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HYPERBOLIC BOUNDARY-VALUE PROBLEM OF MATHEMATICAL PHYSICS IN SEMIBOUNDED PIECEWISE-HOMOGENEOUS SPATIAL ENVIRONMENT

By means of method of integral transforms in combination with the method of main solutions (influence matrices and Green matrices) the exact analytical solution of algorithmic nature of hyperbolic boundary value problem of mathematical physics in semibounded piecewise-homogeneous spatial environment is constructed.

INTRODUCTION. It is known that the actual problems of thermophysics, thermodynamics, theory of elasticity, theory of electrical circuits, theory of vibrations lead to boundary value problems of mathematical physics not only in homogeneous domains if the coefficients of the equations are continuous but also in inhomogeneous and piece-homogeneous domains if the coefficients of the equations are piece-continuous or in particular piece-constant.

Some classes of such boundary value problems were considered in the papers of B. Boley, J. Weiner [1], V. Deineka, I. Sergienko, V. Skopetskiy [4, 13], Yu. Kolyano [5], Ya. Pidstryhach, V. Lomakin, Yu. Kolyano [12], G. Shilin [15], ets., in which there were investigated a number of important mathematical models of mechanics of continuum environment, mechanics of deformable solids, thermomechanics and so on. There are often used methods of numerical analysis, or method of reduction of problems in piecewise-homogeneous environments to the corresponding problems for differential equations with singular coefficients in the form of generalized functions (Dirac δ-function and its derivatives) in homogeneous environment exact solution of which is practically impossible to construct.

However for a rather wide class of problems in piecewise-homogeneous environments it is shown to be effective the method of hybrid integral transforms generated by hybrid differential operators if in each connected component of piecewise-homogeneous environment there are considered different differential operators or differential operators of the same kind but with different sets of coefficients [3, 7–9, 11]. This method makes it possible to construct in analytical form solutions of certain linear boundary value problems of mathematical physics in piecewise-homogeneous environments due to their integral images.

We propose in this paper constructed by means of method of integral and hybrid integral transforms exact analytical solution of hyperbolic boundary value problem in semibounded spatial environment that is described by the Cartesian coordinate system.

FORMULATION OF THE PROBLEM. Let's consider the problem of construction the solution which is bounded in the set $D_3=\left\{(t,x,y,z): t>0; (x,y)\in\Omega_2=(0;+\infty)\times(0;b); z\in I_n^+=\bigcup_{j=1}^{n+1} I_j=\bigcup_{j=1}^{n+1}(l_{j-1};l_j), l_0\ge 0, l_k<l_{k+1}, l_{n+1}=+\infty\right\}$ of separate system of partial differential equations of hyperbolic type [14]

$$\frac{\partial^2 u_j}{\partial t^2}-\left[a_{xj}^2\frac{\partial^2}{\partial x^2}+a_{yj}^2\frac{\partial^2}{\partial y^2}+a_{zj}^2\frac{\partial^2}{\partial z^2}\right]u_j+\chi_j^2 u_j=f_j(t,x,y,z);\quad z\in I_j;\quad j=\overline{1,n+1} \tag{1}$$

with initial conditions

$$u_j\Big|_{t=0}=g_j^1(x,y,z);\quad \frac{\partial u_j}{\partial t}\Big|_{t=0}=g_j^2(x,y,z);\quad z\in I_j;\quad j=\overline{1,n+1}; \tag{2}$$

boundary conditions

$$\left(-\frac{\partial}{\partial x}+p\right)u_j\Big|_{x=0}=\theta_j(t,y,z);\quad \frac{\partial^k u_j}{\partial x^k}\Big|_{x=+\infty}=0;\quad k=0,1;\quad j=\overline{1,n+1}; \tag{3}$$

$$\left(-\frac{\partial}{\partial y}+h_1\right)u_j\Big|_{y=0}=\omega_j^1(t,x,z);\quad \left(\frac{\partial}{\partial y}+h_2\right)u_j\Big|_{y=b}=\omega_j^2(t,x,z);\quad j=\overline{1,n+1}; \tag{4}$$

$$\left(\alpha_{11}^0\frac{\partial}{\partial z}+\beta_{11}^0\right)u_1\Big|_{z=l_0}=g_0(t,x,y);\quad \frac{\partial^k u_{n+1}}{\partial z^k}\Big|_{z=+\infty}=0;\quad k=0,1 \tag{5}$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^k\frac{\partial}{\partial z}+\beta_{j1}^k\right)u_k-\left(\alpha_{j2}^k\frac{\partial}{\partial z}+\beta_{j2}^k\right)u_{k+1}\right]\Big|_{z=l_k}=0;\quad j=1,2;\quad k=\overline{1,n}, \tag{6}$$

here a_{xj}, a_{yj}, a_{zj}, χ_j, p, h, k, α_{js}^k, β_{js}^k are some non-negative constants;

$\tilde{n}_{jk}=\alpha_{2j}^k\beta_{1j}^k-\alpha_{1j}^k\beta_{2j}^k\ne 0;\ c_{1k}c_{2k}>0;\ \alpha_{11}^0\le 0,\ \beta_{11}^0\ge 0;\ \left|\alpha_{11}^0\right|+\left|\beta_{11}^0\right|\ne 0;$

