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Відстань дробового броунівського руху до підпросторів гауссівських мартингалів

У роботі розглянуто наближення дробового броунівського руху гауссівськими мартингалами в інтегральній нормі при $H \in (0,1)$. Розглянуто наближення дробових броунівських рухів з ядрами Мандельброта–ван Несса та Молчана при всіх $H \in (0,1)$ та сталих підінтегральних функціях. Доведено, що при степеневих підінтегральних функціях наближення для ДБР з ядром Молчана краще ніж з ядром Мандельброта–ван Несса.

Ключові слова: дробовий броунівський рух, вінерівський процес, інтегральна норма, ядро Мандельброта–ван Несса, ядро Молчана.

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Distance of fractional Brownian motion to the subspaces of Gaussian martingales

The paper investigates approximation of fractional Brownian motion by Gaussian martingales in the integral norm for all $H \in (0,1)$. Also we consider the approximation of a fractional Brownian motion with Mandelbrot–van Ness and Molchan kernels and constants integrands. It is proved that approximation of the fBm with Molchan kernel is better than of the fBm with Mandelbrot–van Ness kernel whenever integrand is a power function.

Key Words: fractional Brownian motion, Wiener process, integral norm, kernel of Mandelbrot–van Ness, kernel of Molchan.

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1 Introduction

In the paper we continue the investigation of approximation of a fractional Brownian motion by stochastic processes from different classes.

Fractional Brownian motion with Hurst index $H \in \left(\frac{1}{2}, 1\right)$ is a random process with long-term dependence. It is well known that a fractional Brownian motion is neither semimartingale nor a Markov process unless $H = \frac{1}{2}$. And also it is neither martingale nor a process of bounded variation. Semimartingale processes are good models for many problems, from statistics to finance. Therefore, stochastic analysis of semimartingale processes is deeply developed.

However, well studied semimartingale theory demonstrates insufficiency to describe the processes that occur in telecommunication connections, the processes of assets prices with long-term dependence and many others. In this connection, wide range of potential applications of fractional Brownian motion creates an interesting object of study. Fractional Brownian motion is a generalization of ordinary Brownian motion, and is used to simulate natural phenomena, technical and economic phenomena such as changes in weather and climate fluctuations, fluctuations in financial markets, hydromechanical and meteorological processes.

That is why a natural questions arises: is it possible to approximate a fractional Brownian motion in a certain metric by a Markov process, martingale, semimartingale or a process of bounded variation? As for a process of bounded variation and semimartingale, then answer is positive and corresponding results are presented in [1], [2] and [12]. The papers [3, 4, 5, 6, 9, 11] study approximation of a fractional Brownian motion with Gaussian martingales. General questions conc fractional Brownian motion are presented in [8].

In the present paper we continue to consider the approximation of a fractional Brownian motion by Gaussian martingales. In particular, we consider the approximation of a fractional Brownian motion with Mandelbrot–van Ness kernel for all

$H \in (0,1)$, approximation of a fractional Brownian motion with Molchan kernel for $H \in \left(0, \frac{1}{2}\right)$ and approximation of a fractional Brownian motion by Gaussian martingales in the integral norm for all $H \in (0,1)$.

2 Representation of fractional Brownian motion

(a) Representation of fBm via the Wiener process on a finite interval. The (one-sided) fractional Brownian motion (fBm) with Hurst index $H \in (0,\ 1)$ is a Gaussian process $\{B_t^H,\ t \geqslant 0\}$ with zero mean $\mathrm{E}B_t^H = 0$ and covariation $\mathrm{E}B_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, such that $B_0^H = 0$. It is shown in [10] that the fBm $\{B_t^H,\ t \in [0,\ T]\}$ can be represented as ($\Gamma(x)$, $x > 0$ is Gamma function)

$$B_t^H = \int_0^t K(t,s) dW_s, \tag{1}$$

where $\{W_t,\ t \in [0,\ T]\}$ is a Wiener process,

$$K(t,s) = c_H\Big(t^{H-1/2} s^{1/2-H} (t-s)^{H-1/2} - (H-1/2) s^{1/2-H} \int_s^t u^{H-3/2} (u-s)^{H-1/2} du\Big), \tag{2}$$

and

$$c_H = \Big(\frac{2H \cdot \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \cdot \Gamma(2-2H)}\Big)^{1/2}. \tag{3}$$

Let $H \in \left(\frac{1}{2}, 1\right)$. In this case the kernel $K(t,s)$ can be simplified to $K(t,s) = c_H\left(H - \frac{1}{2}\right) \times s^{1/2-H} \int_s^t u^{H-1/2} (u-s)^{H-3/2} du$.

