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REPRESENTATION OF CLASSICAL SOLUTIONS TO A LINEAR WAVE EQUATION WITH PURE DELAY

We consider the linear differential equation of heat conduction delay.
Keywords: dynamical systems, difference equations, stationary points, asymptotic stability, phase portrait.

Introduction

The wave equation is a typical linear hyperbolic second-order partial differential equation which naturally arises when modeling various phenomena of continuum mechanics such as sound, light, water or other kind of waves in acoustics, (electro)magnetics, elasticity and fluid dynamics, etc. [6, 13]. Providing a rather adequate description of physical processes, partial differential equations, or equations with distributed parameters in general, have found numerous applications in mechanics, medicine, ecology, etc. Introducing after-effects such as delay into such equations has gained a lot of attention over several past decades [2, 3, 7, 8]. Mathematical treatment of such systems requires additional carefulness since distributed systems with delay often turn out to be even ill-posed [4, 5, 12].

In the present paper, we consider an initial-boundary value problem for a general linear wave equation with pure delay and constant coefficients in a bounded interval subject to non-homogeneous Dirichlet boundary conditions. To solve the equation, we employ Fourier's separation method as well as the special functions referred to as delay sine and cosine functions which were introduced in [9, 10]. We prove the existence of a unique classical solution on any finite time interval, show its continuous dependence on the data, give its representation as a Fourier series and prove its absolute and uniform convergences under certain conditions on the data.

1. Equation with pure delay

We consider the following linear wave equation in a bounded interval $(0,l)$ with a single delay being a second order partial difference-differential equation for an unknown function η

$$\frac{\partial^2\eta(x,t)}{\partial t^2}=a^2\frac{\partial^2\eta(x,t-\tau)}{\partial x^2}+b\frac{\partial\eta(x,t-\tau)}{\partial x}+d\eta(x,t-\tau)+g(x,t) \tag{1.1}$$

subject to non-homogeneous Dirichlet boundary conditions and initial conditions

$$\eta(0,t)=\theta_1(t),\ \eta(l,t)=\theta_2(t),\ t\ge-\tau,$$

$$\eta(x,t)=\psi(x,t),\ 0\le x\le l,\ -\tau\le t\le 0. \tag{1.2}$$

Since we are interested in studying classical solutions, the following compatibility conditions are required to assure for the smoothness of solution on the boundary of space-time cylinder

$$\eta(x,t)=\theta_1(t),\ \eta(l,t)=\theta_2(t),\ -\tau\le t\le 0.$$

Definition 1.1. Under a classical solution to the problem (1.1), (1.2) we understand a function $\eta\in C([0,l]\times[-\tau,T])$ which satisfies $\partial_{tt}\eta,\partial_{tx}\eta,\partial_{xx}\eta\in C([0,l]\times[-\tau,0])$ as well as $\partial_{tt}\eta,\partial_{tx}\eta,\partial_{xx}\eta\in C([0,l]\times[0,T])$ and, being plugged into Equations (1.1), (1.2), turns them into identity.

Remark 1.2. The previous does not impose any continuity of time derivatives in $t=0$. If the continuity is desired, additional compatibility conditions on the data, including $g(x,t)$, are required.

Let $\|\cdot\|_{k,2}:=\|\cdot\|_{H^{k,2}((0,l))}$ denote the standard Sobolev norm (cf. [1]) and $\|\cdot\|_{-k,2}:=\|\cdot\|_{H^{-k,2}((0,l))}$ denote the norm of corresponding negative Sobolev space. We introduce the norm $\|\cdot\|_X:=\sqrt{\sum_{k=0}^{\infty}\|\cdot\|_{-k,2}^2}$ and define the Hilbert space X as a completion of $L^2((0,l))$ with respect to $\|\cdot\|_X$. Obviously, $X\subset(D((0,l)))'$, i.e., X can be continuously embedded into the space of distributions.

With this notation, we easily see that $A:=a^2\partial_x^2+b\partial_x+d$ (with ∂_x denoting the distributional derivative) is a bounded linear operator on X since

$$\|A\|_{L(X)}=\sup_{\|u\|_X=1}\|Au\|_X=\sup_{\|u\|_X=1}\sqrt{\sum_{k=0}^{\infty}\left\|a^2\partial_x^2u+b\partial_xu+du\right\|_{-k,2}^2}\le\sup_{\|u\|_X=1}\sum_{k=0}^{\infty}\left(a^2\|u\|_{-k-2,2}+b\|u\|_{-k-1,2}+d\|u\|_{-k,2}\right)$$
$$\le\sup_{\|u\|_X=1}\left(a^2\sum_{k=0}^{\infty}\|u\|_{-k-2,2}+b\sum_{k=0}^{\infty}\|u\|_{-k-1,2}+d\sum_{k=0}^{\infty}\|u\|_{-k,2}\right)\le\sup_{\|u\|_X=1}(a^2+b+d)\|u\|_X=a^2+b+d\,.$$

First, we obtain an *a priori* estimate in the distributional space X.

Theorem 1.2. There exists a constant $C>0$, dependent only on a,b,d,l,τ,T, such that the estimate

$$\max_{t\in[0,T]}\left(\|\eta(\cdot,t)\|_X^2+\|\eta_t(\cdot,t)\|_X^2\right)\le C\left(\|\psi(\cdot,0)\|_X^2+\|\psi_t(\cdot,t)\|_X^2\right)+$$
$$C\int_{-\tau}^{0}\|\psi(\cdot,t)\|_X^2+\|\psi_t(\cdot,t)\|_X^2\,dt+C\int_0^T\left(\|g(\cdot,t)\|_X^2+|\theta_1(t)|^2+|\theta_2(t)|^2\right)dt$$

holds true for any classical solution of Equations (1.1), (1.2).