$f(t,x,y,z)=\{f_1(t,x,y,z), f_2(t,x,y,z),\dots, f_{n+1}(t,x,y,z)\}$;

$g^1(x,y,z)=\left\{g_1^1(x,y,z), g_2^1(x,y,z),\dots, g_{n+1}^1(x,y,z\right\}$;

$g^2(x,y,z)=\left\{g_1^2(x,y,z), g_2^2(x,y,z),\dots, g_{n+1}^2(x,y,z\right\}$;

$\theta(t,y,z)=\{\theta_1(t,y,z),\theta_2(t,y,z),\ldots,\theta_{n+1}(t,y,z)\}$;

$\omega^1(t,x,z)=\left\{\omega_1^1(t,x,z),\omega_2^1(t,x,z),\ldots,\omega_{n+1}^1(t,x,z\right\}$;

$\omega^2(t,x,z)=\left\{\omega_1^2(t,x,z),\omega_2^2(t,x,z),\ldots,\omega_{n+1}^2(t,x,z\right\}$;

$g_0(t,x,y)$ are given bounded continuous functions;

$u(t,x,y,z)=\{u_1(t,x,y,z),u_2(t,x,y,z),\ldots,u_{n+1}(t,x,y,z)\}$ is the unknown function.

Be noted that: 1) in the case of $\chi_j^2\equiv 0$ equations (1) are classic three-dimensional nonhomogeneous wave equations (oscillation equations) for orthotropic spatial environment; 2) in the case of $\alpha_{11}^k=0,\ \beta_{11}^k=1;\ \alpha_{12}^k=0,\ \beta_{12}^k=1;\ \alpha_{21}^k=E_1^k$, $\beta_{21}^k=0;\ \alpha_{22}^k=E_2^k,\ \beta_{22}^k=0$, here E_1^k, E_2^k are Young modulus, $k=\overline{1,n}$, conjugate conditions (6) coincide with the terms of the ideal mechanical contact.

Therefore, in these cases, the problem which is considered is a mathematical model of forced oscillatory processes in semibounded piecewise-homogeneous spatial environment $\Omega_3=\left\{(x,y,z):(x,y)\in\Omega_2;z\in I_n^+\right\}$.

THE MAIN PART. Let's suppose that the solution of hyperbolic initial boundary value problem of conjugation (1)–(6) exists and given and unknown functions satisfy the conditions of applicability of involved integral transforms [10, 11].

Let's apply the integral Fourier transform on the Cartesian semiaxis $(0;+\infty)$ relative to the variable x [10] to the problem (1)–(6):

$$F_{+x}\left[g(x)\right]=\int\limits_0^{+\infty}g(x)K_x(x,\sigma)dx\equiv\tilde g(\sigma), \tag{7}$$

$$F_{+x}^{-1}\left[\tilde g(\sigma)\right]=\int\limits_0^{+\infty}\tilde g(\sigma)K_x(x,\sigma)d\sigma\equiv g(x), \tag{8}$$

$$F_{+x}\left[\frac{d^2g}{dx^2}\right]=-\sigma^2\tilde g(\sigma)+K_x(0,\sigma)\left(-\frac{dg}{dx}+pg\right)\Bigg|_{x=0}, \tag{9}$$

here the kernel of transform is

$$K_x(x,\sigma)=\sqrt{\frac{2}{\pi}}\frac{\sigma\cos(\sigma x)+p\sin(\sigma x)}{\sqrt{\sigma^2+p^2}}.$$

The integral operator F_{+x} due to the rule (7) because of identity (9) for nitial boundary value problem (1)–(6) assigns to the problem of constructing of bounded solution on the set $D_3^{'}=\left\{(t,y,z);t>0;y\in(0;b);z\in I_n^+\right\}$ of separate system of differential equations

$$\frac{\partial^2\tilde u_j}{\partial t^2}-\left[a_{yj}^2\frac{\partial^2}{\partial y^2}+a_{zj}^2\frac{\partial^2}{\partial z^2}\right]\tilde u_j+\left(a_{xj}^2\sigma^2+\chi_j^2\right)\tilde u_j=\tilde F_j(t,\sigma,y,z);\quad z\in I_j;\quad j=\overline{1,n+1} \tag{10}$$

with initial conditions

$$\tilde u_j\big|_{t=0}=g_j^1(\sigma,y,z)\,,\ \frac{\partial\tilde u_j}{\partial t}\bigg|_{t=0}=\tilde g_j^2(\sigma,y,z);\ \ z\in I_j;\ \ j=\overline{1,n+1}; \tag{11}$$

boundary conditions

$$\left(-\frac{\partial}{\partial y}+h_1\right)\tilde u_j\bigg|_{y=0}=\omega_j^1(t,\sigma,z);\ \left(\frac{\partial}{\partial y}+h_2\right)\tilde u_j\bigg|_{y=b}=\omega_j^2(t,\sigma,z);\ \ z\in I_j;\ j=\overline{1,n+1} \tag{12}$$

