$$-\left(1-\alpha_2\right)\alpha_3\alpha_4...\alpha_{n-1}\left(A_2^TH_3+H_3A_2\right)-...$$
$$-\left(1-\alpha_{n-2}\right)\alpha_{n-1}\left(A_2^TH_{n-1}-H_{n-1}A_2\right)-\left(1-\alpha_{n-1}\right)\left(A_2^TH_n+H_nA_2\right).$$

Для последней подсистемы получаем

$$C_n\left(\alpha_1,\alpha_2,...\alpha_{n-1}\right)=-\alpha_1\alpha_2\alpha_3...\alpha_{n-1}\left(A_n^TH_1+H_1A_n\right)-\left(1-\alpha_1\right)\alpha_2\alpha_3...\alpha_{n-1}\left(A_n^TH_2+HA_n\right)-$$
$$-\left(1-\alpha_2\right)\alpha_3\alpha_4...\alpha_{n-1}\left(A_n^TH_3+H_3A_n\right)-...-\left(1-\alpha_{n-2}\right)\alpha_{n-1}\left(A_n^TH_{n-1}-H_{n-1}A_n\right)+\left(1-\alpha_{n-1}\right)C_n.$$

Обозначим

$$C_{ij}=H_j-B_i^TH_jB_i,\quad i\neq j.$$

Тогда

$$C_1\left(\alpha_1,\alpha_2,...\alpha_{n-1}\right)=\alpha_1\alpha_2\alpha_3...\alpha_{n-1}C_1-\left(1-\alpha_1\right)\alpha_2\alpha_3...\alpha_{n-1}C_{12}-\left(1-\alpha_2\right)\alpha_3\alpha_4...\alpha_{n-1}C_{13}-...$$
$$-\left(1-\alpha_{n-2}\right)\alpha_{n-1}C_{1,n-1}-\left(1-\alpha_{n-1}\right)C_{1,n},$$
$$C_2\left(\alpha_1,\alpha_2,...\alpha_{n-1}\right)=-\alpha_1\alpha_2\alpha_3...\alpha_{n-1}C_{21}+\left(1-\alpha_1\right)\alpha_2\alpha_3...\alpha_{n-1}C_2-\left(1-\alpha_2\right)\alpha_3\alpha_4...\alpha_{n-1}C_{23}-...$$
$$-\left(1-\alpha_{n-2}\right)\alpha_{n-1}C_{2,n-1}-\left(1-\alpha_{n-1}\right)C_{2,n}.$$

………………………………………………………………………………..

$$C_n\left(\alpha_1,\alpha_2,...\alpha_{n-1}\right)=-\alpha_1\alpha_2\alpha_3...\alpha_{n-1}C_{n,1}-\left(1-\alpha_1\right)\alpha_2\alpha_3...\alpha_{n-1}C_{n,2}-\left(1-\alpha_2\right)\alpha_3\alpha_4...\alpha_{n-1}C_{n3}-...$$
$$-\left(1-\alpha_{n-2}\right)\alpha_{n-1}C_{n,n-1}+\left(1-\alpha_{n-1}\right)C_n.$$

И получаем утверждение теоремы 2.2.

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ПРО СТІЙКІСТЬ ЛІНІЙНИХ СИСТЕМ З ПЕРЕМИКАННЯМ

У даній роботі будуть розглядатися лінійні диференціальні системи з лінійними законами перемикання. Одержано умови стійкості їх рішень.
Ключові слова: стійкість, різницеві системи, перемикання, метод Ляпунова.

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STABILITY OF LINEAR SYSTEMS WITH SWITCHING

In this paper we consider linear differential systems with linear laws change. Stability conditions of their decisions.
Keywords: stability, difference systems, switching, Lyapunov method.

УДК 517.929

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ADAPTIVE CONTROL OF LYAPUNOV EXPONENTS

A new approach to adaptive control of local Lyapunov exponents is considered. A numerical optimization algorithm to determine the spectrum of Lyapunov exponents from the observed noise time series of a single variable is proposed. The approach is tested on non-linear lattice with known Lyapunov spectra.