Proof. Let η be a classical solution to Equations (1.1), (1.2). We define

$$w(x,t) := \eta(x,t) \text{ for } -\tau \le t \le 0,$$

$$w(x,t) := \eta(x,t) - \theta_1(t) - \frac{x}{l}\left[\theta_2(t) - \theta_1(t)\right] \text{ for } t > 0.$$

Then $w(x,t)$ satisfies homogeneous Dirichlet boundary conditions and solves the equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = Aw(x,t-\tau) + f(x,t) \tag{1.3}$$

in the extrapolated space X with

$$f(x,t) = g(x,t) + b\left(\theta_2(t) - \theta_1(t)\right) + \theta_1(t) + \frac{dx}{l}\left(\theta_2(t) - \theta_1(t)\right).$$

We multiply the equation with $w_t(\cdot,t)$ in the scalar product of X and use Young's inequality to get the estimate

$$\begin{aligned} \partial_t \|w_t(\cdot,t)\|_X^2 &= \left\langle Aw(\cdot,t-\tau), w_t(\cdot,t)\right\rangle_X + \left\langle f(\cdot,t), w_t(\cdot,t)\right\rangle_X \\ &\le \|w(\cdot,t-\tau)\|_X^2 + \left(1 + \|A\|_{L(X)}^2\right)\|w_t(\cdot,t)\|_X^2 + \|f(\cdot,t)\|_X^2. \end{aligned} \tag{1.4}$$

As in [11], we introduce the history variable

$$z(x,t,s) := w(x,t-s) \text{ for } (x,t,s) \in [0,l]\times[0,T]\times[0,\tau]$$

and obtain

$$z_t(\cdot,t,s) + z_s(\cdot,t,s) = 0.$$

Multiplying these identities with $w(\cdot,t)$ in X and performing a partial integration, we find

$$\partial_t \int_0^\tau \|z(\cdot,t,s)\|_X^2 ds = -\int_0^\tau \partial_s \|z(\cdot,t,s)\|_X^2 ds = \|w(\cdot,t)\|_X^2 - \|w(\cdot,t-\tau)\|_X^2. \tag{1.5}$$

Adding Equations (1.4) and (1.5) to the trivial identity

$$\partial_t \|w(\cdot,t)\|_X^2 \le \|w(\cdot,t)\|_X^2 + \|w_t(\cdot,t)\|_X^2,$$

we obtain

$$\partial_t \left\{ \|w(\cdot,t)\|_X^2 + \|w_t(\cdot,t)\|_X^2 + \int_{-\tau}^0 \|w(\cdot,t,s)\|_X^2 ds \right\} \le \left(2 + \|A\|_{L(X)}^2\right)\|w(\cdot,t)\|_X^2 + \|w_t(\cdot,t)\|_X^2 + \|f(\cdot,t)\|_X^2.$$

Thus, we have shown

$$\partial_t E(t) \le \left(2 + \|A\|_{L(X)}^2\right) E(t) + \|f(\cdot,t)\|_X^2, \tag{1.6}$$

where

$$E(t) := \|w(\cdot,t)\|_X^2 + \|w_t(\cdot,t)\|_X^2 + \int_{-\tau}^0 \|z(\cdot,t,s)\|_X^2 ds.$$

From Equation (1.6) we conclude

$$E(t) \le E(0) + \left(2 + \|A\|_{L(X)}^2\right)\int_0^t E(s)ds + \int_0^t \|f(\cdot,s)\|_X^2 ds.$$

Using now the integral form of Gronwall's inequality, we obtain

$$\begin{aligned} E(t) &\le E(0) + \int_0^t \|f(\cdot,s)\|_X^2 ds + \int_0^t \exp\left\{\left(2 + \|A\|_{L(X)}^2\right)(t-s)\right\}\left(E(0) + \int_0^s \|f(\cdot,\xi)\|_X^2 d\xi\right) ds \\ &\le \left(\tilde{C}E(0) + \int_0^T \|f(\cdot,s)\|_X^2 ds\right) \end{aligned} \tag{1.7}$$

for certain $\tilde{C} > 0.$ Taking into account

$$c_2\left(\|w(\cdot,t)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2\right) \le \|\eta(\cdot,t)\|_X^2 \le C_2\left(\|w(\cdot,t)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2\right),$$

$$c_2\left(\|f(\cdot,t)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2\right) \le \|g(\cdot,t)\|_X^2 \le C_2\left(\|f(\cdot,t)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2\right)$$

for some constants $c_1, c_2, C_1, C_2 > 0$ and exploiting the definition of $E(t)$, the proof is a direct consequence of Equation (1.7).

Corollary 1.2. Solutions of Equations (1.1), (1.2) are unique. The solution map

$$\left(\psi, g, \theta_1, \theta_2\right) \mapsto \eta$$

is well-defined, linear and continuous in the norms from Theorem 1.1.

Remark 1.3. It was essential to consider the weak space X. If the space corresponding to the usual wave equation is used, i.e., $(\eta,\eta_t) \in H_0^1\left((0,l)\right)\times L^2\left((0,l)\right)$, there follows from [5] that Equation (1.1), (1.2) is an ill-posed problem due to the lack of continuous dependence on the data even in the homogeneous case.