$$\left(\alpha_{11}^0\frac{\partial}{\partial z}+\beta_{11}^0\right)\tilde u_1\bigg|_{z=l_0}=\tilde g_0(t,\sigma,y);\ \ \frac{\partial^k\tilde u_{n+1}}{\partial z^k}\bigg|_{z=+\infty}=0;\ \ k=0,1; \tag{13}$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^k\frac{\partial}{\partial z}+\beta_{j1}^k\right)\tilde u_k-\left(\alpha_{j2}^k\frac{\partial}{\partial z}+\beta_{j2}^k\right)\tilde u_{k+1}\right]\Bigg|_{z=l_k}=0;\ \ j=1,2;\ \ k=\overline{1,n}\,, \tag{14}$$

here

$$F_j(t,\sigma,y,z)=f_j(t,\sigma,y,z)+a_{xj}^2K_x(0,\sigma)\theta_j(t,y,z);\ j=\overline{1,n+1}.$$

Let's apply to the problem (10)-(14) a finite integral Fourier transform on Cartesian segment $\left[0;b\right]$ relative to the variable y [10]:

$$\Lambda_{yk}\left[g(y)\right]=\int\limits_0^b g(y)\nu_k(y)dy\equiv g_k\,, \tag{15}$$

$$\Lambda_{yk}^{-1}\left[g_k\right]=\sum_{k=1}^{\infty}g_k\frac{\nu_k(y)}{\left\|\nu_k\right\|^2}\equiv g(y), \tag{16}$$

$$\Lambda_{yk}\left[\frac{d^2g}{dy^2}\right]=-\gamma_k^2 g_k+\nu_k(0)\left(-\frac{dg}{dy}+h_1 g\right)\Bigg|_{y=0}+\nu_k(b)\left(\frac{dg}{dy}+h_2 g\right)\Bigg|_{y=b}, \quad (17)$$

here the kernel of transform is

$$\nu_k(y)=\frac{\gamma_k\cos(\gamma_k y)+h_1\sin(\gamma_k y)}{\sqrt{\gamma_k^2+h_1^2}},\quad \|\nu_k\|^2\equiv\int_0^b\nu_k^2(y)dy=\frac{b}{2}+\frac{(h_1+h_2)(\gamma_k^2+h_1h_2)}{2(\gamma_k^2+h_1^2)(\gamma_k^2+h_2^2)},$$

$\{\gamma_k\}_{k=1}^{\infty}$ is monotonically increasing sequence of real various positive roots of transcendental equation $ctg(\gamma b)=\dfrac{\gamma^2-h_1h_2}{\gamma(h_1+h_2)}$, which form the discrete spectrum.

The integral operator Λ_{yk} due to the rule (15) because of identity (17) for initial boundary value problem (10)–(14) assigns to the problem of constructing of bounded solution on the set $D_3^{'}=\{(t,y,z);t>0;y\in(0;b);z\in I_n^+\}$ of separate system of differential equations

$$\frac{\partial^2\tilde{u}_{jk}}{\partial t^2}-a_{zj}^2\frac{\partial^2\tilde{u}_{jk}}{\partial z^2}+\left(a_{xj}^2\sigma^2+a_{yj}^2\gamma_k^2+\chi_j^2\right)\tilde{u}_{jk}=\tilde{G}_{jk}(t,\sigma,z);\quad z\in I_j;\ j=\overline{1,n+1} \quad (18)$$

with initial conditions

$$\tilde{u}_{jk}\big|_{t=0}=g_{jk}^1(\sigma,z);\quad \frac{\partial\tilde{u}_{jk}}{\partial t}\bigg|_{t=0}=\tilde{g}_{jk}^2(\sigma,z);\quad z\in I_j;\ j=\overline{1,n+1}; \quad (19)$$

boundary conditions

$$\left(\alpha_{11}^0\frac{\partial}{\partial z}+\beta_{11}^0\right)\tilde{u}_{1k}\bigg|_{z=l_0}=\tilde{g}_{0k}(t,\sigma);\quad \frac{\partial^k\tilde{u}_{n+1,k}}{\partial z^s}\bigg|_{z=+\infty}=0;\ s=0,1; \quad (20)$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^s\frac{\partial}{\partial z}+\beta_{j1}^s\right)\tilde{u}_{sk}-\left(\alpha_{j2}^s\frac{\partial}{\partial z}+\beta_{j2}^s\right)\tilde{u}_{s+1,k}\right]\bigg|_{z=l_s}=0;\ s=\overline{1,n}, \quad (21)$$

here

$$\tilde{G}_{jk}(t,\sigma,z)=\tilde{F}_{jk}(t,\sigma,z)+a_{yj}^2\nu_k(0)\tilde{\omega}_j^1(t,\sigma,z)+a_{yj}^2\nu_k(b)\tilde{\omega}_j^2(t,\sigma,z);\quad j=\overline{1,n+1}.$$