(b) Mandelbrot–van Ness representation of fBm. The (two-sided) fractional Brownian motion (fBm) with Hurst index $H \in (0,\ 1)$ is a Gaussian process $\{B_t^H,\ t \in \mathbb{R}\}$ with zero mean $\mathrm{E}B_t^H = 0$ and covariation $\mathrm{E}B_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, such that $B_0^H = 0$. It is shown in [7] that the fBm $\{B_t^H,\ t \in [0,\ T]\}$ can be represented as

$$B_t^H = \int_{-\infty}^t K_H(t,s) dW_s, \tag{4}$$

where $\{W_t,\ t \in [0,\ T]\}$ is a Wiener process,

$$K_H(t,s) = c_H^{(1)}((t-u)_+^{\alpha} - (-u_+^{\alpha})),$$

with $\alpha = H - \frac{1}{2}$ and

$$c_H^{(1)} = \frac{\Big(2H \sin \pi H \cdot \Gamma(2H)\Big)^{1/2}}{\Gamma(H + \frac{1}{2})}. \tag{5}$$

Remark 1. *Constants c_H and $c_H^{(1)}$ from* (3) *and* (5) *are equal.*

Proof. We need to prove the following equality.

$$\frac{2H \cdot \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \cdot \Gamma(2-2H)} = \frac{2H \sin \pi H \cdot \Gamma(2H)}{\Gamma^2(H + \frac{1}{2})}.$$

By identical transformation of both side we get

$$\frac{\Gamma(\frac{3}{2} - H)}{\Gamma(2-2H)} = \frac{\sin \pi H \cdot \Gamma(2H)}{\Gamma(H + \frac{1}{2})},$$

or

$$\frac{(\frac{1}{2} - H)\Gamma(\frac{1}{2} - H)}{(1-2H)\Gamma(1-2H)} = \frac{\sin \pi H \cdot \Gamma(2H)}{\Gamma(H + \frac{1}{2})},$$

$$\frac{\pi}{\sin(\frac{1}{2} - H)\pi} = \frac{2 \sin \pi H \cdot \pi}{\sin 2\pi H},$$

which is reduced to the evident identity

$$\frac{1}{\cos \pi H} = \frac{1}{\cos \pi H}.$$

□

3 Approximation of a fractional Brownian motion by Gaussian martingales in the integral norm for $H \in (0,1)$

At first we approximate fBm with the help of Gaussian martingales in the quadratic integral norm.

Theorem 3.1. *Consider the value*

$$\inf_{a \in L_2[0,1]} \int_0^1 \Big(\int_0^t (K(t,s) - a(s))^2 \, ds\Big) dt. \tag{6}$$

This infimum is achieved on the function

$$a_0(s) = \frac{\int_s^1 K(t,s) dt}{1-s}. \tag{7}$$

Function $a_0 \in L_2[0,1]$.

Proof.

$$\inf_{a\in L_2[0,1]}\int_0^1\Big(\int_0^t (K(t,s)-a(s))^2\,ds\Big)dt =$$

$$=\inf_{a\in L_2[0,1]}\int_0^1 \mathrm{E}\Big(B_t^H-\int_0^t a(s)dW_s\Big)^2 dt.$$

$$\int_0^1\int_0^t (K(t,s)-a(s))^2\,dsdt =$$

$$=\int_0^1\int_0^t K^2(t,s)dsdt-2\int_0^1\int_0^t K(t,s)a(s)dsdt+$$

$$+\int_0^1\int_0^t a^2(s)dsdt =$$

$$=\int_0^1\int_0^t K^2(t,s)dsdt-2\int_0^1 a(s)\int_s^1 K(t,s)dtds+$$

$$+\int_0^1 a^2(s)(1-s)ds.$$

Minimum is achieved for

$$a_0(s)=\frac{\int_s^1 K(t,s)dt}{1-s},$$

whenever $\frac{\int_s^1 K(t,s)dt}{1-s}\in L_2[0,1]$.