Next, we want to establish conditions on the data allowing for the existence of a classical solution. Performing the substitution

$$\xi(x,t) = e^{-\frac{b}{2a^2}x}\xi(x,t) \tag{1.8}$$

with a new unknown function $\xi(x,t)$ (cp. [11]), the initial boundary value problem (1.1), (1.2) can be written in following simplified form with a self-adjoint operator on the right-hand side

$$\frac{\partial^2\xi(x,t)}{\partial t^2} = a^2\frac{\partial\xi(x,t-\tau)}{\partial x^2} + c\xi(x,t-\tau) + f(x,t),\ c = d - \frac{b^2}{4a^2} \tag{1.9}$$

complemented by the following boundary and initial conditions

$$\xi(0,t) = \mu_1(t),\ \mu_1(t) = \theta_1(t),\ \xi(l,t) = \mu_2(t),\ \mu_2(t) = e^{\frac{b}{2a^2}l}\theta_2(t)\ \ t \geq -\tau\ , \tag{1.10}$$

$$\xi(x,t) = \phi(x,t),\ \phi(x,t) = e^{\frac{b}{2a^2}x}\psi(x,t),\ f(x,t) = e^{\frac{b}{2a^2}x}g(x,t),\ 0 \le x \le l,\ -\tau \le t \le 0\ . \tag{1.11}$$

The solution will be determined in the form

$$\xi(x,t) = \xi_0(x,t) + \xi_1(x,t) + G(x,t)\ .$$

Here, $G(x,t)$ is an arbitrary function with $\partial_{tt}G, \partial_{tx}G, \partial_{xx}G \in C([0,l]\times[-\tau,T])$ satisfying the boundary conditions

$$G(x,0) = \mu_1(t),\ G(x,l) = \mu_2(t)\ .$$

Assuming $\mu_1, \mu_2 \in C^2([-\tau,T])$, we let

$$G(x,t) = \mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]. \tag{1.12}$$

– $\xi_0(x,t)$ solves the homogeneous equation

$$\frac{\partial^2\xi_0(x,t)}{\partial t^2} = a^2\frac{\partial\xi_0(x,t-\tau)}{\partial x^2} + c\xi_0(x,t-\tau) \tag{1.13}$$

subject to homogeneous boundary and non-homogeneous initial conditions

$$\xi_0(0,t) \equiv 0,\ \xi_0(l,t) \equiv 0,\ t \ge -\tau,$$

$$\xi_0(x,t) = \Phi(x,t),\ \Phi(x,t) = \phi(x,t) - G(x,t),\ \ -\tau \le t \le 0,\ 0 \le x \le l\ . \tag{1.14}$$

In particular, with the function $G(x,t)$ selected as in Equation (1.12), we obtain

$$\Phi(x,t) = \phi(x,t) - \mu_1(t) - \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]. \tag{1.15}$$

– $\xi_1(x,t)$ solves the non-homogeneous equation

$$\frac{\partial^2\xi_1(x,t)}{\partial t^2} = a^2\frac{\partial\xi_1(x,t-\tau)}{\partial x^2} + c\xi_1(x,t-\tau) + F(x,t) \tag{1.16}$$

with

$$F(x,t) = a^2\frac{\partial^2}{\partial x^2}G(x,t-\tau) + cG(x,t-\tau) - \frac{\partial^2}{\partial t^2}G(x,t) \tag{1.17}$$

subject to homogeneous boundary and initial conditions. For $G(x,t)$ from Equation (1.12), we have

$$F(x,t) = f(x,t) + c\left\{\mu_1(t-\tau) + \frac{x}{l}\left[\mu_2(t-\tau) - \mu_1(t-\tau)\right]\right\} - \left\{\mu_1''(t) + \frac{x}{l}\left[\mu_2''(t) - \mu_1''(t)\right]\right\}. \tag{1.18}$$

2. Homogeneous equation. In this section, we obtain a formal solution to the initial-boundary value problem (1.13) with initial and boundary conditions given in Equations (1.10), (1.11). We exploit Fourier's separation method to determine $\xi_0(x,t)$ in the product form $\xi_0(x,t) = X(x)T(t)$. After plugging this ansatz into Equation (1.13), we find

$$X(x)T''(t) = a^2X''(x)T(t-\tau) + cX(x)T(t-\tau)\ .$$

Hence,

$$X(x)\left[T''(t) - cT(t-\tau)\right] = a^2X''(x)T(t-\tau)\ .$$

By formally separating the variables, we deduce

$$\frac{X''(x)}{X(x)} = \frac{T''(t) - cT(t-\tau)}{a^2T(t-\tau)} = -\lambda^2\ .$$

Thus, the equations can be decoupled as follows

$$T''(t) + \left(a^2\lambda^2 - c\right)T(t-\tau) = 0\ ,\quad X''(x) + \lambda^2X(x) = 0\ . \tag{2.1}$$

These are linear second order (delay) ordinary differential equations with constant coefficients.

