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A.D. ANTIA, E.E. ITUEN

Theoretical Physics Group, Department of Physics, Faculty of Science, University of Uyo
(Nigeria; e-mail: antiacauchy@yahoo.com, akaninyeneantia@uniuyo.edu.ng)

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NONRELATIVISTIC TREATMENT OF SCHRÖDINGER PARTICLES UNDER INVERSELY QUADRATIC HELLMANN PLUS RING-SHAPED POTENTIALS

We have solved approximately the Schrödinger equation with the inversely quadratic Hellmann plus ring-shaped potential in the framework of the Nikiforov–Uvarov method. The energy eigenvalues and corresponding wave functions of the radial and angular parts are obtained in terms of Jacobi polynomials. In special cases, our result reduces to the cases of three well-known potentials such as the Coulomb potential, inversely quadratic Yukawa potential, and Hartman potential. The energy eigenvalues are evaluated as well. Our numerical results can be useful for other physical systems.

K e y w o r d s: Schrödinger wave equation, inversely quadratic Hellmann potential, ring-shaped potential, Nikiforov–Uvarov method, approximation scheme.

1. Introduction

The exact bound-state solutions of the Schrödinger equation with physically significant potentials play a major role in quantum mechanics [1–5]. Interestingly, one of the important tasks in theoretical physics is to obtain the exact solution of the Schrödinger equation for special potentials [6–10]. Some of these potentials are known to play important roles in many fields of physics such as molecular, solid-state, and chemical physics [11–13].

The purpose of this work is to present the solution of the Schrödinger equation with the inversely quadratic Hellmann [14] plus ring-shaped potential [15] of the form

$$V(r,\theta) = -\frac{V_0}{r} + \frac{V_1}{r^2}e^{-\alpha r} + \\ + \frac{\hbar^2}{2\mu}\left(\frac{\gamma + \beta\cos^2\theta + \Im\cos^4\theta}{r^2\cos^2\theta\sin^2\theta}\right), \quad (1)$$

where r represents the internuclear distance, V_0 and V_1 are the strengths of the Coulomb and Yukawa potentials, respectively, α is the screening parameter, and γ, β, $\Im$ are arbitrary constants.

The Hellmann potential is used to study the systematization of energy eigenvalues of the two particles interacting through the superposition of the Coulomb and Yukawa potentials (SCYP). The potential in Eq. (1) with V_1 positive was first suggested by Hellmann [16, 17] and has been used by various authors to represent the electron-core [18, 19] or the electron-ion [20, 21] interactions. The ring-shaped potential has been studied in the fields of nuclear physics and quantum chemistry and can be used for interactions between a deformed pair of nuclei and ring-shaped molecules such as benzenes [15, 22].

Ita [23] solved the Schrödinger equation for the Hellmann potential and obtained the energy eigenvalues and their corresponding wave functions, by using the expansion method and the Nikiforov–Uvarov one. Hamzari and Rajabi [24] used the parametric Nikiforov–Uvarov method to obtain the tensor coupling and relativistic spin and pseudospin symmetries of the Dirac equation with the Hellmann potential. Antia *et al.* [15] solved the nonrelativistic Schrödinger equation with the Hulthen–Yukawa plus ring-shaped potential, by using the Nikiforov–Uvarov method and obtained the energy eigenvalues and corresponding wave functions of the polar and angle-dependent parts, respectively. However, not much has been achieved in the area of solving the radial Schrödinger equation for any angular momentum quantum number l with the inversely quadratic Hellmann plus ring-shaped potential with the use of the Nikiforov–Uvarov method. The inversely quadratic Hellmann potential would give a better realistic results than the Hellmann potential, and some other physically well-known potentials could be deduced from the combined inversely quadratic Hellmann potential, rather than from the single Hellmann potential.

The potential in Eq. (1) can be expressed as

$$V(r,\theta) = -V_r(r) + \frac{\hbar^2}{2\mu}\frac{V_0(\theta)}{r^2}, \tag{2}$$

where

$$V_r(r) = \frac{V_0}{r} - \frac{V_1}{r^2}e^{-\alpha r}; \tag{3}$$

$$V_\theta(\theta) = \frac{\gamma + \beta\cos^2\theta + \Im\cos^4\theta}{\cos^2\theta\sin^2\theta}. \tag{4}$$