Check now that $a_0\in L_2[0,1]$. Let $C_\alpha := \alpha c_H$.

a) For $H>\frac{1}{2}$ we have that $0<\alpha=H-\frac{1}{2}<\frac{1}{2}$. Taking into account that $s\leqslant u\leqslant 1$, $u^\alpha\leqslant 1$, we get

$$a_0(s)=\frac{C_\alpha s^{-\alpha}\int_s^1(\int_s^t u^\alpha(u-s)^{\alpha-1}du)dt}{1-s}=$$

$$=\frac{C_\alpha s^{-\alpha}\int_s^1 u^\alpha(u-s)^{\alpha-1}(1-u)du}{1-s}\leqslant$$

$$\leqslant C_\alpha s^{-\alpha}\int_s^1 u^\alpha(u-s)^{\alpha-1}du\leqslant$$

$$\leqslant\frac{C_\alpha s^{-\alpha}(1-s)^\alpha}{\alpha}=\frac{C_\alpha}{\alpha}s^{-\alpha}(1-s)^\alpha\in L_2[0,1]$$

b) For $H<\frac{1}{2}$ we have that $-\frac{1}{2}<\alpha=H-\frac{1}{2}<0$ and

$$a_0(s)=\frac{C_\alpha}{\alpha(1-s)}\int_s^1\Big(t^\alpha s^{-\alpha}(t-s)^\alpha-$$

$$-\alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^\alpha du\Big)dt.$$

Taking into account that $t^\alpha\leqslant s^\alpha$, we obtain

$$|a_0(s)|\leqslant\frac{C_\alpha s^{-\alpha}}{\alpha(1-s)}\int_s^1 t^\alpha(t-s)^\alpha dt+$$

$$+\frac{C_\alpha s^{-\alpha}}{1-s}\int_s^1\int_s^t u^{\alpha-1}(u-s)^\alpha dudt\leqslant$$

$$\leqslant\frac{C_\alpha}{\alpha(1-s)}\int_s^1(t-s)^\alpha dt+$$

$$+\frac{C_\alpha s^{-\alpha}}{1-s}\int_s^1 u^{\alpha-1}(u-s)^\alpha(1-u)du\leqslant$$

$$\leqslant\frac{C_\alpha(1-s)^{\alpha+1}}{\alpha(1+\alpha)}+$$

$$+\frac{C_\alpha s^{-\alpha}}{1-s}s^{2\alpha}\int_1^{1/s} x^{\alpha-1}(x-1)^\alpha dx\cdot(1-s)=$$

$$=\frac{C_\alpha(1-s)^{\alpha+1}}{\alpha(1+\alpha)}+$$

$$+C_\alpha s^{-\alpha}s^{2\alpha}\int_1^{1/s} x^{\alpha-1}(x-1)^\alpha dx,$$

and integral $\int_1^\infty x^{\alpha-1}(x-1)^\alpha dx$ converges. □

4 Approximation of a fractional Brownian motion with Mandelbrot–van Ness kernel for $H \in (0,1)$

Now we establish the upper bound for c_H.

Lemma 1. *For* $0 < H < 1$ *the following inequality holds*

$$c_H^2 \leqslant 2H. \tag{8}$$

Proof. Note that $\ln\Gamma(x)$ is convex function for $x \in (0,+\infty)$. Therefore, for any $x > 0$, $y > 0$ Jensen inequality holds:

$$\ln\Gamma\Big(\frac{x+y}{2}\Big) \leqslant \frac{1}{2}\ln\Gamma(x) + \frac{1}{2}\ln\Gamma(y).$$

Transforming identically, we get

$$e^{2\ln\Gamma\left(\frac{x+y}{2}\right)} \leqslant e^{\ln\Gamma(x)} \cdot e^{\ln\Gamma(y)},$$

$$\Gamma^2\Big(\frac{x+y}{2}\Big) \leqslant \Gamma(x)\cdot\Gamma(y).$$

For $x = 2-2H$, $y = 1$ we obtain

$$\Gamma^2\Big(\frac{3}{2} - H\Big) \leqslant \Gamma(2-2H)\cdot\Gamma(1). \tag{9}$$

For $x = \frac{3}{2} - H$, $y = H + \frac{1}{2}$ we obtain

$$\Gamma^2(1) \leqslant \Gamma\Big(\frac{3}{2} - H\Big)\cdot\Gamma\Big(H + \frac{1}{2}\Big). \tag{10}$$