Due to the zero boundary conditions for ξ_0, the boundary conditions for the second equation in (2.1) will also be homogeneous, i.e.,

$$X(0) = 0,\ X(l) = 0\ .$$

Therefore, we obtain a Sturm-Liouville problem admitting nontrivial solutions only for the eigennumbers

$$\lambda^2 = \lambda_n^2 = \left(\frac{\pi n}{l}\right)^2, \quad n = 1,2,3,...$$

and the corresponding eigenfunctions

$$X_n(x) = \sin\frac{\pi n}{l}x, \quad n = 1,2,3,...$$

Assuming

$$\left(\frac{\pi}{l}a\right)^2 - c > 0,$$

we denote

$$\omega_n = \sqrt{\left(\frac{\pi n}{l}a\right)^2 - c}, \quad n = 1,2,3,...$$

and consider the first equation in (2.1), i.e.,

$$T''(t) + \omega_n^2 T(t-\tau) = 0, \quad n = 1,2,3,... . \tag{2.2}$$

The initial conditions for each of the equations in (2.2) can be obtained by expanding the initial data into a Fourier series with respect to the eigenfunction basis of the second equation in (2.1)

$$\Phi(x,t) = \sum_{n=1}^{\infty}\Phi_n(t)\sin\frac{\pi n}{l}x, \quad \Phi_t(x,t) = \sum_{n=1}^{\infty}\Phi_n'(t)\sin\frac{\pi n}{l}x, \quad n = 1,2,... ,$$

$$\Phi_n(t) = \frac{2}{l}\int_0^l\left[\phi(s,t) - G(s,t)\right]\sin\frac{\pi n}{l}s ds, \quad \Phi_n'(t) = \frac{2}{l}\int_0^l\left[\phi_t(s,t) - G_t(s,t)\right]\sin\frac{\pi n}{l}s ds. \tag{2.3}$$

Let us further determine the solution of the Cauchy problem associated with each of the equations in (2.2) subject to the initial conditions from (2.3).

First, we briefly present some useful results from the theory of second order delay differential equations with pure delay obtained in [9]. The authors considered a linear homogeneous second order ordinary delay differential equation

$$\ddot{x}(t) + \omega^2 x(t-\tau) = 0, \ t \geq 0, \ \tau > 0, \ x(t) = \beta(t), \ -\tau \leq t \leq 0. \tag{2.4}$$

They introduced two special functions referred to as delay cosine and sine functions. Exploiting these functions, a unique solution to the initial value problem (2.4) was obtained.

Definition 2.1. Delay cosine is the function given as

$$\cos_\tau\{\omega,t\} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 - \omega^2\dfrac{t^2}{2!}, & 0 \leq t < \tau, \\ \cdots & \cdots \\ 1 - \omega^2\dfrac{t^2}{2!} + \omega^4\dfrac{(t-\tau)^4}{4!} + \cdots + \\ \quad + (-1)^k\omega^{2k}\dfrac{[t-(k-1)\tau]^{2k}}{(2k)!}, & (k-1)\tau \leq t < k\tau \end{cases} \tag{2.5}$$

with $2k$-order polynomials on each of the intervals $(k-1)\tau \leq t < k\tau$ continuously adjusted at the nodes $t = k\tau, \ k = 0,1,2,...$.

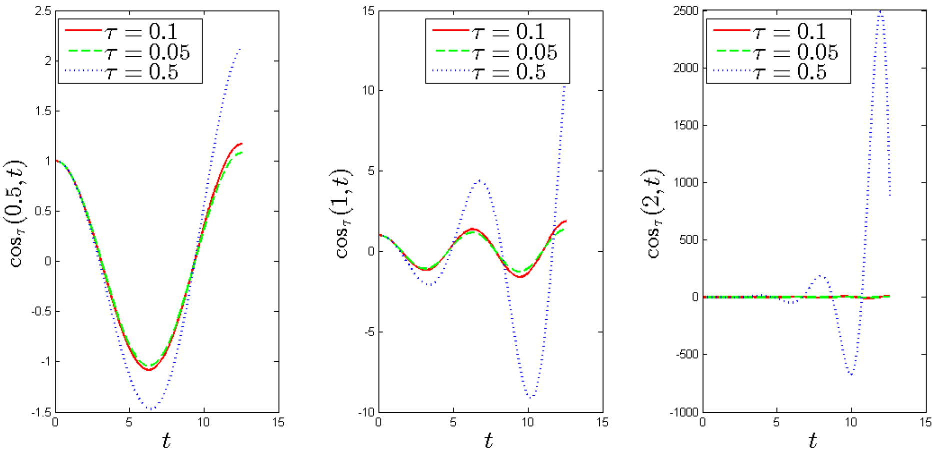

Figure 2.2. Delay cosine function

Definition 2.1. Delay sine is the function given as

$$\sin_\tau\{\omega,t\} = \begin{cases} 0, & -\infty < t < -\tau, \\ \omega(t+\tau), & -\tau \le t < 0, \\ \omega(t+\tau) - \omega^3 \dfrac{t^3}{3!}, & 0 \le t < \tau, \\ \dots & \dots \\ \omega(t+\tau) - \omega^3 \dfrac{t^3}{3!} + \cdots + (-1)^k \omega^{2k+1} \dfrac{[t-(k-1)\tau]^{2k+1}}{(2k+1)!}, & (k-1)\tau \le t < k\tau \end{cases} \tag{2.6}$$

with $(2k+1)$-order polynomials on each of the intervals $(k-1)\tau \le t < k\tau$ continuously adjusted at the nodes $t = k\tau,\ k = 0,1,2,\dots$.