2. Generalized Parametric Nikiforov–Uvarov (NU) Method

The NU method was presented by Nikiforov and Uvarov [25] and has been employed to solve second-order differential equations such as the Schrödinger wave equation (SWE), Klein–Gordon equation (KGE), Dirac equation (DE), *etc.* The SWE

$$\psi''(r) + [E - V(r)]\,\psi(r) = 0 \tag{5}$$

can be solved by transforming it into a hypergeometric-type equation, by using the transformation $s = s(x)$, and the resulting equation can be expressed as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{6}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most the second degree, $\tilde{\tau}(s)$ is the first-degree polynomial, and $\psi(s)$ is a function of the hypergeometric type. The parametric generalization of the NU method is given by the generalized hypergeometric-type equation as [26]

$$\psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi'(s) + \frac{1}{s^2(1-c_3 s)^2}\left[\xi_1 s^2 + \xi_2 s - \xi_3\right]\psi(s) = 0. \tag{7}$$

Equation (7) is solved by comparing with Eq. (6), and the following polynomials are obtained:

$$\tilde{\tau}(s) = (c_1 - c_2 s), \quad \sigma(s) = s(1 - c_3 s), \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{8}$$

According to the NU method, the energy eigenvalues and eigen functions, respectively, satisfy the following system of equations:

$$c_2 n - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0, \tag{9}$$

$$\psi(s) = N_{nl}s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{11}}{c_3}} \times P_n^{\left(c_{10}-1,\, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s), \tag{10}$$

where

$$\left.\begin{aligned}
&c_4 = \tfrac{1}{2}(1 - c_1),\ c_5 = \tfrac{1}{2}(c_2 - c_3),\\
&c_6 = c_5^2 + \xi_1,\\
&c_7 = 2c_4c_5 - \xi_2,\ c_8 = c_4^2 + \xi_3,\\
&c_9 = c_3c_7 + c_3^2c_8 + c_6,\\
&c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},\\
&c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right),\\
&c_{12} = c_4 + \sqrt{c_8},\ c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right),
\end{aligned}\right\} \tag{11}$$

and P_n is the orthogonal Jacobi polynomial.

3. Factorization Method

In spherical coordinates, the Schrödinger equation with noncentral potential of Eq. (2) can be written as [27]

$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\theta}{\partial\phi}\right]\psi(r,\theta,\phi) + \left(-V_r(r) + \frac{\hbar^2}{2\mu}\frac{V_\theta(\theta)}{r^2}\right)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi). \tag{12}$$

The wave function in Eq. (12) may be defined as

$$\psi(r,\theta,\phi)=\frac{R(r)}{r}\Theta(\theta)\Phi(\phi). \quad (13)$$

By decomposing the spherical wave function in Eq. (12) and using Eq. (3), the following equations are obtained:

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E+V_r(r)\right)-\frac{\lambda}{r^2}\right]R(r)=0, \quad (14)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\left[\lambda-\frac{m^2}{\sin^2\theta}-V_\theta(\theta)\right]\Theta(\theta)=0, \quad (15)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+m^2\Phi(\theta)=0, \quad (16)$$

where $\lambda=l(l+1)$ and m^2 are the separation constants. The solution of Eq. (16) is well known [26]. Equations (14) and (15) are the radial and angular parts of the Schrödinger equation, respectively, which are the subject of discussion in the preceding sections.

4. Solutions of the Radial Schrödinger Equation

For the eigenvalues and corresponding eigenfunctions of the radial part of the Schrödinger equation, the substitution of Eq. (3) into Eq. (4) gives

$$\frac{d^2R}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E+\frac{V_0}{r}-\frac{V_1e^{-\alpha r}}{r^2}\right)-\frac{\lambda}{r^2}\right]R(r)=0. \quad (17)$$

Equation (17) has no analytic or exact solution for $l\neq0$ due to the centrifugal term, but can be solved. Hence, the approximation scheme [27] is invoked for the evaluation of the potential barrier. The variation of the approximation in Eq. (18) with the potential range is presented in Fig. 1. From the graph, it can be seen that the approximation (f1) is suitable for short-range potentials, as can be seen for $\alpha=0.1$. We have

$$\frac{1}{r^2}\approx\frac{\alpha^2}{(1-e^{-\alpha r})^2}, \quad (18)$$

$$\frac{1}{r}\approx\frac{\alpha}{(1-e^{-\alpha r})}. \quad (19)$$

Substituting Eqs. (18) and (19) into Eq. (17) yields

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E+\frac{\alpha V_0}{1-e^{-\alpha r}}-\frac{\alpha^2V_1e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right)-\frac{\alpha^2\lambda}{(1-e^{-\alpha r})^2}\right]R(r)=0. \quad (20)$$