Let's apply to the problem (18)–(21) the integral Fourier transform on the Cartesian semiaxis $(l_0;+\infty)$ whith n points of conjugation relative to the variable z [11]:

$$F_{n,+}[g(z)]=\int_{l_0}^{+\infty}g(z)V(z,\beta)\sigma(z)dz\equiv\tilde{g}(\beta), \quad (22)$$

$$F_{n,+}^{-1}[\tilde{g}(\beta)]=\frac{2}{\pi}\int_0^{+\infty}\tilde{g}(\beta)V(z,\beta)\Omega_n(\beta)d\beta\equiv g(z), \quad (23)$$

$$\begin{gathered}F_{n,+}\left[\sum_{j=1}^n a_{zj}^2\theta(z-l_{j-1})\theta(l_j-z)\frac{d^2g}{dz^2}+a_{z,n+1}^2\theta(z-l_n)\frac{d^2g}{dz^2}\right]=-\beta^2\tilde{g}(\beta)-\sigma_1a_{z1}^2(\alpha_{11}^0)^{-1}V_1(z,l_0)\times\\ \times\left(\alpha_{11}^0\frac{d}{dz}+\beta_{11}^0g\right)\bigg|_{z=l_0}-\sum_{j=1}^{n+1}\kappa_j^2\int_{l_{j-1}}^{l_j}g(z)V_j(z,\beta)\sigma_jdz.\end{gathered} \quad (24)$$

In formulas (22)-(24) there are values and functions:

$$V(z,\beta)=\sum_{k=1}^n V_k(z,\beta)\theta(z-l_{k-1})\theta(l_k-z)+V_{n+1}(z,\beta)\theta(z-l_n);\quad \sigma(z)=\sum_{k=1}^n\sigma_k\theta(z-l_{k-1})\theta(l_k-z)+\sigma_{n+1}\theta(z-l_n);$$

$$\Omega_n(\beta)=\frac{\beta}{b_{n+1}(\beta)\omega_n(\beta)};\quad V_m(z,\beta)=\prod_{j=m}^n c_{2j}a_{z,j+1}^{-1}b_{j+1}(\beta)G_m(z,\beta);\quad m=\overline{1,n};$$

$$V_{n+1}(z,\beta)=\omega_{n2}(\beta)\cos\left(\frac{b_{n+1}z}{a_{z,n+1}}\right)-\omega_{n1}(\beta)\sin\left(\frac{b_{n+1}z}{a_{z,n+1}}\right);\ \sigma_k=\prod_{j=k}^n\frac{c_{1j}\cdot a_{z,n+1}}{c_{2j}\cdot a_{zj}^2};\ \sigma_n=\frac{c_{1n}\cdot a_{z,n+1}}{c_{2n}\cdot a_{2n}};\ \sigma_{n+1}=\frac{1}{a_{z,n+1}};$$

$$G_k(z,\beta)=\omega_{k-1,2}(\beta)\cos\left(\frac{b_kz}{a_{zk}}\right)-\omega_{k-1,1}(\beta)\sin\left(\frac{b_kz}{a_{zk}}\right);\ k=\overline{1,n};\ b_j(\beta)=\left(\beta^2+k_j^2\right)^{1/2};\ j=\overline{1,n+1};$$

$$\omega_n(\beta)=\omega_{n1}^2(\beta)+\omega_{n2}^2(\beta);\quad \omega_{01}(q_1l_0)=-\nu_{11}^{01}(q_1l_0);\quad \omega_{02}(q_1l_0)=-\nu_{11}^{02}(q_1l_0);$$

$$\omega_{jm}(\beta)=\omega_{j-1,2}(\beta)\psi_{1m}^j(q_jl_j;q_{j+1}l_j)-\omega_{j-1,1}(\beta)\psi_{2m}^j(q_jl_j;q_{j+1}l_j);$$

$$\psi_{jm}^k(q_kl_k;q_{k+1}l_k)=\nu_{11}^{kj}(q_kl_k)\nu_{22}^{km}(q_{k+1}l_k)-\nu_{21}^{kj}(q_kl_k)\nu_{12}^{km}(q_{k+1}l_k);$$

$$v_{ij}^{k1}(q_s l_m) \equiv \left(\alpha_{ij}^k \frac{d}{dz} + \beta_{ij}^k\right)\cos(q_s z)\Big|_{z=l_m} = -\alpha_{ij}^k q_s \sin(q_s l_m) + \beta_{ij}^k \cos(q_s l_m);\ \ v_{ij}^{k2}(q_s l_m) \equiv \left(\alpha_{ij}^k \frac{d}{dz} + \beta_{ij}^k\right)\sin(q_s z)\Big|_{z=l_m} =$$

$$= \alpha_{ij}^k q_s \cos(q_s l_m) + \beta_{ij}^k \sin(q_s l_m);\quad m = 1,2;$$

$\theta(x)$ is the Heaviside step function [16].