Keywords: adaptive control, Lyapunov exponent, optimization, modeling

1. Optimization approach to estimation of Lyapunov exponents

Let us consider an observed trajectory $x(t)$, which can be considered as a solution of a certain dynamical system:

$$\dot{x}=F(x), \tag{1}$$

where $u\in U\subset R^n$, $x\in M$ – smooth manifold, and is defined in a *d*-dimensional space. On the other hand, the evolution of the tangent vector γ in a tangent space at $x(t)$ is represented by linearizing Eq. (1),

$$\dot{\gamma} = S\left(x\left(t\right)\right)\cdot\gamma\ , \tag{2}$$

where $S = DF = \partial F \partial x$ is the Jacobian matrix of F. The solution of the linear nonautonomous Eq. (2) can be obtained as

$$\gamma\left(t\right) = A^t \cdot \gamma\left(0\right)\ , \tag{3}$$

where A^t is the linear operator which maps tangent vector $\gamma(0)$ to $\gamma(t)$. The mean exponential rate of divergence of the tangent vector γ is defined as follows:

$$\lambda\left(x\left(0\right),\gamma\left(0\right)\right) = \lim_{t\to\infty}\frac{1}{t}\ln\frac{\left\|\gamma\left(t\right)\right\|}{\left\|\gamma\left(0\right)\right\|}\ , \tag{4}$$

where $\|...\|$ denotes a norm with respect to some Riemannian metric. Furthermore, there is a d-dimensional basis $\{e_i\}$ of $\gamma(0)$, for which λ takes values $\lambda_i\left(x\left(0\right)\right) = \lambda\left(x\left(0\right),e_i\right)$. These can be ordered by their magnitudes $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d$, and are the spectrum of Lyapunov characteristic exponents. These exponents are independent of $x\left(0\right)$ if the system is ergodic [1].

We often have no knowledge of the nonlinear equations of the system which produces the observed time series. Moreover, even if we know the equations of motion, such as the Navier-Stokes equations for fluid systems, it is a hard task to derive the mode-truncated equations with finite degrees of freedom from partial differential equations (which is the infinite-dimensional system) and reproduce the same phenomena as the experiment from them. However, there is a possibility of estimating a linearized flow map A^t of tangent space from the observed data.

Let $\{x_j\}$ $\left(j = 1,2,...\right)$ denote a time series of some geomagnetic index measured at the discrete time interval Δt, i.e., $x_j = x\left(t_0 + \left(j-1\right)\Delta t\right)$. Consider a small ball of radius r is centered at the orbital point x_j, and find any set of points $\{x_{k_i}\}$ $\left(i = 1,2,...,N\right)$, included in this ball, i.e.,

$$\left\{y^i\right\} = \left\{x_{k_i} - x_j \mid \left\|x_{k_i} - x_j\right\| \leq r\right\}\ , \tag{5}$$

where y_i is the displacement vector between x_{k_i} and x_j. We used a usual Euclidean norm defined as follows: $\|w\| = \left(w_1^2 + w_2^2 + ... + w_d^2\right)^{1/2}$ for some vector $w = \left(w_1, w_2, ..., w_d\right)$. After the evolution of a time interval $\tau = m\Delta t$, the orbital point x_j will proceed to x_{j+m} and neighboring points $\{x_{k_i}\}$ to $\{x_{k_i+m}\}$. The displacement vector $y^i = x_{k_i} - x_j$ is mapped to

$$\left\{z^i\right\} = \left\{x_{k_i+m} - x_{j+m} \mid \left\|x_{k_i} - x_j\right\| \leq \varepsilon\right\}\ , \tag{6}$$

If the radius r is small enough for the displacement vector y^i and regarded as a good approximation of tangent vectors in the tangent space, evolution of y^i to z^i can be represented by some matrix A_j, as

$$z^i = A_j y^i\ , \tag{7}$$

The matrix A_j is an approximation of the flow map A^τ at x_j in Eq. (3). Let us proceed to the optimal estimation of the linearized flow map A_j from the data sets $\{y^i\}$ and $\{z^i\}$. A plausible procedure for optimal estimation is the least-square-error algorithm, which minimized the average of the squared error norm between z^i and $A_j y^i$ with respect to all components of the matrix A_j as follows [1]:

$$\min_{A_j} S = \min_{A_j}\frac{1}{N}\sum_{i=1}^{N}\left\|z^i - A_j y^i\right\|^2\ . \tag{8}$$