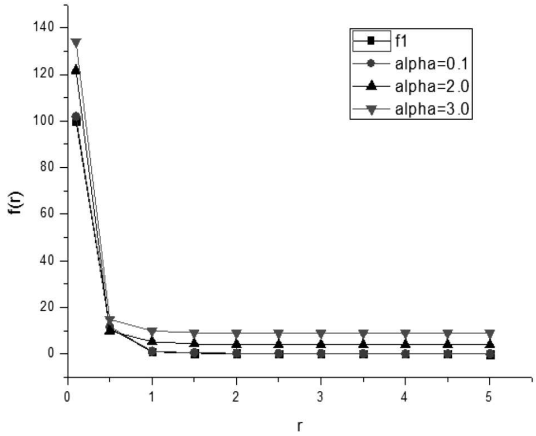

Fig. 1. Comparison of the potential barrier f1= $\frac{1}{r^2}$ with the approximation for $(\alpha)=0.1$, 2.0, and 3.0

By the transformation $s=e^{-\alpha r}$, Eq. (20) is reduced to

$$\frac{d^2R(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR}{ds}+\frac{1}{s^2(1-s)^2}\times\left[-\varepsilon s^2+(2\varepsilon-A'-B')s-(\varepsilon-A'+\lambda)\right]R(s)=0, \quad (21)$$

where the following dimensionless quantities have been defined as

$$-\varepsilon=\frac{2\mu E}{\hbar^2\alpha^2};\quad A'=\frac{2\mu V_0}{\alpha\hbar^2};\quad B'=\frac{2\mu V_1}{\hbar^2}. \quad (22)$$

Comparing Eq. (21) with the NU parameters, we obtain

$$\left.\begin{aligned}
&c_1=c_2=c_3=1,\\
&\xi_1=\varepsilon,\quad \xi_2=2\varepsilon-A'-B',\\
&\xi_3=\varepsilon-A'+\lambda,\\
&c_4=0,\quad c_5=-\frac{1}{2},\quad c_6=\varepsilon+\frac{1}{4},\\
&c_7=B'+A'-2\varepsilon,\\
&c_8=\varepsilon-A'+\lambda,\quad c_9=B'+\lambda+\frac{1}{4},\\
&c_{10}=1+2\sqrt{\varepsilon-A'+\lambda},\\
&c_{11}=2\left(1+\sqrt{B'+\lambda+\frac{1}{4}}+\sqrt{\varepsilon-A'+\lambda}\right),\\
&c_{12}=\sqrt{\varepsilon-A'+\lambda},\\
&c_{13}=\sqrt{B'+\lambda+\frac{1}{4}}+\sqrt{\varepsilon-A'+\lambda}-\frac{1}{2}.
\end{aligned}\right\} \quad (23)$$

Thus, the energy eigenvalue from Eq. (9) is expressed as

$$\varepsilon = \frac{\left[A' - B' + (l+1)^2 - (n+1)^2 + + (2n+1)\sqrt{B' + \left(l+\frac{1}{2}\right)^2}\right]^2}{\left[(2n+1) + 2\sqrt{B' + \lambda + \frac{1}{4}}\right]^2} + \left(l^2 + l + A'\right). \tag{24}$$

The energy spectrum for the system is expressed explicitly by the substitution of Eq. (22) into Eq. (24):

$$E_{nl} = -\frac{2\mu}{\hbar^2}\frac{\left[V_0 - \alpha V_1 + \frac{\hbar^2\alpha\in}{2\mu}\right]^2}{\left[2n+1+2\sqrt{\frac{2\mu V_1}{\hbar^2} + \lambda + \frac{1}{4}}\right]^2} + \left(l^2 + l + \frac{2\mu V_0}{\alpha\hbar^2}\right), \tag{25}$$

where

$$\in = (l+1)^2 - (n+1)^2 + (2n+1)\sqrt{\frac{2\mu V_1}{\hbar^2} + \left(l+\frac{1}{2}\right)^2}.$$

From Eq. (10), the corresponding wave function of the system is obtained as

$$R(s) = N_{nl} S^z (1-s)^{\frac{1}{2}-v} P_n^{2z,\,2v}(1-2s) \tag{26}$$

or its equivalent (by using the transformation $s = e^{-\alpha r}$)

$$R(r) = N_{nl} e^{-\alpha z r}\left(1 - e^{-\alpha r}\right)^{\frac{1}{2}-v} P_n^{2z,\,2v}(1 - 2e^{-\alpha r}), \tag{27}$$

where

$$z = \sqrt{\varepsilon - A' + l(l+1)}, \quad v = \sqrt{B' + \left(l + \frac{1}{2}\right)^2}.$$

5. Solution of the Polar Part

In this case, the eigenvalue and the eigenfunctions of the polar part of the Schrödinger equation can be expressed with the use of Eqs. (4) and (15):