Multiplying left- and right-hand sides of inequalities (9) and (10) we obtain that

$$\Gamma^2\Big(\frac{3}{2} - H\Big) \leqslant \Gamma(2-2H)\cdot\Gamma\Big(\frac{3}{2} - H\Big)\cdot\Gamma\Big(H + \frac{1}{2}\Big),$$

whence

$$\Gamma\Big(\frac{3}{2} - H\Big) \leqslant \Gamma(2-2H)\cdot\Gamma\Big(H + \frac{1}{2}\Big).$$

Since

$$c_H^2 = \frac{2H\cdot\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\cdot\Gamma(2-2H)},$$

we get the claimed inequality. Note that equality in (8) is achieved when $H = \frac{1}{2}$. □

Consider $\mathrm{E}(B_t^H - M_t)^2$, $0 \leqslant t \leqslant 1$, where B_t^H is fractional Brownian motion admitting the representation (4) with Hurst index $0 < H < 1$, M_t is square-integrable martingale of the form $M_t = \int_0^t a(s)dW_s$. Suppose that the condition $\int_0^1 a^2(s)ds < \infty$ holds, i.e. $a \in L_2[0,1]$. Let's find $\min\limits_{a\in L_2[0,1]} \max\limits_{0\leqslant t\leqslant 1} \mathrm{E}(B_t^H - M_t)^2$ in some partial cases.

Consider the simplest case. Let the function $a(s) = a$ be a constant.

Theorem 4.1. *The following equality holds*

$$\min_a \max_{0\leqslant t\leqslant 1} \mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 = 1 - \frac{c_H^2}{(\alpha+1)^2} \tag{11}$$

and the value of $a_{\min}$ *that supplies the minimal value of* (11), *equals* $a_{\min} = \frac{c_H}{\alpha+1}$.

Proof. Consider

$$\mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 = \mathrm{E}(B_t^H)^2 + \mathrm{E}\Big(\int_0^t a(s)dW_s\Big)^2 - 2\mathrm{E}B_t^H\int_0^t a(s)dW_s = t^{2H} + \int_0^t a^2(s)ds - 2\int_0^t K_H(t,s)a(s)ds.$$

Assuming that $a(s) = a$ we get

$$\mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 = t^{2H} + a^2t - c_H 2a\int_0^t ((t-u)_+^\alpha - (-u_+^\alpha))du.$$

Evidently,

$$\int_0^t ((t-u)_+^\alpha - (-u_+^\alpha))du = \frac{t^{\alpha+1}}{\alpha+1},$$

whence

$$\mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 = t^{2H} + a^2t - 2ac_H\frac{t^{\alpha+1}}{\alpha+1} =: f(t).$$

Differentiating function f with respect to t, we get

$$(2\alpha+1)t^{2\alpha} - 2ac_H t^{\alpha} + a^2 = 0$$

Changing the variable $t^{2\alpha} =: y$, we obtain a quadratic equation

$$(2\alpha+1)y^2 - 2ac_H y + a^2 = 0. \tag{12}$$

The discriminant of the quadratic equation (12) equals

$$D = 4a^2c_H^2 - 4a^2(2\alpha+1) = \\ = (2a)^2(c_H^2 - (2\alpha+1)).$$

According to Lemma 1, this discriminant is negative.

Therefore,

$$\max_{0\leqslant t\leqslant 1} f(t) = f(1) =$$

$$= 1 + a^2 - \frac{2ac_H}{\alpha+1}. \tag{13}$$

The right-hand side of (13) achieves its minimal value with

$$a_{\min} = \frac{c_H}{\alpha+1}.$$

Therefore, $\min_a \max_{0\leqslant t\leqslant 1} \mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 = \frac{c_H^2}{(\alpha+1)^2} - \frac{2c_H^2}{(\alpha+1)^2} + 1 = 1 - \frac{c_H^2}{(\alpha+1)^2}$.

Consider more general case that corresponds to power functions with a constant exponent β, i.e., now $a(t) = at^{\beta}$, $a > 0$.

5 Approximation of a fractional Brownian motion with Molchan kernel for $H \in (0, \frac{1}{2})$

The following lemma can be proved by direct calculations.