Let's write the system of differential equations (18) and the initial conditions (19) in matrix form:

$$\begin{bmatrix} \left(\frac{\partial^2}{\partial t^2} - a_{z1}^2 \frac{\partial^2}{\partial z^2} + q_1^2(\sigma,\gamma_k)\right)\tilde{u}_{1k}(t,\sigma,z) \\ \left(\frac{\partial^2}{\partial t^2} - a_{z2}^2 \frac{\partial^2}{\partial z^2} + q_2^2(\sigma,\gamma_k)\right)\tilde{u}_{2k}(t,\sigma,z) \\ \dots\dots\dots \\ \left(\frac{\partial^2}{\partial t^2} - a_{z,n+1}^2 \frac{\partial^2}{\partial z^2} + q_{n+1}^2(\sigma,\gamma_k)\right)\tilde{u}_{n+1,k}(t,\sigma,z) \end{bmatrix} = \begin{bmatrix} \tilde{G}_{1k}(t,\sigma,z) \\ \tilde{G}_{2k}(t,\sigma,z) \\ \dots\dots \\ \tilde{G}_{n+1,k}(t,\sigma,z) \end{bmatrix}, \tag{25}$$

$$\begin{bmatrix} \tilde{u}_{1k}(t,\sigma,z) \\ \tilde{u}_{2k}(t,\sigma,z) \\ \dots\dots \\ \tilde{u}_{n+1,k}(t,\sigma,z) \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} \tilde{g}_{1k}^1(\sigma,z) \\ \tilde{g}_{2k}^1(\sigma,z) \\ \dots\dots \\ \tilde{g}_{n+1,k}^1(\sigma,z) \end{bmatrix};\ \ \frac{\partial}{\partial t}\begin{bmatrix} \tilde{u}_{1k}(t,\sigma,z) \\ \tilde{u}_{2k}(t,\sigma,z) \\ \dots\dots \\ \tilde{u}_{n+1,k}(t,\sigma,z) \end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix} \tilde{g}_{1k}^2(\sigma,z) \\ \tilde{g}_{2k}^2(\sigma,z) \\ \dots\dots \\ \tilde{g}_{n+1,k}^2(\sigma,z) \end{bmatrix}, \tag{26}$$

here $q_j^2(\sigma,\gamma_k) = a_{xj}^2\sigma^2 + a_{yj}^2\gamma_k^2 + \chi_j^2;\quad j = \overline{1,n+1}.$

The integral operator $F_{n,+}$ which operates by the formula (22) let's represent as an operator matrix-row

$$F_{n,+}[\cdots] = \left[\int_{l_0}^{l_1} \dots V_1(z,\beta)\sigma_1 dz \int_{l_1}^{l_2} \dots V_2(z,\beta)\sigma_2 dz \dots \int_{l_n}^{+\infty} \dots V_{n+1}(z,\beta)\sigma_{n+1} dz\right] \tag{27}$$

and apply to the problem (25), (26) due to matrices multiplication rule.

As a result of the identity (24), we get a Cauchy problem

$$\sum_{j=1}^{n+1}\left(\frac{d^2}{dt^2} + \beta^2 + q_j^2(\sigma,\gamma_k) + k_j^2\right)\tilde{\tilde{u}}_{jk}(t,\sigma,\beta) = \sum_{j=1}^{n+1}\tilde{\tilde{G}}_{jk}(t,\sigma,\beta) - \sigma_1 a_{z1}^2\left(\alpha_{11}^0\right)^{-1} V_1(l_0,\beta)\tilde{g}_{0k}(t,\sigma), \tag{28}$$

$$\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jk}\Big|_{t=0} = \sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jk}^1(\sigma,\beta);\ \ \frac{d}{dt}\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jk}\Big|_{t=0} = \sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jk}^1(\sigma,\beta), \tag{29}$$

here $\tilde{\tilde{u}}_{jk}(t,\sigma,\beta) = \int_{l_{j-1}}^{l_j} \tilde{u}_{jk}(t,\sigma,z)V_j(z,\beta)\sigma_j dz;\ j = \overline{1,n+1};\ \tilde{\tilde{G}}_{jk}(t,\sigma,\beta) = \int_{l_{j-1}}^{l_j} \tilde{G}_{jk}(t,\sigma,z)V_j(z,\beta)\sigma_j dz;\ j = \overline{1,n+1};$

$$\tilde{\tilde{g}}_{jk}^1(\sigma,\beta) = \int_{l_{j-1}}^{l_j} \tilde{g}_{jk}^1(\sigma,z)V_j(z,\beta)\sigma_j dz;\ j = \overline{1,n+1};\ \ \tilde{\tilde{g}}_{jk}^2(\sigma,\beta) = \int_{l_{j-1}}^{l_j} \tilde{g}_{jk}^2(\sigma,z)V_j(z,\beta)\sigma_j dz;\ \ j = \overline{1,n+1};\ \ l_{n+1} = +\infty.$$

Let's suppose, without reducing of generality that $\max\left\{q_1^2, q_2^2, ..., q_{n+1}^2\right\} = q_1^2$ and we put everywhere $k_j^2 = q_1^2 - q_j^2\ \left(j = \overline{1,n+1}\right)$.