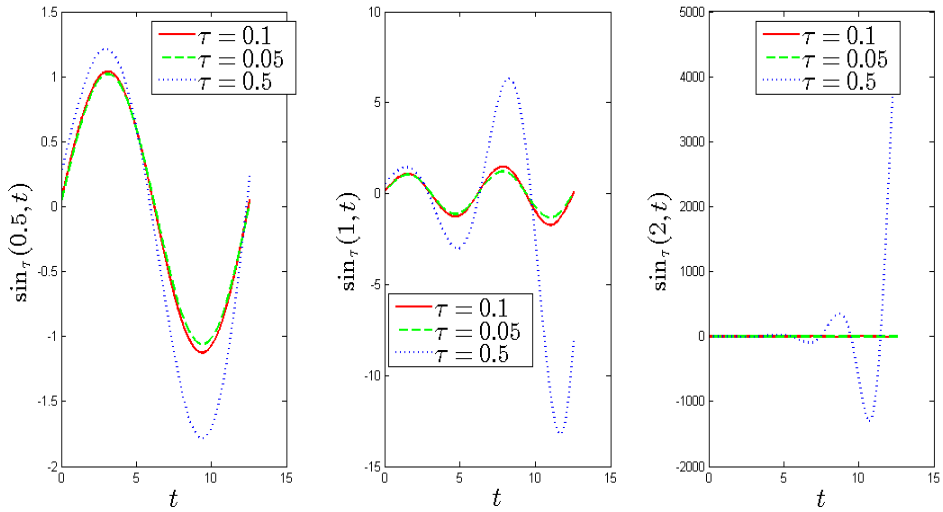

Figure 2.2. Delay sine function

There has further been proved that delay cosine uniquely solves the linear homogeneous second order ordinary delay differential equation with pure delay subject to the unit initial conditions $x(t) \equiv 1$, $-\tau \le t \le 0$, and the delay sine in its turn solves Equation (2.4) subject to the initial conditions $x(t) \equiv \omega(t+\tau)$, $-\tau \le t \le 0$.

Using the facts above, the unique solution of the Cauchy problem was represented in the integral form. In particular, the solution $x(t)$ to the homogeneous delay differential equation (2.4) with the initial conditions $x(t) = \beta(t)$, $-\tau \le t \le 0$ for an arbitrary $\beta \in C^2([-\tau,0])$ was shown to be given as

$$x(t) = \beta(-\tau)\cos_\tau\{\omega,t\} + \frac{1}{\omega}\beta'(-\tau)\sin_\tau\{\omega,t\} + \frac{1}{\omega}\int_{-\tau}^{0}\sin_\tau\{\omega,t-\tau-s\}\beta''(s)\,ds. \tag{2.7}$$

Turning back to the delay differential equation (2.2) with the initial conditions (1.4), we obtain their unique solution in the form

$$T_n(t) = \Phi_n(-\tau)\cos_\tau\{\omega_n,t\} + \frac{1}{\omega_n}\Phi_n'(-\tau)\sin_\tau\{\omega_n,t\} + \frac{1}{\omega_n}\int_{-\tau}^{0}\sin_\tau\{\omega_n,t-\tau-s\}\Phi_n''(s)\,ds. \tag{2.8}$$

Thus, assuming sufficient smoothness of the data to be specified later, the solution $\xi_0(x,t)$ to the homogeneous equation (1.13) satisfying homogeneous boundary and non-homogeneous initial conditions $\xi(x,t) = \Phi(x,t)$, $-\tau \le t \le 0$, $0 \le x \le l$, reads as

$$\xi_0(x,t) = \sum_{n=1}^{\infty}\Bigg\{\Phi_n(-\tau)\cos_\tau\{\omega_n,t\} + \frac{1}{\omega_n}\Phi_n'(-\tau)\sin_\tau\{\omega_n,t\} + \\ + \frac{1}{\omega_n}\int_{-\tau}^{0}\sin_\tau\{\omega_n,t-\tau-s\}\Phi_n''(s)\,ds\Bigg\}\sin\frac{\pi n}{l}x, \tag{2.9}$$

$$\Phi_n(t) = \frac{2}{l}\int_0^l[\phi(s,t) - G(s,t)]\sin\frac{\pi n}{l}s\,ds,\quad n = 1,2,3,\dots.$$

3. Non-homogeneous equation. Next, we consider the non-homogeneous Equation (1.16) with the right-hand side from Equation (1.18) subject to homogeneous initial and boundary conditions

$$\frac{\partial^2\xi_1(x,t)}{\partial t^2} = a^2\frac{\partial \xi_1(x,t-\tau)}{\partial x^2} + c\xi_1(x,t-\tau) + F(x,t),\ F(x,t) = f(x,t) + c\left\{\mu_1(t-\tau) + \frac{x}{l}[\mu_2(t-\tau) - \mu_1(t-\tau)]\right\} - \\ - \left\{\mu_1''(t) + \frac{x}{l}[\mu_2''(t) - \mu_1''(t)]\right\}.$$

The solution will be constructed as a Fourier series with respect to the eigenfunctions of the Sturm-Liouville problem from the previous section, i.e.,

$$\xi_1(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{\pi n}{l}x . \tag{3.1}$$

Plugging (3.1) into Equation (1.6) and comparing the time-dependent Fourier coefficients, we obtain a system of countably many second order delay differential equations

$$T_n''(t)+\omega_n^2T_n(t-\tau)=F_n(t),\ F_n(t)=\frac{2}{l}\int_0^l F(s,t)\sin\frac{\pi n}{l}sds . \tag{3.2}$$