Denoting $\left(k,l\right)$ component of matrix A_j by $a_{kl}\left(j\right)$ and applying condition (8), one obtains $d\times d$ equations to solve $\partial S / \partial a_{kl}\left(j\right) = 0$. One will easily obtain the following expression for A_j:

$$A_j V = C, \quad \left(V\right)_{kl} = \frac{1}{N}\sum_{i=1}^{N} y^{ik} y^{il}\ , \tag{9}$$

$$\left(C\right)_{kl} = \frac{1}{N}\sum_{i=1}^{N} z^{ik} y^{il}\ ,$$

where V and C are $d\times d$ matrices, called covariance matrices, and y^{ik} and z^{ik} are the k **.**

Let us consider a stochastic optimization problem for custom function

$$F(A_i) = E\left\{S(A_i,\omega\right\},$$

where ω is an uncertain quantity element of a probability space, and $E\{\bullet\}$ denotes the expectation operations. The problem of minimizing the cost functional F subject to constraints on matrix A_j can be viewed as a deterministic optimization problem. In the case where the function F $F(\cdot,\omega)$ is a differentiable function of A_j for each ω, it can be shown under quite general assumption that the gradient of the function F exists for each x and is given by

$$\nabla F(A_i) = F\left\{\nabla f(A_j,\omega)\right\}\ . \tag{10}$$

To conclude, by using the method we could obtain good estimates of the Lyapunov spectrum from the observed time series in a very systematic way. Because of the ability of the method to measure several Lyapunov exponents, positive, zero, and even negative ones, other important characteristic invariants such as fractal dimension of attractors of Kolmogorov entropy are obtainable with great ease. It is hoped that the method has wide applicability to systems whose dynamical equations are not available. By definition, chaotic systems display sensitive dependence on initial conditions: two initially close trajectories can diverge exponentially in the phase space with a rate given by the largest Lyapunov exponent [3].

2. Control of Lyapunov exponents. Let us consider the system

$$\dot{x} = f(x, u(t))$$

where $x = (x_1, x_2, ..., x_n)$ are the state variables and $u(t)$ is the parameter, whose value determines the nature of the dynamics

$$\dot{u} = \gamma(S^* - S),$$

where S^* is the target value of some variable S, and the value of γ indicates the stiffness of control.

For the maintenance of a stable fixed point in a discrete dynamical system, the procedure is as follows. The nonlinear system evolves according to the appropriate equation

$$x_{n+1} = f(u, x_n),$$

where u is the parameter to be controlled. If x^* is the required value of x, then the additional equation (for $S \equiv x$)

$$u_{t+1} = u_t + \gamma(x^* - x_t)$$

has the desired effect of tuning the value of u in the way, the dynamics of the combined equation gives $x \to x^*$ over a wide range of initial conditions.

For a one-dimensional discrete dynamical system, the Lyapunov exponent is defined through

$$\lambda = \lim_{N \to \infty} \sum_{i=0}^{N-1} \ln\left|f'(u, x_i)\right|$$

The control equation takes the form

$$u_{t+1} = u_t + \gamma(\lambda^* - \lambda_t)$$

where $\lambda_t = \ln\left|f'(u, x_t)\right|$ is the instantaneous value of the Lyapunov exponent implementation of the methodology in, say, the logistic equation, is direct and the relevant equation are

$$x_{t+1} = u_t x_t (1 - x_t),$$
$$\lambda_t = \ln\left|u_t(1 - 2x_t)\right|,$$
$$u_{t+1} = u_t + \gamma(\lambda^* - \lambda_t)$$

The presented adaptive algorithm, can be used to achieve desired chaotic behavior in nonlinear controlled dynamical systems.