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \left[\lambda - \frac{m^2}{\sin^2\theta} - \frac{\gamma + \beta\cos^2\theta + \Im\cos^4\theta}{\cos^2\theta\sin^2\theta}\right]\Theta(\theta) = 0. \tag{28}$$

Using the transformation $q = \cos^2\theta$, Eq. (28) is reduced to

$$\frac{d^2\Theta(q)}{dq^2} + \frac{(1-3q)}{2q(1-q)}\frac{d\Theta(q)}{dq} + \frac{1}{4q^2(1-q)^2} \times \left[-(\lambda+\Im)q^2 + \left(\lambda - m^2 - \beta\right)q - \gamma\right]\Theta(q) = 0. \tag{29}$$

Comparing Eq. (29) with the NU parameters, we obtain

$$\left.\begin{aligned}
&c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}, \quad c_3 = 1,\\
&\xi_1 = \frac{1}{4}(\lambda+\Im), \quad \xi_2 = \frac{1}{4}(\lambda - m^2 - \beta),\\
&\xi_3 = \frac{\gamma}{4},\\
&c_4 = \frac{1}{4}, \quad c_5 = -\frac{1}{4}, \quad c_6 = \frac{1}{16} + \frac{1}{4}(\lambda+\Im),\\
&c_7 = -\frac{1}{8} - \frac{1}{4}(\lambda - m^2 - \beta), \quad c_8 = \frac{1}{16} + \frac{\gamma}{4},\\
&c_9 = \frac{1}{4}(m^2 + \gamma + \beta + \Im),\\
&c_{10} = 1 + \sqrt{\gamma + \frac{1}{4}},\\
&c_{11} = 2\left(\sqrt{m^2 + \gamma + \beta + \Im} + \sqrt{\gamma + \frac{1}{4}}\right),\\
&c_{12} = \frac{1}{4} + \frac{1}{2}\sqrt{\gamma + \frac{1}{4}},\\
&c_{13} = -\frac{1}{4} - \frac{1}{2}\left(\sqrt{m^2 + \gamma + \beta + \Im} + \sqrt{\gamma + \frac{1}{4}}\right).
\end{aligned}\right\} \tag{30}$$

Substituting Eq. (30) into Eq. (9) gives the relation for λ as

$$\lambda = 4\left(n + \frac{1}{2}\right)^2 + 2(2n+1)\left(\sqrt{m^2+\gamma+\beta+\Im} + \sqrt{\gamma + \frac{1}{4}}\right) + 2\sqrt{(m^2+\gamma+\beta+\Im)\left(\gamma + \frac{1}{4}\right)} + m^2 + \gamma + \beta, \tag{31}$$

where $\lambda = l(l+1)$.

The corresponding wave function of the angle-dependent part is obtained by substituting Eq. (30) into Eq. (10) as:

$$\Theta(q) = N_m s^{\frac{1}{4}+\frac{1}{2}\sqrt{\gamma+\frac{1}{4}}}(1-s)^{\frac{1}{2}\sqrt{m^2+\gamma+\beta+\Im}} \times P_n^{\sqrt{\gamma+\frac{1}{4}},\,\sqrt{m^2+\gamma+\beta+\Im}}(1-2q) \tag{32}$$

or its equivalent (by using the transformation $q = \cos^2\theta$)

$$\Theta(\theta) = N_m (\cos\theta)^{\frac{1}{2}\sqrt{\gamma+\frac{1}{4}}} (\sin\theta)^{\sqrt{m^2+\gamma+\beta+\Im}} \times \times P_n^{\sqrt{\gamma+\frac{1}{4}},\,\sqrt{m^2+\gamma+\beta+\Im}} (-\cos 2\theta), \quad (33)$$

where N_m is the normalization constant.

6. Effect of the Angle-Dependent Part on the Radial Solution

The total energy of the inversely quadratic Hellmann plus a ring-shaped potential is obtained by considering the effect of the angle-dependent part on the radial solutions. Substituting Eq. (31) into Eq. (25) yields the energy spectrum for this system as

$$E_{nlm} = -\frac{2\mu}{\hbar^2}\frac{\left[V_0 - \alpha V_1 + \frac{\hbar^2\alpha\in}{2\mu}\right]^2}{[(2n+1)+2\delta]^2}, \quad (34)$$

where

$$\delta = \sqrt{\begin{array}{l}\frac{2\mu V_1}{\hbar^2} + 4\left(n+\frac{1}{2}\right)^2 + \\ +2(2n+1)\left(\sqrt{m^2+\gamma+\beta+\Im} + \sqrt{\gamma+\frac{1}{4}}\right) + \\ +2\sqrt{(m^2+\gamma+\beta+\Im)\left(\gamma+\frac{1}{4}\right)} \\ +m^2+2\gamma+\beta+\frac{1}{4}\end{array}}.$$