In [9], the initial value problem for the non-homogeneous delay differential equation

$$x''(t)+\omega^2x(t-\tau)=f(t),\ t\ge 0,\ \tau>0$$

with homogeneous initial conditions $x(t)\equiv 0$, $-\tau\le t\le 0$ was shown to be uniquely solved by

$$x(t)=\int_0^t \sin_\tau\{\omega,t-\tau-s\}f(s)ds . \tag{3.3}$$

Exploiting Equation (3.3), the equations in (3.2) subject to zero initial conditions are uniquely solved by

$$T_n(t)=\int_0^t \sin_\tau\{\omega_n,t-\tau-s\}F_n(s)ds , \tag{3.4}$$

Therefore, the non-homogeneous partial delay differential equation with homogeneous initial and boundary conditions formally reads as

$$\xi_1(x,t)=\sum_{n=1}^{\infty}\left\{\int_0^t \sin_\tau\{\omega_n,t-\tau-s\}F_n(s)ds\right\}\sin\frac{\pi n}{l}x,\ F_n(t)=\frac{2}{l}\int_0^l F(s,t)\sin\frac{\pi n}{l}sds . \tag{3.5}$$

General case solution. The solution in the general case can thus formally be represented as the following series

$$\begin{aligned}\xi(x,t)=\sum_{n=1}^{\infty}\Bigg\{&\Phi_n(-\tau)\cos_\tau\{\omega_n,t\}+\frac{1}{\omega_n}\Phi_n'(-\tau)\sin_\tau\{\omega_n,t\}+\\&+\frac{1}{\omega_n}\int_{-\tau}^0 \sin_\tau\{\omega_n,t-\tau-s\}\Phi_n''(s)ds\Bigg\}\sin\frac{\pi n}{l}x+\\&+\sum_{n=1}^{\infty}\left\{\int_0^t \sin_\tau\{\omega_n,t-\tau-s\}F_n(s)ds\right\}\sin\frac{\pi n}{l}x+G(x,t) .\end{aligned} \tag{3.6}$$

Convergence of the Fourier series. Further, we present the conditions guaranteeing that the series converges to the classical solution of Problem (1.9)–(1.11) in the sense of Definition 1.1.

Theorem 3.1. Let $T>0$, $\tau>0$ and $m:=\left\lceil \frac{T}{\tau}\right\rceil$. Further, let the data functions $\phi(x,t)$, $\mu_1(t)$, $\mu_2(t)$ and $f(x,t)$ be such that their Fourier coefficients $\Phi_n(t)$ and $F_n(t)$ given in Equations (2.3) and (3.5) satisfy the conditions

$$\lim_{n\to\infty}\left(\left|\Phi_n(-\tau)\right|+\left|\Phi_n'(-\tau)\right|\right)n^{2m+3+\alpha}=0,\ \lim_{n\to\infty}\max_{s\in[-\tau,0]}\left|\Phi_n''(s)\right|n^{2m+3+\alpha}=0,$$

$$\lim_{n\to\infty}\max_{k=1,\cdots,m}\ \max_{(k-1)\tau\le t\le\max\{k\tau,T\}}\left|F_n(t)\right|n^{2k+3+\alpha}=0 \tag{3.7}$$

for an arbitrary, but fixed $\alpha>0$. Let $\left(\frac{\pi}{l}a\right)^2>c$.

Then the classical solution to problem (1.9)–(1.11) can be represented as an absolutely and uniformly convergent Fourier series given in Equation (3.6). The latter series is a twice continuously differentiable function with respect to both variables. Its derivatives of order less or equal two with respect to $0\le x\le l$, $0\le t<T$ can be obtained by a term-wise differentiation of the series and the resulting series are also absolutely and uniformly convergent.

Proof. We regroup the series from Equation (3.6) into the following sum

$$\xi(x,t)=S_1(x,t)+S_2(x,t)+S_3(x,t)+G(x,t),$$

where

$$S_1(x,t)=\sum_{n=1}^{\infty}A_n(t)\sin\frac{\pi n}{l}x,\ S_2(x,t)=\sum_{n=1}^{\infty}B_n(t)\sin\frac{\pi n}{l}x,\ S_3(x,t)=\sum_{n=1}^{\infty}C_n(t)\sin\frac{\pi n}{l}x,$$

$$A_n(t)=\Phi_n(-\tau)\cos_\tau\{\omega_n,t\}+\frac{1}{\omega_n}\Phi_n'(-\tau)\sin_\tau\{\omega_n,t\},$$

$$B_n(t)=\frac{1}{\omega_n}\int_{-\tau}^0 \sin_\tau\{\omega_n,t-\tau-s\}\Phi_n''(s)ds,\ C_n(t)=\frac{1}{\omega_n}\int_{-\tau}^0 \sin_\tau\{\omega_n,t-\tau-s\}F_n(s)ds,$$

$$\omega_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c},\ n=1,2,3,\ldots .$$

1. First, we consider the coefficient functions $A_n(t)$. For an arbitrary $t \in [0,T]$ with $(k-1)\tau \le t < k\tau$, we find

$$A_n(t) = \Phi_n(-\tau)\cos_\tau\{\omega_n, t\} + \frac{1}{\omega_n}\Phi'_n(-\tau)\sin_\tau\{\omega_n, t\} =$$

$$= \left\{1 - \left(\frac{\pi n}{l}a\right)^2 \frac{(t-\tau)^2}{2!} + \ldots + (-1)^k \left(\frac{\pi n}{l}a\right)^{2k} \frac{\left[t-(k-1)\tau\right]^{2k}}{(2k)!}\right\} \times \Phi_n(-\tau) +$$

$$+ \left\{(t+\tau) - \left(\frac{\pi n}{l}a\right)^2 \frac{t^3}{3!} + \ldots + (-1)^k \left(\frac{\pi n}{l}a\right)^{2k} \frac{\left[t-(k-1)\tau\right]^{2k+1}}{(2k+1)!}\right\} \times \Phi'_n(-\tau).$$

If $\Phi_n(-\tau)$ and $\Phi'_n(-\tau)$ are such that the condition

$$\lim_{n\to+\infty}\left(\left|\Phi_n(-\tau)\right| + \left|\Phi'_n(-\tau)\right|\right) n^{2k+3+\alpha} = 0$$

holds true, the series $S_1(x,T)$ as well as its derivatives of order less or equal 2 converge absolutely and uniformly. Note that a single differentiation with respect to x corresponds, roughly speaking, to a multiplication with n.