Cauchy problem (28), (29) takes the form

$$\frac{d^2\tilde{\tilde{u}}_k}{dt^2} + \Delta^2(\sigma,\gamma_k,\beta)\tilde{\tilde{u}}_k = \tilde{\tilde{G}}_k(t,\sigma,\beta) - \sigma_1 a_{z1}^2(\alpha_{11}^0)^{-1} V_1(l_0,\beta)\tilde{g}_{0k}(t,\sigma), \tag{30}$$

$$\tilde{\tilde{u}}_k\Big|_{t=0} = \tilde{\tilde{g}}_k^1(\sigma,\beta),\ \ \frac{d\tilde{\tilde{u}}_k}{dt}\Big|_{t=0} = \tilde{\tilde{g}}_k^2(\sigma,\beta), \tag{31}$$

here
$$\tilde{\tilde{u}}_k(t,\sigma,\beta) = \sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jk}(t,\sigma,\beta);\qquad \Delta^2(\sigma,\gamma_k,\beta) = \beta^2 + a_{x1}^2\sigma^2 + a_{y1}^2\gamma_k^2 + \chi_1^2;$$

$$\tilde{\tilde{G}}_k(t,\sigma,\beta) = \sum_{j=1}^{n+1}\tilde{\tilde{G}}_{jk}(t,\sigma,\beta);\qquad \tilde{\tilde{g}}_k^1(\sigma,\beta) = \sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jk}^1(\sigma,\beta);\qquad \tilde{\tilde{g}}_k^2(\sigma,\beta) = \sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jk}^2(\sigma,\beta).$$

Directly is checked that the only solution to the problem (30), (31) is the function

$$\tilde{\tilde{u}}_k(t,\sigma,\beta) = \frac{\sin(\Delta(\sigma,\gamma_k,\beta)t)}{\Delta(\sigma,\gamma_k,\beta)}\tilde{\tilde{g}}_k^2(\sigma,\beta) + \frac{d}{dt}\frac{\sin(\Delta(\sigma,\gamma_k,\beta)t)}{\Delta(\sigma,\gamma_k,\beta)}\tilde{\tilde{g}}_k^1(\sigma,\beta) + \int_0^t \frac{\sin(\Delta(\sigma,\gamma_k,\beta)(t-\tau))}{\Delta(\sigma,\gamma_k,\beta)}\times \tag{32}$$

$$\times\left[\tilde{\tilde{G}}_k(\tau,\sigma,\beta) - \sigma_1 a_{z1}^2(\alpha_{11}^0)^{-1} V_1(l_0,\beta)\tilde{g}_{0k}(\tau,\sigma)\right]d\tau.$$

A superposition of operators $F_{n,+}$ and $F_{n,+}^{-1}$ is the identity operator so we represent the operator $F_{n,+}^{-1}$ as the operator matrix-column

$$F_{n,+}^{-1}[\ldots]=\frac{2}{\pi}\begin{bmatrix}\int_0^{+\infty}\ldots V_1(z,\beta)\Omega_n(\beta)d\beta\\ \int_0^{+\infty}\ldots V_2(z,\beta)\Omega_n(\beta)d\beta\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \int_0^{+\infty}\ldots V_{n+1}(z,\beta)\Omega_n(\beta)d\beta\end{bmatrix} \quad (33)$$

Let's apply operator matrix-column (33) to matrix-element $\left[\tilde{\tilde{u}}_k(t,\sigma,\beta)\right]$ due to matrices multiplication rule, if the function $\tilde{\tilde{u}}_k(t,\sigma,\beta)$ is defined by formula (32). We get the only analytical solution of initial-boundary value problem (18)–(21):

$$\tilde{u}_{jk}(t,\sigma,z)=\frac{2}{\pi}\int_0^{+\infty}\left[\frac{\sin(\Delta(\sigma,\gamma_k,\beta)t)}{\Delta(\sigma,\gamma_k,\beta)}\tilde{\tilde{g}}_k^2(\sigma,\beta)+\frac{\partial}{\partial t}\frac{\sin(\Delta(\sigma,\gamma_k,\beta)t)}{\Delta(\sigma,\gamma_k,\beta)}\tilde{\tilde{g}}_k^1(\sigma,\beta)\right]V_j(z,\beta)\Omega_n(\beta)d\beta+ \quad (34)$$

$$+\int_0^t\frac{2}{\pi}\int_0^{+\infty}\frac{\sin(\Delta(\sigma,\gamma_k,\beta)(t-\tau))}{\Delta(\sigma,\gamma_k,\beta)}\left[\tilde{\tilde{G}}_k(\tau,\sigma,\beta)-\sigma_1a_{z1}^2(\alpha_{11}^0)^{-1}V_1(l_0;\beta)\tilde{g}_{0k}(\tau,\sigma)\right]V_j(z,\beta)\Omega_n(\beta)d\beta d\tau;\quad j=\overline{1,n+1}.$$