Finally, we write the total wave function for the system as

$$\psi(r,\theta,\phi) = \frac{N_{nm}}{\sqrt{2\pi}}\frac{1}{r}e^{-\alpha zr+im\phi}\left(1-e^{-\alpha zr}\right)^{\frac{1}{2}-v} \times \times P_n^{(2z,\,2z)}\left(1-2e^{-\alpha zr}\right) \times \times (\cos\theta)^{\frac{1}{2}+\sqrt{\gamma+\frac{1}{4}}} (\sin\theta)^{\sqrt{m^2+\gamma+\beta+\Im}} \times \times P_n^{\left(\sqrt{\gamma+\frac{1}{4}},\sqrt{m^2+\gamma+\beta+\Im}\right)}(-\cos 2\theta), \quad (35)$$

where N_{mn} is a normalization constant.

7. Discussions

The numerical data of our results are presented in Tables 1–5 which discuss the energy spectrum of this system for the variation of the screening parameter (α) with different quantum states n and l.

We have also reported the behavior of the energy for the quantum state n with $V_0 = -0.1$, $V_1 = 0.2$, $\mu = 0.2$, $\gamma = \beta = 0.1$, $m = \hbar = 1$, and $\Im = 0.1$ for $\alpha = 0.01$, 0.02, 0.03, 0.04, and 0.05 in Figs. 2–6.

By choosing appropriate values in the inversely quadratic Hellmann plus ring-shaped potential, we have three (3) well-known potentials and their corresponding eigenvalues.

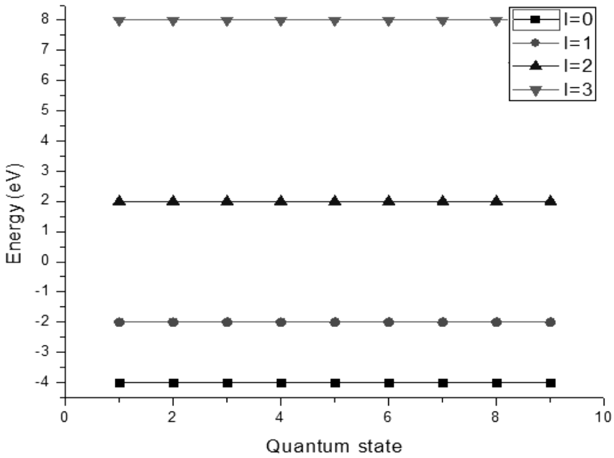

Fig. 2. Energy versus the quantum state for $\alpha = 0.01$, $\gamma = \beta = \Im = 0.1$

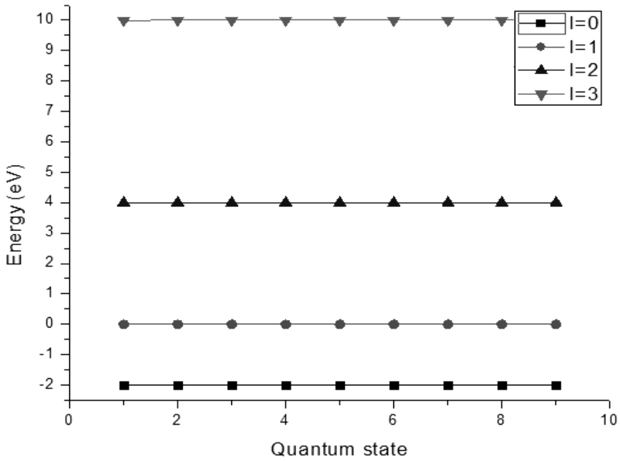

Fig. 3. Energy versus the quantum state for $\alpha = 0.02$, $\gamma = \beta = \Im = 0.1$

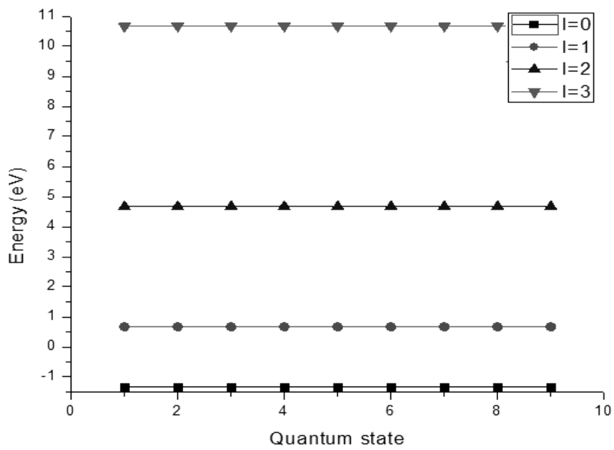

Fig. 4. Energy versus the quantum state for $\alpha = 0.03$, $\gamma = \beta = \Im = 0.1$