2. Next, we consider the coefficients $B_n(t)$. For an arbitrary $t \in [0,T]$ with $(k-1)\tau \le t < k\tau$, we perform the substitution $t - \tau - s = \xi$ and exploit the mean value theorem to estimate

$$\left|B_n(t)\right| = \left|\frac{1}{\omega_n}\int_{t-\tau}^{t} \sin_\tau\{\omega_n, \xi\}\Phi''_n(t-\tau-\xi)\,d\xi\right| \le \tau \max_{-\tau\le s\le 0}\left|\Phi''_n(s)\right| \times$$

$$\times \max_{j=k-1,k}\max_{t-\tau\le s\le t}\left|(s-\tau) - \left(\frac{\pi n}{l}a\right)^2\frac{s^3}{3!} + \ldots + (-1)^j\left(\frac{\pi n}{l}a\right)^{2j}\frac{\left[s-(j-1)\tau\right]^{2j+1}}{(2j+1)!}\right|.$$

Applying the theorem on differentiation under the integral sign to $B_n(t)$ and taking into account that $\sin_\tau\left\{\frac{\pi n}{l}a, t\right\}$ is twice weakly differentiable for $t \ge 0$, its derivatives are polynomials of order lower than those of $\sin_\tau\left\{\frac{\pi n}{l}a, t\right\}$ and their convolution with Φ''_n is continuous, analogous estimates can be obtained for $B'_n(t)$ and $B''_n(t)$ which, in their turn, follow to be also continuous functions.

Now, if the condition

$$\lim_{n\to\infty}\max_{s\in[-\tau,0]}\left|\Phi''_n(s)\right| n^{2k+3+\alpha} = 0$$

is satisfied, the series $S_2(x,t)$ as well as its derivatives of order less or equal 2 converge absolutely and uniformly.

3. Finally, we look at the Fourier coefficients $C_n(t)$. Again, for an arbitrary time moment $t \in [0,T]$ with $(k-1)\tau \le t < k\tau$, $0 \le k \le m$, we substitute $t - \tau - \xi = s$. Once again, using the mean value theorem, we estimate

$$\left|C_n(t)\right| = \left|\frac{1}{\omega_n}\int_{t-\tau}^{t}\sin_\tau\left\{\frac{\pi n}{l}a, \xi\right\}F_n(t-\tau-\xi)\,d\xi\right| \le \tau \max_{t-\tau\le s\le t}\left|\Phi''_n(s)\right| \times$$

$$\times \max_{j=k-1,k}\max_{t-\tau\le s\le t}\left|(s-\tau) - \left(\frac{\pi n}{l}a\right)^2\frac{s^3}{3!} + \ldots + (-1)^j\left(\frac{\pi n}{l}a\right)^{2j}\frac{\left[s-(j-1)\tau\right]^{2j+1}}{(2j+1)!}\right|.$$

As before, $C_n(t)$ can be shown to be twice continuously differentiable. If now

$$\lim_{n\to\infty}\max_{k=1,\cdots,m}\max_{(k-1)\tau\le t\le \max\{k\tau,T\}}\left|F_n(t)\right| n^{2k+3+\alpha} = 0,$$

is satisfied, then both $S_3(x,t)$ its derivatives of order less or equal 2 converge absolutely and uniformly.

Since all three conditions are guaranteed by the assumptions of the Theorem due to $k \le n$, the proof is finished. □ **Remark 3.2.** From the practical point of view, the rapid decay condition on the Fourier coefficients of the data given in Equation (3.7) means a sufficiently high Sobolev regularity of the data and corresponding higher order compatibility conditions on the boundary (cf. [11]).

References

1. Adams, R.A., Fournier, J.J.F. Sobolev Spaces, 2nd ed., Pure and Applied Mathematics, vol. 140, Academic Press Inc, 2003 – pp. 1-320.
2. Azizbayov, E.I., Khusainov, D.Ya. The solution of an equation with delay // Bulletin of the Kyiv National University. Series: Cybernetics, 12, 2012. – pp. 4-14 (in Russian).
3. Bátkai, A., Piazzera, S. Semigroups for delay equations, Research Notes in Mathematics, 10, A.K. Peters: Wellesley MA, 2005 – pp. 1-259.
4. Datko, R. Two examples of ill-posedness with respect to time delays revisited // IEEE Trans. Automatic Control, VOl. 42, 1997 – pp. 511-515.
5. Dreher, M., Quintanilla, R., Racke, R. Ill-posed problems in thermomechanics // Applied Mathematics Letter, 22(9), 2009 – pp. 1374-1379.
6. Eck, Ch., Garcke, H., Knabber, P. Mathematische Modellierung. Springer-Verlag Berlin Heidelberg, 2008 – pp. 1-503.
7. Els'gol'ts, L.E., Norkin, S.B. Introduction to the Theory and Application of Differential Equations with Deviating Arguments. Mathematics in Science and Engineering, Vol. 105, Academic Press, 1973 – pp. 1-357.
8. Hale, J. K. Theory of Functional Differential Equations. – Applied Mathematical Sciences Series, Vol. 3, 1977 – pp. 1-365.
9. Khusainov, D.Ya., Diblík, J., Růžičková, M., Lukáčová, J. Representation of a solution of the Cauchy problem for an oscillating system with pure delay // Nonlinear Oscillations, Vol. 11, Is. 2, 2008 – pp 276-285.