Lemma 2. *Let $f \in C^3[-2,1]$, then*

$$-f(-2)+3f(-1)-3f(0)+f(1) = \int_{-2}^{1} f'''(t)g(t)\,dt,$$

where $g(t) = \begin{cases} 0{,}5(t+2)^2, & -2\leqslant t\leqslant -1,\\ -t^2 - t + 0{,}5, & -1\leqslant t\leqslant 0,\\ 0{,}5(1-t)^2, & 0\leqslant t\leqslant 1.\end{cases}$

Note that $0\leqslant g(t)\leqslant 0{,}75$ for $t\in[-2,1]$.

Using this result, we establish the following bounds for c_H.

Lemma 3. *We have the inequalities*

$$c_H^2\left(\frac{\pi\alpha}{\sin(\alpha\pi)}\right)^2 > 2H \quad \text{for } 0 < H < \frac{1}{2},$$

$$c_H^2\left(\frac{\pi\alpha}{\sin(\alpha\pi)}\right)^2 < 2H \quad \text{for } \frac{1}{2} < H < 1. \tag{14}$$

Proof. Remind that

$$c_H = \sqrt{\frac{2H\Gamma(1{,}5-H)}{\Gamma(H+0{,}5)\Gamma(2-2H)}}, \qquad \alpha = H - \frac{1}{2}.$$

Transform the left-hand side of the inequality (14). From the classical formula

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$$

we immediately get that

$$\Gamma(1+\alpha)\Gamma(1-\alpha) = \frac{\pi\alpha}{\sin(\pi\alpha)}.$$

Therefore

$$c_H^2\left(\frac{\pi\alpha}{\sin(\alpha\pi)}\right)^2 = \frac{2H\Gamma(1-\alpha)^3\Gamma(1+\alpha)}{\Gamma(1-2\alpha)}.$$

So we need to prove the inequalities

$$\frac{\Gamma(1-\alpha)^3\Gamma(1+\alpha)}{\Gamma(1-2\alpha)} > 1 \qquad \text{for } 0 < H < \frac{1}{2},$$

$$\text{and } \frac{\Gamma(1-\alpha)^3\Gamma(1+\alpha)}{\Gamma(1-2\alpha)} < 1 \qquad \text{for } \frac{1}{2} < H < 1.$$

Consider function $f(t) = \ln\Gamma(1+t\alpha)$ with constant α satisfying $|\alpha| < 0{,}5$. Function $f(t)$ is infinitely differentiable on the interval $[-2,2]$, and $f(0) = 0$. Denote $\frac{df(t)}{dt} = (\ln\Gamma(1+t\alpha))' = \frac{\alpha\Gamma'(1+t\alpha)}{\Gamma(1+t\alpha)} =: \psi(1+t\alpha)$. Note that $\psi''(1+t\alpha) < 0$. Therefore the value

$3\ln\Gamma(1-\alpha) + \ln\Gamma(1+\alpha) - \ln\Gamma(1-2\alpha) = -f(-2) + 3f(-1) - 3f(0) + f(1) = \int_{-2}^{1} f'''(t)g(t)\,dt = \alpha^3\int_{-2}^{1}\psi''(1+t\alpha)g(t)\,dt$ has the same sign as $-\alpha^3$, whence the proof follows. □

Now we establish an auxiliary result.

Lemma 4. *Let $\beta > 0, H \in (0,1)$ and K be Molchan kernel.*

Then
$$\int_0^t s^\beta K(t,s)ds =$$
$$= \frac{c_H(\beta+1)B(\alpha+1,\beta-\alpha+1)}{\alpha+\beta+1}t^{\alpha+\beta+1}.$$

Доведення. It is well-known that if $\alpha > -1, \beta > -1, t > 0$ then

$$\int_0^t u^\alpha (t-u)^\beta du = B(\alpha+1,\, \beta+1)t^{\alpha+\beta+1}. \quad (15)$$

