0$ with the finite value ($I(0) = 0.1$) and then oscillating. From the asymptotic expression for $Ai(-\xi)$, it follows that the intensity averaged over the oscillations decreases with a according to the law

$$\bar{I}(a) = \frac{1}{4\pi}\sqrt{\frac{2M}{a}} \quad \text{at } a \gg \frac{E}{2M}. \quad (34)$$

If one assumes that, during the quantum epoch, intensity (34) decreases in time as τ^{-2} like the energy density in general relativity, then the scale factor should increase in time according to a power law $a \sim \tau^4$. Such a growth of a corresponds to an inflationary model, in which it is supposed that the scale factor increases more slowly than in the exponential regime [24].

6. Analogies

Intensities (27) and (33) obtained from exact solutions of Eq. (18) allow us to draw analogy with known phenomena, which are described by the equations of quantum mechanics and optics. We will consider some of these analogies in detail.

6.1. Oscillating wave packet

Formula (27) gives the intensity distribution of matter in stationary states. Let us find how this intensity changes with the proper time in the expanding QGS.

The eigenfunction of the ground state of an oscillator (26)

$$f_0(a - \langle a\rangle) = \frac{1}{\pi^{1/4}}\, e^{-\frac{1}{2}(a-\langle a\rangle)^2} \quad (35)$$

has the form of a normalized minimum packet, whose center of gravity is displaced in the positive a direction by an amount $\langle a\rangle = M$ [11, 12]. We assume that such a state corresponds to the time instant $\tau = 0$, $\psi(a, \tau = 0) = f_0(a - \langle a\rangle)$. If the equivalent classical system evolves in time τ with a power-law scale factor, $a = \beta\tau^\alpha$, where α and β are some positive constants, then the time phase in Eq. (17) at $T_0 = 0$ takes the form

$$\frac{2}{3}E_n T = \left[\frac{1}{2}(1 - M^2) + n\right]\omega\tau, \quad (36)$$

where the frequency

$$\omega = \frac{4}{3(1-\alpha)a} \quad (37)$$

depends on a. From the requirement $\omega > 0$, we get the following restriction: $a > 0$ and $0 \leq \alpha < 1$. Under these conditions, we have $\lim_{\tau\to 0}\omega\tau = 0$. Considering only the sum with respect to n in Eq. (16) (continuous spectrum is absent), using representation (17) with $T_0 = 0$, and the explicit form of function (35), as well as the representation of the phase (36), one can calculate the wave function $\psi(a, \tau) = \psi_k(a, T(\tau))$. As a result, the intensity distribution $I(a, \tau) = M|\psi(a, \tau)|^2$ of matter with the mass M in a wave packet moving in time appears to equal

$$I(a, \tau) = \frac{M}{\sqrt{\pi}} \exp\left\{-\left[a - (1 + \cos\omega\tau)M\right]^2\right\}. \quad (38)$$

If the mass of matter (dust) goes to zero, then the wave function tends to the eigenfunction corresponding to the lowest energy of radiation $\frac{1}{2}\omega$ at $n = 0$, while the intensity does not depend on time and goes to zero as $I(a,\tau) \to (M/\sqrt{\pi})\exp(-a^2)$ at $M \to 0$. If the mass $M \neq 0$, then the condition $E > 0$ is ensured by including the stationary states with the quantum numbers $n \neq 0$ in the packet. In the case $n \gg 1$, the main contribution to the packet is made by the eigenfunction with $n = \frac{1}{2}M^2$, and the intensity is described by expression (38).

In Fig. 4, we show intensity $I(a,\tau)$ (38) for the parameters $M = 5$ and $\alpha = 0$ in Eq. (37). The modes with $n = 12$ and $n = 13$ make the most important contribution to $I(a,\tau)$. Matter is distributed in a and τ in the form of periodic structures like petals or stretched bubbles and displaced to their edges. These structures are limited by the value $a = 2M$ with respect to a (as in Fig. 2), and their number increases with time.

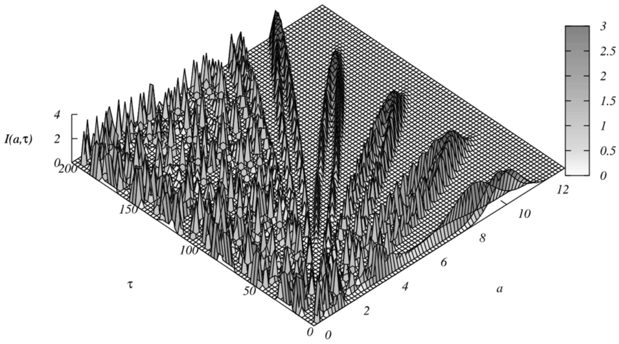

Fig. 4. Intensity distribution (38) for $M = 5$, $\alpha = 0$

6.2. Diffraction

For the spatially flat QGS formed by dust and filled with radiation, described by Eq. (18) with k = 0 and $M = \text{const}$, an analogy with the motion of a particle in a uniform external field along the coordinate a with the energy E under the action of the force $F = 2M$ is obvious [25]. Therefore, we consider a more interesting analogy with the distribution of the light intensity in the neighborhood of the point, where its ray is tangent to the caustic [26]. Eq. (33) yields the asymptotic expression for $I(a)$ at $\left(a + \frac{E}{2M}\right) \gg 1$. We rewrite it in the form

$$I \approx \frac{2A}{\sqrt{-x}} \sin^2\left(\frac{2}{3}(-x)^{3/2}\sqrt{\frac{2\kappa^2}{R}} + \frac{\pi}{4}\right), \qquad (39)$$

where we denote

$$A = \frac{\sqrt{2M}}{4\pi}, \quad x = -\left(a + \frac{E}{2M}\right), \\ \kappa^2 = \frac{E}{2a}, \quad R = \frac{E}{2aM}. \qquad (40)$$

The introduced quantities have a clear physical meaning. The amplitude A is the intensity far from the caustic, which would be obtained from geometric optics neglecting the diffraction effects, x is the distance from the point of observation along the normal to the caustic, which takes positive values for points on the normal in the direction of its center of curvature, κ^2 is the energy of the ray of light, R is the radius of curvature of the caustic at the point of observation.