3. Control of Lyapunov exponents in nonlinear lattice. A coupled map lattice is a *N*-dimensional network of interconnected units where each unit evolves in time through a map (or recurrence equation) of the discrete form [3–5]:

$$X^{k+1} = F(X^k), \tag{11}$$

where X^k denotes the field value (*N*-dimensional vector) at the indicated time *k*. In the case of a globally coupled map, with a global (mean field) coupling factor ***e***, the dynamics can be rewritten as:

$$x_n^{k+1} = (1-\varepsilon) f_n\left[x_n^k\right] + \frac{\varepsilon}{L} \sum_{j=1}^{L} f_j\left[x_j^k\right], \tag{12}$$

where *n* and *j* are the labels of lattice sites ($j \neq n$). The term *L* indicates over how many neighbors we are averaging and it is sometimes referred to as *coordination number*. The local *N*-dimensional map is assumed to be chaotic. Completely synchronous chaotic states are possible with this model when corresponding *N*-dimensional manifolds are attracting or stable. The criterion for stability of this synchronization manifold has been derived in [4]. Further stability analysis of synchronized periodic orbits in coupled map lattices can be found in [5]. Varying ***e*** and *L* we can change the extent of spatial correlations, from systems with local interactions to systems with long-range interactions. These systems typically exhibit spatially and/or temporally chaotic behavior, the control of which is very desirable because of its potential real-life applications. Several strategies have been proposed to control the collective spatiotemporal dynamics of such systems. In this paper we first describe adaptive feedback control strategies for coupled map lattice systems and then describe an optimization technique for choosing optimal feedback parameters.

Experimental studies in rodent models of epilepsy have used EEG recordings from four to six electrodes placed in frontal and temporal regions of the animal brain. We have therefore chosen a CML model with five non-identical logistic maps. The system parameters $б_1 \ldots б_5$ were chosen randomly as 3.9, 3.97, 3.95, 3.965 and 3.96. The coupling term ***e*** was varied from a value of 0.10 to 0.14 to study the dynamical behavior in both the spatial and temporal regimes. Figure 1 shows the changes in spatiotemporal patterns as we increase the value of the parameter ***e***. For illustration purposes we have only shown the amplitude and Lyapunov exponent profiles of the single cell (cell 1). The remaining cells exhibit a similar pattern. As we increase the value of ***e*** gradually as shown in Figure 1D, the amplitude plot, shown in Figure 1A becomes more ordered and we can also see a drop in the Lyapunov exponents (calculated as a running mean) from the same time series, suggesting a more ordered state as illustrated in Figure 1B. Figure 1C shows the mean Lyapunov exponent profile calculated over all 5 cells in the CML [5]. We can observe a gradual fall in the values of this global measure with increasing values of coupling.

Figure 2 illustrates the feedback control strategy also referred to as 'dynamic feedback control' in literature described, for a target *л** = 0.3. Since there can be several values of the controlled parameter *б* (corresponding to several different attractors) which gives the desired value of the Lyapunov exponent, the actual value of the controlled parameter takes depends on the stiffness of control, and initial conditions. The fluctuations in the controlled parameter are proportional to the value of the stiffness, converging to a single value for small stiffness while exhibiting large variations for higher values of stiffness.

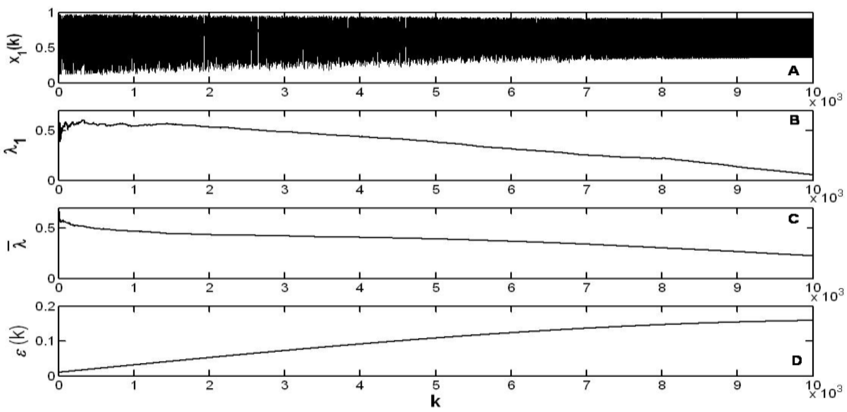

Figure 1. (A) Amplitude spectrum as a function of time; (B) Lyapunov exponent profile of the single cell; (C) Mean Lyapunov exponent profile (L=5) estimated from a five cell CML; (D) parameter e as a function of time