10. Khusainov, D.Ya., Ivanov, A.F., Kovarzh, I.V., The solution of wave equation with delay // Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics & Mathematics, Vol. 4, 2006 – pp. 243-248 (in Ukrainian with English summary).
11. Khusainov, D.Ya., Pokojovy, M., Azizbayov, E.I. On classical Stability for a linear 1D heat equation with one constant delay // to appear in Journal Appl. Math., No.1, 2013.
12. Racke, R. Instability of coupled systems with delay // Commun. Pure Appl. Anal, Vol. 11, Is. 5, 2012 – pp. 1753-1773.
13. Tikhonov, A.N., Samarskii, A.A. Equations of Mathematical Physics. Dover Publications, 1990 – pp. 1-765.

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ПРЕДСТАВЛЕННЯ КЛАСИЧНОГО РОЗВ'ЯЗКУ ЛІНІЙНОГО ХВИЛЬОВОГО РІВНЯННЯ З ЧИСТИМ ЗАПІЗНЮВАННЯМ

Розглянуто лінійне диференціальне рівняння теплопровідності з запізнюванням.
Ключові слова: динамічна система, різницеві рівняння, точки спокою, асимптотична стійкість, фазовий портрет.

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ПРЕДСТАВЛЕНИЕ КЛАССИЧЕСКОГО РЕШЕНИЯ ЛИНЕЙНОГО ВОЛНОВОГО УРАВНЕНИЯ С ЧИСТЫМ ЗАПАЗДЫВАНИЕМ

Рассмотрено линейное дифференциальное уравнение теплопроводности с запаздыванием.
Ключевые слова: динамическая система, разностные уравнения, точки покоя, асимптотическая устойчивость, фазовый портрет.

УДК 519.87

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APPROACH FOR SOLVING OF TRANSPORTATION PROBLEM WITH FUZZY RESOURCES

In this paper, a method is proposed to find the fuzzy optimal solution of fuzzy transportation model by representing all the parameters as triangle fuzzy numbers. To illustrate the proposed method a fuzzy transportation problem is solved by using the proposed method and the results are obtained. The proposed method is easy to understand, and to apply for finding the fuzzy optimal solution of fuzzy transportation models in real life situations.

Keywords: fuzzy transportation problem, triangle fuzzy numbers, optimal solution.

INTRODUCTION

The transportation problem which, is one of network integer programming problems is a problem that deals with distributing any commodity from any group of 'sources' to any group of destinations or 'sinks' in the most cost effective way with a given 'supply' and 'demand' constraints . Depending on the nature of the cost function, the transportation problem can be categorized into linear and nonlinear transportation problem.

Transportation problem is a linear programming (LP) problem stemmed from a network structure consisting of a finite number of nodes and arcs attached to them. In a typical problem a production is to be transported from m sources to n destinations and their capacities are a_1, a_2, ... a_m and b_1 ,b_2, ... b_n, respectively. There is a penalty C_{ij} and variable X_{ij} associated with transporting unit of production and unknown quantity to be shipped from source *i* to destination *j*.

Efficient algorithms have been developed for solving the transportation problem when the cost coefficients and the supply and demand quantities are known exactly. However, there are cases that these parameters may not be presented in a precise manner. For example, the unit shipping cost may vary in a time frame. The supplies and demands may be uncertain due to some uncontrollable factors.

Bellman and Zadeh [1] proposed the concept of decision making in fuzzy environment. Lai and Hwang [2] considered the situation where all parameters are fuzzy. In 1979, Isermann [3] introduced algorithm for solving this problem which provides effective solutions. The Ringuest and Rinks [4] proposed two iterative algorithms for solving linear, multi criteria transportation problem. S.Chanas and D.Kuchta [6] the approach based on interval and fuzzy coefficients had been elaborated. Tien Fuling [7] applied the method of interactive fuzzy multi-objective linear programming to transportation planning decisions. A new approach called fuzzy modified computational procedure to find the optimal solution was discussed in [8]. The new arithmetic operations of trapezoidal fuzzy numbers are employed to get the fuzzy optimal solutions. In this work, the fuzzy transportation problems using triangle fuzzy numbers are discussed. Here after, we have to propose the method of fuzzy modified distribution to be finding out the optimal solution for the total fuzzy transportation minimum cost.

There are also studies discussing the fuzzy transportation problem. Chanas et al. [6] investigate the transportation problem with fuzzy supplies and demands and solve them via the parametric programming technique in terms of the Bellman and Zadeh criterion. Their method is to derive the solution which simultaneously satisfies the constraints and the goal to a maximal degree. In this paper fuzzy transportation problem is discussed with constraints where the supply and demand are triangle fuzzy numbers. This paper aims to find out the best compromise solution among the set of feasible solutions for fuzzy transportation problem.