To the functions $\tilde{u}_{jk}(t,\sigma,z)$ which are defined by formulas (34), we apply the inverse operators Λ_{yk}^{-1} due to the rule (16) and F_{+x}^{-1} due to the rule (8). As a result of simple transformations, we get functions

$$u_j(t,x,y,z)=\sum_{k=1}^{n+1}\int_0^t\int_0^{+\infty}\int_0^b\int_{l_{k-1}}^{l_k}E_{jk}(t-\tau,x,\xi,y,\eta,z,\zeta)f_k(\tau,\xi,\eta,\zeta)\sigma_k d\xi d\eta d\zeta d\tau+$$

$$+\frac{\partial}{\partial t}\sum_{k=1}^{n+1}\int_0^{+\infty}\int_0^b\int_{l_{k-1}}^{l_k}E_{jk}(t,x,\xi,y,\eta,z,\zeta)g_k^1(\xi,\eta,\zeta)\sigma_k d\xi d\eta d\zeta+$$

$$+\sum_{k=1}^{n+1}\int_0^{+\infty}\int_0^b\int_{l_{k-1}}^{l_k}E_{jk}(t,x,\xi,y,\eta,z,\zeta)g_k^2(\xi,\eta,\zeta)\sigma_k d\xi d\eta d\zeta+\int_0^t\int_0^{+\infty}\int_0^b W_j(t-\tau,x,\xi,y,\eta,z)g_0(\tau,\xi,\eta)d\xi d\eta d\tau+$$

$$+a_{xj}^2\sum_{k=1}^{n+1}\int_0^t\int_0^b\int_{l_{k-1}}^{l_k}W_{xjk}(t-\tau,x,y,\eta,z,\zeta)\theta_k(\tau,\eta,\zeta)\sigma_k d\eta d\zeta d\tau+a_{yj}^2\sum_{k=1}^{n+1}\int_0^t\int_0^{+\infty}\int_{l_{k-1}}^{l_k}[W_{xjk}^1(t-\tau,x,\xi,y,z,\zeta)\omega_k^1(\tau,\xi,\zeta)+$$

$$+W_{yjk}^2(t-\tau,x,\xi,y,z,\zeta)\omega_k^2(\tau,\xi,\zeta)]\sigma_k d\xi d\zeta d\tau;\quad j=\overline{1,n+1}, \quad (35)$$

which define the unique solution of hyperbolic initial boundary value problem of conjugation (1)–(6).

In formulas (35) the components

$$E_{jk}(t,x,\xi,y,\eta,z,\zeta)=\frac{2}{\pi}\sum_{r=1}^{\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{\sin(\Delta(\sigma,\gamma_r,\beta)t)}{\Delta(\sigma,\gamma_r,\beta)}V_j(z,\beta)V_k(\zeta,\beta)\Omega_n(\beta)K_x(x,\sigma)K_x(\xi,\sigma)\frac{v_r(y)v_r(\eta)}{\|v_r\|^2}d\sigma d\beta;\quad j,k=\overline{1,n+1}$$

of influence matrix (influence function) and components

$$W_j(t,x,\xi,y,\eta,z)=-\sigma_1a_{z1}^2\left(\alpha_{11}^0\right)^{-1}E_{j1}(t,x,\xi,y,\eta,z,l_0),$$

$$W_{xjk}(t,x,y,\eta,z,\zeta)=E_{j1}(t,x,0,y,\eta,z,\zeta),$$

$$W_{xjk}^1(t,x,\xi,y,z,\zeta)=E_{jk}(t,x,\xi,y,0,z,\zeta),$$

$$W_{xjk}^2(t,x,\xi,y,z,\zeta)=E_{jk}(t,x,\xi,y,b,z,\zeta)$$

of Green matrices take part of the problem under consideration.

Using the properties of influence functions $E_{jk}(t,x,\xi,y,\eta,z,\zeta)$ and Green functions $W_j(t,x,\xi,y,\eta,z)$, $W_{xjk}(t,x,y,\eta,z,\zeta)$, $W_{yjk}^s(t,x,\xi,y,z,\zeta)$, $(s=1,2)$ can be verified directly that functions $u_j(t,x,y,z)$ defined by formulas (35) satisfy the equation (1), the initial conditions (2), the boundary conditions (3), (4), (5) and conjugate conditions (6) in terms of the theory of generalized functions [16].

The uniqueness of the solution (35) follows from its structure (integral image) and uniqueness of the main solutions (influence matrices and Green matrices) of initial boundary value problem of conjugation (1)–(6).

By methods from [2, 6] it can be proved that under appropriate conditions of initial data of problem, formulas (35) define a limited classical solution of the considered hyperbolic initial boundary value problem of conjugation.

Remark 1. In the case of $a_{xj}^2=a_{yj}^2=a_{zj}^2\equiv a_j^2>0$ formulas (35) define the structure of solution of hyperbolic boundary value problem (1)–(6) in isotropic $(n+1)$-layer semibounded spatial environment.

Remark 2. Parameters $p, h_j \, (j = 1, 2)$ make it possible to allocate from formulas (35) solutions of initial boundary value problems (1)–(6) in the case of setting on the surfaces $x = 0; y = 0, y = b$ the boundary conditions of the 1st, 2nd and 3rd kind and their possible combinations.