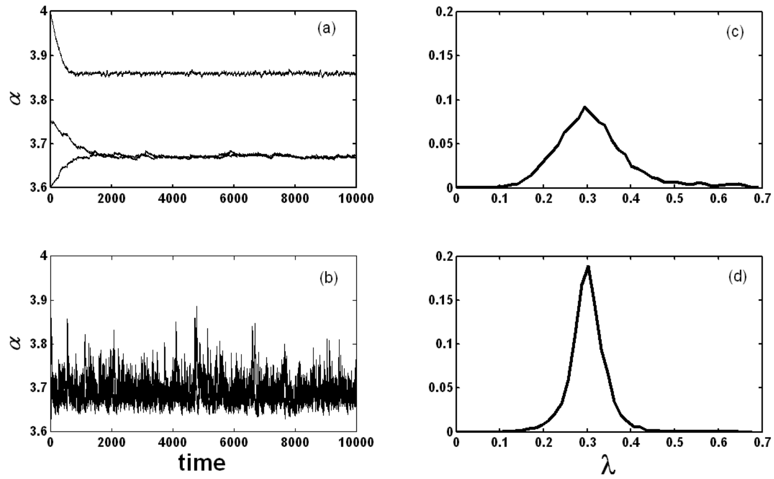

Figure 2. Multiplicative control: of the parameter б as a function of iteration step for л* = 0.3, and stiffness:
a) g = 0.001, and b) g = 0.02. The different curves correspond to different initial б.
Probability distributions of finite step Lyapunov exponents for б0 = 4.0 and stiffness (c) g = 0.001, and d) g = 0.01

A coupled map lattice system can be used to model the dynamical evolution of Lyapunov exponents in a complex system (Figure 3). The algorithm involves generating an error function between the target Lyapunov exponent profile of the complex system and some nonlinear transformation of estimated lattice Lyapunov exponent values. The error is used to generate an optimized feedback input to the lattice. Such a learning algorithm can be used in developing realistic model of complex system dynamics and hence make the models more useful in the study and control of such complex systems.

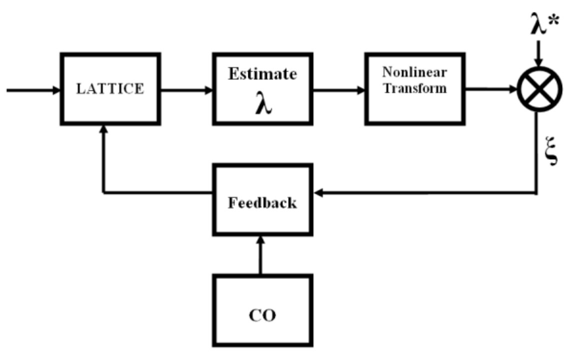

Figure 3. Proposed adaptive learning algorithm for a coupled map lattice via optimized feedback control to emulate the target dynamics of any complex network. CO refers to the constrained optimization block. o refers to error generated from nonlinearly transformed estimates of local Lyapunov exponents and target Lyapunov exponents

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АДАПТИВНЕ КЕРУВАННЯ ПОКАЗНИКАМИ ЛЯПУНОВА

Запропоновано підхід до адаптивного керування локальними показниками Ляпунова. Запропоновано чисельний алгоритм визначення спектра показників Ляпунова по спостережуваному зашумленому часовому ряду. Підхід протестовано на прикладі нелінійної гратки з відомим спектром показників Ляпунова.
Ключові слова: адаптивне керування, показники Ляпунова, оптимізація, моделювання.

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АДАПТИВНОЕ УПРАВЛЕНИЕ ПОКАЗАТЕЛЯМИ ЛЯПУНОВА

Предложен подход к адаптивному управлению локальными показателями Ляпунова. Предложен численный алгоритм определения спектра показателей Ляпунова по наблюдаемому зашумленному временному ряду. Подход протестирован на примере нелинейной решетки с известным спектром показателей Ляпунова.
Ключевые слова: адаптивное управление, показатели Ляпунова, оптимизация, моделирование.