Table 1. **Energy eigenvalues for $\alpha = 0.01$, $\gamma = \beta = \Im = 0.1$**

n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)
0	**0**	–4.00003941795	**0**	**1**	–2.00000063929	**0**	**2**	1.99986713564	**0**	**3**	7.99933196254
1	**0**	–4.00001081396	**1**	**1**	–2.00000009396	**1**	**2**	1.99997297614	**1**	**3**	7.99987165077
2	**0**	–4.00000825138	**2**	**1**	–2.00000020401	**2**	**2**	1.99999296872	**2**	**3**	7.99995919089
3	**0**	–4.00000810975	**3**	**1**	–2.00000104089	**3**	**2**	1.99999847429	**3**	**3**	7.99998497105
4	**0**	–4.00000846039	**4**	**1**	–2.00000207363	**4**	**2**	1.99999989407	**4**	**3**	7.99999446965
5	**0**	–4.00000891734	**5**	**1**	–2.00000309137	**5**	**2**	1.99999990370	**5**	**3**	7.99999824216
6	**0**	–4.00000937375	**6**	**1**	–2.00000402583	**6**	**2**	1.99999939084	**6**	**3**	7.99999965422
7	**0**	–4.00000979769	**7**	**1**	–2.00000486190	**7**	**2**	1.99999869449	**7**	**3**	7.99999999900
8	**0**	–4.00001018145	**8**	**1**	–2.00000560339	**8**	**2**	1.99999795457	**8**	**3**	7.99999983758

Table 2. **Energy eigenvalues for $\alpha = 0.02$, $\gamma = \beta = \Im = 0.1$**

n	l	E_n(eV)	n	l	E_n(eV) $\times 10^{-4}$	n	l	E_n(eV)	n	l	E_n(eV)
0	**0**	–2.00002908239	**0**	**1**	–0.767909777420	**0**	**2**	3.99908691506	**0**	**3**	9.99653587990
1	**0**	–2.00001698512	**1**	**1**	–0.0941669722480	**1**	**2**	3.99983481289	**1**	**3**	9.99936929299
2	**0**	–2.00002022451	**2**	**1**	–0.00118693247332	**2**	**2**	3.99995708196	**2**	**3**	9.99980331965
3	**0**	–2.00002439458	**3**	**1**	–0.0164875706656	**3**	**2**	3.99998958758	**3**	**3**	9.99992758202
4	**0**	–2.00002819083	**4**	**1**	–0.0562720642688	**4**	**2**	3.99999865721	**4**	**3**	9.99997284357
5	**0**	–2.00003144718	**5**	**1**	–0.0993419214400	**5**	**2**	3.99999993444	**5**	**3**	9.99999090209
6	**0**	–2.00003420848	**6**	**1**	–0.139754779654	**6**	**2**	3.99999834501	**6**	**3**	9.99999789583
7	**0**	–2.00003655524	**7**	**1**	–0.176045815708	**7**	**2**	3.99999571090	**7**	**3**	9.99999992302
8	**0**	–2.00003856320	**8**	**1**	–0.208168599484	**8**	**2**	3.99999277130	**8**	**3**	9.99999959770

Table 3. **Energy eigenvalues for $\alpha = 0.03$ and $\gamma = \beta = \Im = 0.1$**

n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)
0	**0**	–1.33335364859	**0**	**1**	0.666386889637	**0**	**2**	4.66427468295	**0**	**3**	10.6582270968
1	**0**	–1.33335787686	**1**	**1**	0.666632668388	**1**	**2**	4.66624614687	**1**	**3**	10.6651535633
2	**0**	–1.33337080992	**2**	**1**	0.666665365434	**2**	**2**	4.66655744918	**2**	**3**	10.6661974958
3	**0**	–1.33338276002	**3**	**1**	0.666664270858	**3**	**2**	4.66663943432	**3**	**3**	10.6664939274
4	**0**	–1.33339278255	**4**	**1**	0.666655748024	**4**	**2**	4.66666269818	**4**	**3**	10.6666015305
5	**0**	–1.33340105579	**5**	**1**	0.666646005266	**5**	**2**	4.66666662596	**5**	**3**	10.6666445136
6	**0**	–1.33340791288	**6**	**1**	0.666636742357	**6**	**2**	4.66666345384	**6**	**3**	10.6666613162
7	**0**	–1.33341365184	**7**	**1**	0.666628392749	**7**	**2**	4.66665767001	**7**	**3**	10.6666663929
8	**0**	–1.33341850808	**8**	**1**	0.666620996774	**8**	**2**	4.66665108738	**8**	**3**	10.6666659176