Remark 3. Parameters $\alpha_{11}^0, \beta_{11}^0$ make it possible to allocate from formulas (35) solutions of boundary value problems in the case of setting on the surface $z = l_0$ the boundary condition of the 1st kind $(\alpha_{11}^0 = 0, \beta_{11}^0 = 1)$, 2nd kind $(\alpha_{11}^0 = -1, \beta_{11}^0 = 0)$ and 3rd kind $(\alpha_{11}^0 = -1, \beta_{11}^0 \equiv h > 0)$.

Remark 4. The analysis of the solution (35) according to the analytical expression of functions $f_1(t, x, y, z)$, $g_j^1(x, y, z)$, $g_j^2(x, y, z)$, $g_0(t, x, y)$, $\theta_1(t, y, z)$, $\omega_j^1(t, x, z)$, $\omega_j^2(t, x, z)$, $j = \overline{1, n+1}$ is carried out directly from the general structures.

CONCLUSIONS. By means of method of integral and hybrid integral transforms of Fourier type in combination with method of main solutions (influence matrices and Green matrices) the integral image of exact analytical solution of hyperbolic boundary value problem of mathematical physics at semibounded piecewise-homogeneous spatial environment that is described by Cartesian coordinate system is obtained. The resulting solution is of algorithmic nature, continuously depends on the parameters and data of the problem and can be used in further theoretical studies and practice of engineering calculations of real processes that are modeled by hyperbolic boundary-value problems of mathematical physics of inhomogeneous environments.

REFERENCES

1. Боли Б., Уэйнер Дж. Теория температурных напряжений. – М.: Мир, 1964.
2. Гельфанд И. М., Шилов Г. Е. Некоторые вопросы теории дифференциальных уравнений. – М.: Физматгиз, 1958.
3. Громик А. П., Конет І. М., Ленюк М. П. Температурні поля в кусково-однорідних просторових середовищах. – Кам'янець-Подільський: Абетка-Світ, 2011.
4. Дейнека В. С., Сергиенко И. В., Скопецкий В. В. Модели и методы решения задач с условиями сопряжения. – К.: Наук. думка, 1998.
5. Коляно Ю. М. Методы теплопроводности и термоупругости неоднородного тела. – К.: Наук. думка, 1992.
6. Конет І. М. Інтегральні зображення розв'язків крайових і мішаних задач для диференціальних рівнянь з частинними похідними в кусково-однорідних середовищах: автореф. дис. на здобуття наук. ступ. докт. фіз.-мат. наук: спец. 01.01.02 «Диференціальні рівняння». – К.: КНУ ім. Т.Шевченка, 2008.
7. Конет І. М. Стаціонарні та нестаціонарні температурні поля в ортотропних сферичних областях. – К.: Ін-т математики НАН України, 1998.
8. Конет І. М., Ленюк М. П. Стаціонарні та нестаціонарні температурні поля в циліндрично-кругових областях. – Чернівці: Прут, 2001.
9. Конет І. М., Ленюк М. П. Температурні поля в кусково-однорідних циліндричних областях. – Чернівці: Прут. 2004.
10. Ленюк М. П. Интегральные преобразования с разделенными переменными (Фурье, Ханкеля). – К., 1983. – 60 с. – (Препр./АН УССР. Ин-т математики; 83.4).
11. Ленюк М. П. Температурні поля в плоских кусково-однорідних ортотропних областях. – К.: Ін-т математики НАН України, 1997.
12. Подстригач Я. С.. Ломакин В. А., Коляно Ю. М. Термоупругость тел неоднородной структуры. – М.: Наука, 1984.
13. Сергиенко И. В., Скопецкий В. В., Дейнека В. С. Математическое моделирование и исследование процессов в неоднородных средах. – К.: Наук. думка, 1991.
14. Тихонов А. Н., Самарский А. А. Уравнения математической физики. – М.: Наука, 1972.
15. Шилин Г. Ф. Инженерные алгоритмы решения стационарных задач теплопроводности в составных телах. – Иркутск: Изд-во ИГУ, 1983.
16. Шилов Г. Е. Математический анализ. Второй специальный курс. – М.: Наука. 1965.

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ГИПЕРБОЛИЧЕСКАЯ КРАЕВАЯ ЗАДАЧА МАТЕМАТИЧЕСКОЙ ФИЗИКИ В ПОЛУОГРАНИЧЕННОЙ КУСОЧНО-ОДНОРОДНОЙ ПРОСТРАНСТВЕННОЙ СРЕДЕ

Методом интегральных преобразований в сочетании с методом главных решений (матриц влияния и матриц Грина) построено точное аналитическое решение алгоритмического характера гиперболической краевой задачи математической физики в полуограниченной кусочно-однородной пространственной среде.

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ГІПЕРБОЛІЧНА КРАЙОВА ЗАДАЧА МАТЕМАТИЧНОЇ ФІЗИКИ В НАПІВОБМЕЖЕНОМУ КУСКОВО-ОДНОРІДНОМУ ПРОСТОРОВОМУ СЕРЕДОВИЩІ

Методом інтегральних перетворень у поєднанні з методом головних розв'язків (матриць впливу та матриць Гріна) побудовано точний аналітичний розв'язок алгоритмічного характеру гіперболічної крайової задачі математичної фізики в напівобмеженому кусково-однорідному просторовому середовищі.