Equation (39) describes the intensity for the ray of light at large negative values of x. In the radiation-dominated epoch, $a \ll E/(2M)$, the radius of curvature is $R \gg 1$. This means that the wave surface is practically flat. In the matter-dominated epoch, $a \gg E/(2M)$, we have $R \ll 1$. For $R \sim 0$, the wave surface is practically spherically symmetric, and its center of the caustic coincides with the focus. Far from the focus, the averaged intensity decreases as $a^{-1/2}$ with account for the diffraction effects (see Eq. (34)). The diffraction in the QGS can be explained by the scattering of electromagnetic waves of radiation on massive dust particles playing the role of opaque bodies (screens). The observed diffraction picture is similar to that depicted in Fig. 3.

7. Final Remarks

In conclusion, we note some interesting properties of the QGS under consideration. Because of the fact that frequency (37) is inversely proportional to the variable a, in the spatially closed QGS, only one cycle of oscillations of intensity (27) is possible. In such a system, the dissipation of energy occurs with increase of a. This makes a difference between the spatially closed quantum system with gravitation and the corresponding classical universe, which allows oscillations of the scale factor $a(\tau)$ in time with endlessly repeated passage through the singular state $a = 0$ (so-called oscillatory models [27]).

The next property deals with the time-energy uncertainty relation. Let us find the form of this relation for the QGS model with Eqs. (1) and (2). Let A be an observable which does not depend on time explicitly. Then, from the Heisenberg equation for the mean value $\langle \mathrm{A} \rangle$,

$$\frac{d\langle \mathrm{A} \rangle}{dT} = \frac{1}{i}\langle [\mathrm{A}, \mathrm{H}] \rangle, \tag{41}$$

where H is the Hamiltonian of the system written in units of the lapse function (for details, see [11, 18]), and from the uncertainty relation in the general form [28],

$$\Delta\mathrm{A}\,\Delta\mathrm{E} \geq \frac{1}{2}|\langle [\mathrm{A}, \mathrm{H}] \rangle|, \tag{42}$$

where $\Delta\mathrm{A}$ and $\Delta\mathrm{E}$ are the root-mean-squares of A and of H, respectively, using Eq. (3), we find

$$\tau_{\mathrm{A}} \frac{\Delta\mathrm{E}}{|a|} \geq \frac{\hbar}{2}, \tag{43}$$

where

$$\tau_{\mathrm{A}} = \left| \Delta\mathrm{A} \left(\frac{d\langle \mathrm{A} \rangle}{d\tau} \right)^{-1} \right| \tag{44}$$

is a time characteristic of the evolution of the statistical distribution of A (i.e. the time necessary for this statistical distribution to be considerably modified). The quantity $\Delta\mathrm{E}/|a|$ is the statistical fluctuation of the result of the energy measurement. In the limit $|a| \to 0$, this fluctuation becomes infinitely large, while a time characteristic τ_{A} can acquire any value in accordance with the uncertainty relation (43). Thus, near the initial cosmological singularity, the notions of time and energy lose their meaning (they cannot be measured).

Let $\mathrm{A} = a$. Then, for a stationary state, one has $\langle a \rangle = M = \mathrm{const}$ and $d\langle \mathrm{A} \rangle / d\tau = 0$. In this case, $\tau_{\mathrm{A}} = \infty$, and $\Delta\mathrm{E} = 0$.

Possible modifications of the uncertainty relations caused by the spacetime curvature are discussed in the literature (see, e.g., Refs. [29–32]). For example, in Ref. [29], the statistical fluctuation of the spatial variable is identified with the distance between the two near points (the coordinate distance). In such a "naive" analysis for quantum gravity, the uncertainty relation between the curvature and the metric can be obtained. But it is noted that the consequences of it are unclear. The further discussion of Unruh's uncertainty principle can be found in Ref. [33] and references therein.

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ПОВЕДІНКА ГРАВІТАЦІЙНОЇ СИСТЕМИ
В ЕПОХУ, БЛИЗЬКУ ДО ПЛАНКІВСЬКОЇ

Р е з ю м е

Досліджено еволюцію квантової гравітаційної системи (КГС) з максимально-симетричною геометрією в епоху, близьку до планківської. Вектор стану КГС задовольняє систему хвильових рівнянь, які описують еволюцію квантової системи у часі в просторі квантових полів. Показано, що для стріли часу, спрямованої з минулого в майбутнє, вектор стану описує КГС, яка стискається для від'ємних значень космічного масштабного фактора та розширюється для його додатних значень. Обчислено розподіл інтенсивності матерії для двох моделей, що допускають точний розв'язок, а саме для просторово замкненої та плоскої КГС, які утворені пилом та випромінюванням. Наведені аналогії з відомими явищами з квантової механіки та оптики.