Table 4. **Energy eigenvalues for $\alpha = 0.04$ and $\gamma = \beta = \Im = 0.1$**

n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)
0	**0**	–1.00001311658	**0**	**1**	0.999390402554	**0**	**2**	4.99543043930	**0**	**3**	10.9844056132
1	**0**	–1.00003348919	**1**	**1**	0.999926161299	**1**	**2**	4.99920697808	**1**	**3**	10.9972244616
2	**0**	–1.00006000762	**2**	**1**	0.999996248368	**2**	**2**	4.99979407039	**2**	**3**	10.9991417192
3	**0**	–1.00008320607	**3**	**1**	0.999996717957	**3**	**2**	4.99994801453	**3**	**3**	10.9996840071
4	**0**	–1.00010223558	**4**	**1**	0.999982052062	**4**	**2**	4.99999201699	**4**	**3**	10.9998805304
5	**0**	–1.00011774319	**5**	**1**	0.999964727004	**5**	**2**	4.99999997824	**5**	**3**	10.9999590765
6	**0**	–1.00013048696	**6**	**1**	0.999948127677	**6**	**2**	4.99999471730	**6**	**3**	10.9999899151
7	**0**	–1.00014108750	**7**	**1**	0.999933130094	**7**	**2**	4.99998457184	**7**	**3**	10.9999994085
8	**0**	–1.00015001611	**8**	**1**	0.999919837505	**8**	**2**	4.99997290278	**8**	**3**	10.9999987971

Table 5. **Energy eigenvalues for $\alpha = 0.05$ and $\gamma = \beta = \Im = 0.1$**

n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)	n	l	E_n(eV)
0	**0**	–0.800007486349	**0**	**1**	1.19893374777	**0**	**2**	5.19255418414	**0**	**3**	11.1750714291
1	**0**	–0.800043822099	**1**	**1**	1.19987106204	**1**	**2**	5.19871730652	**1**	**3**	11.1955819879
2	**0**	–0.800087817595	**2**	**1**	1.19999253011	**2**	**2**	5.19966694558	**2**	**3**	11.1986359900
3	**0**	–0.800125732731	**3**	**1**	1.19999569254	**3**	**2**	5.19991532819	**3**	**3**	11.1994978213
4	**0**	–0.800156549885	**4**	**1**	1.19997328491	**4**	**2**	5.19998661362	**4**	**3**	11.1998098434
5	**0**	–0.800181509358	**5**	**1**	1.19994623102	**5**	**2**	5.19999999131	**5**	**3**	11.1999345909
6	**0**	–0.800201930707	**6**	**1**	1.19992018048	**6**	**2**	5.19999213542	**6**	**3**	11.1999836928
7	**0**	–0.800218862209	**7**	**1**	1.19989660745	**7**	**2**	5.19997641638	**7**	**3**	11.1999989700
8	**0**	–0.800233087271	**8**	**1**	1.19987570534	**8**	**2**	5.19995821753	**8**	**3**	11.1999982363

7.1. Coulomb potential

Setting $\gamma = \beta = \Im = V_1 = 0$ and $\alpha = 0$, the potential turns to be the Coulomb potential [28] with the corresponding energy as

$$E(\text{coulomb}) = -\frac{\mu \mathrm{V}_0^2}{2\hbar^2 \mathrm{n}'^2} + \left(\mathrm{l}^2 + \mathrm{l} + \frac{2\mu \mathrm{V}_0}{\hbar^2}\right),$$

where $l = 2n + m + 1$ and $n' = n + l + 1$.

7.2. Inversely quadratic Yukawa potential

Setting $\gamma = \beta = \Im = 0, V_0 = 0$ and mapping $V_1 \to -V_1$, the potential is reduced to the inversely quadratic Yukawa potential [29]

$$V(r) = -\frac{V_1}{r^2} e^{-\alpha r},$$

and the energy eigenvalue is

$$E_{nlm} = -\frac{\mu\alpha^2}{2\hbar^2} \frac{\left[V_1 + \frac{\hbar^2 \alpha}{2\mu} \in\right]^2}{\left(3n + m + \frac{2\mu V_1}{\hbar^2} + 2\right)^2}.$$

7.3. Hartmann potential

Setting $\gamma = \Im = V_1 = \alpha = 0$ and mapping $V_0 \to -V_0$, the potential in Eq. (1) is reduced to the Hartmann potential of the form [30]

$$V(r, \theta) = -\frac{V_0}{r} + \frac{\hbar^2}{2\mu} \frac{\beta}{r^2 \sin^2 \theta},$$

and the corresponding energy eigenvalues of this potential can be obtained by substituting these parameters into Eq. (18).

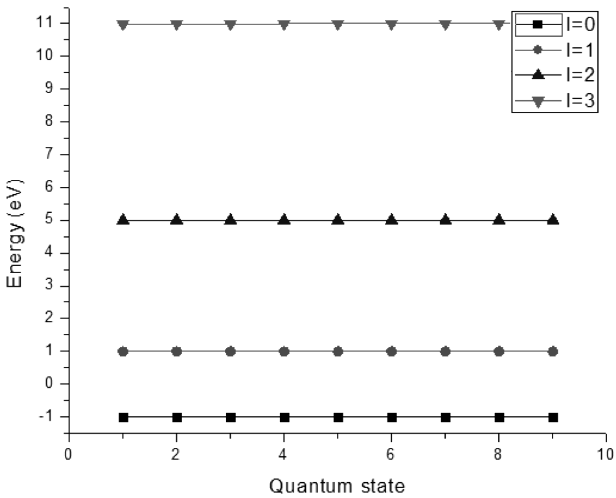

Fig. 5. Energy versus the quantum state for $\alpha = 0.04$, $\gamma = \beta = \Im = 0.1$

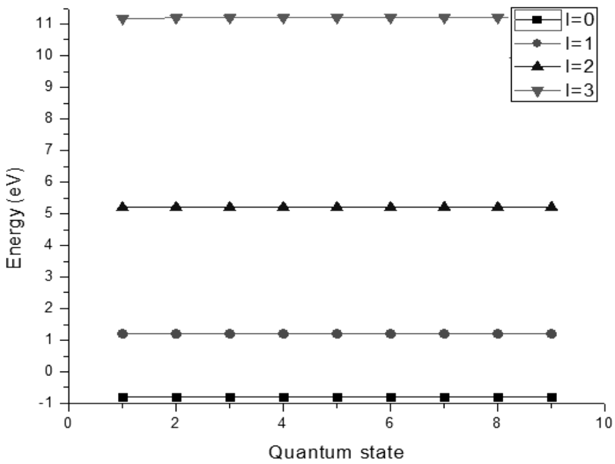

Fig. 6. Energy versus the quantum state for $\alpha = 0.05$, $\gamma = \beta = \Im = 0.1$

8. Conclusion

In this work, we have solved the non-relativistic Schrödinger equation with the inversely quadratic Hellmann plus ring-shaped potential in the framework of the Nikiforov–Uvarov method, by using a suitable approximation scheme to evaluate the centrifugal term. The bound-state energy eigenvalues and the corresponding wavefunctions of the radial and angular parts are obtained, respectively. The wave functions are expressed in terms of Jacobi polynomials. Under a special choice of the potential parameters, the potential under investigation is reduced to the well-known ones such as the Coulomb, inversely quadratic Yukawa, and Hartmann potentials. Their corresponding energy eigenvalues are evaluated. Our results would have interesting applications to the study of ring-shaped molecules and some diatomic molecules. That can also be used to study the systematization of a two-body interaction. The results could also have many applications to atomic, molecular, and nuclear physics, and to other related areas in physics and in chemistry.

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А.Д. Антья, Е.Е. Ітьен

НЕРЕЛЯТИВІСТСЬКИЙ РОЗГЛЯД ШРЕДІНГЕРІВСЬКИХ ЧАСТИНОК У ОБЕРНЕНОКВАДРАТИЧНОМУ ПОТЕНЦІАЛІ ХЕЛЛМАНА І ПОТЕНЦІАЛІ КІЛЬЦЕВОЇ ФОРМИ

Р е з ю м е

Методом Никифорова–Уварова отримано наближене рішення рівняння Шредінгера з оберненоквадратичним потенціалом Хеллмана і потенціалом кільцевої форми. Власні значення енергії і відповідні радіальні і кутові компоненти хвильових функцій отримані з використанням поліномів Якобі. Зокрема, наш результат зводиться до випадків з трьома відомими потенціалами: кулонівський, оберненоквадратичний потенціал Юкави і потенціал Хартмана. Оцінено власні значення енергії. Наші чисельні результати можуть бути застосовані для інших фізичних систем.