We have that $\int_0^t s^\beta K(t,s)ds = c_H(I_1 - I_2)$, where

$$I_1 = \int_0^t t^\alpha s^{\beta-\alpha}(t-s)^\alpha ds = \text{(formula (15))}$$
$$= B(\beta-\alpha+1,\alpha+1)t^{\alpha+\beta+1}.$$
$$I_2 = \int_0^t \alpha s^{\beta-\alpha}\Big(\int_s^t u^{\alpha-1}(u-s)^\alpha du\Big)ds =$$
$$= \alpha\int_0^t u^{\alpha-1}\Big(\int_0^u s^{\beta-\alpha}(u-s)^\alpha ds\Big)du =$$
$$= \alpha\int_0^t u^{\alpha+\beta}B(\alpha+1,\beta-\alpha+1)du =$$
$$= \frac{\alpha B(\alpha+1,\beta-\alpha+1)}{\alpha+\beta+1}t^{\alpha+\beta+1},$$

whence the proof follows. □

Consider $\mathrm{E}(B_t^H - M_t)^2$, $0 \leqslant t \leqslant 1$, where B_t^H is fractional Brownian motion admitting the representation (1) with the kernel $K(t,s) = c_H\Big(t^\alpha s^{-\alpha}(t-s)^\alpha - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^\alpha du\Big)$ and with Hurst index $0 < H < \frac{1}{2}$. Let M_t be the square-integrable martingale of the form $M_t = \int_0^t a(s)dW_s$, and let the condition $\int_0^1 a^2(s)ds < \infty$ hold, i.e. $a \in L_2[0,1]$. Our goal is to find $\min_{a\in L_2[0,1]}\max_{0\leqslant t\leqslant 1}\mathrm{E}(B_t^H - M_t)^2$ in some partial cases.

Consider the simplest case. Let function $a(s) = a > 0$ be a constant.

Theorem 5.1. *The following equality holds*

$$\min_{a>0}\max_{0\leqslant t\leqslant 1}\mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 =$$
$$= 1 - \Big(\frac{c_H}{\alpha+1}\frac{\alpha\pi}{\sin\alpha\pi}\Big)^2, \quad (16)$$

and the value of $a_{\min}$ where the minimum in (16) is achieved, equals $a_{\min} = \frac{c_H}{\alpha+1}\frac{\alpha\pi}{\sin\alpha\pi}$.

Proof. In the case when $a(s) = a > 0$ and $\alpha < 0$ we have that

$$\mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 = \mathrm{E}(B_t^H)^2 +$$
$$+\mathrm{E}\Big(\int_0^t a(s)dW_s\Big)^2 - 2\mathrm{E}B_t^H\int_0^t a(s)dW_s =$$
$$= t^{2H} + \int_0^t a^2(s)ds - 2\int_0^t K(t,s)a(s)ds =$$
$$= t^{2H} + a^2 t - 2a\int_0^t K(t,s)ds.$$

From Lemma 4 we have that

$$\int_0^t K(t,s) = c_H\frac{B(1+\alpha,1-\alpha)}{\alpha+1} = c_H\frac{\alpha\pi}{\sin\alpha\pi}\frac{t^{\alpha+1}}{\alpha+1}$$

Therefore,

$$\mathrm{E}\Big(B_t^H - \int_0^t a(s)dW_s\Big)^2 = t^{2\alpha+1} -$$
$$-2ac_H\frac{\alpha\pi}{\sin\alpha\pi}\frac{t^{\alpha+1}}{\alpha+1} + a^2 t =: f(t,a).$$

Differentiate function f in t and equate the derivative to zero:

$$\frac{\partial f(t,a)}{\partial t} =$$
$$= t^{2\alpha}(2\alpha+1) - 2ac_H\frac{\alpha\pi}{\sin\alpha\pi}t^\alpha + a^2 = 0. \quad (17)$$

The discriminant of quadratic equation (17) equals

$$D = 4a^2c_H^2\Big(\frac{\alpha\pi}{\sin\alpha\pi}\Big)^2 - 4a^2(2\alpha+1) =$$

$$= (2a)^2\Big(c_H^2\Big(\frac{\alpha\pi}{\sin\alpha\pi}\Big)^2 - (2\alpha+1)\Big) > 0$$

for $\alpha < 0$ according to Lemma 3.

Therefore, the roots of equation (17) equal

$$t^\alpha = \frac{1}{2\alpha+1}(\alpha c_H\frac{\alpha\pi}{\sin\alpha\pi}\pm$$
$$\pm a\sqrt{c_H^2\Big(\frac{\alpha\pi}{\sin\alpha\pi}\Big)^2 - (2\alpha+1)}) =$$
$$= \frac{a}{2H}(A \pm \sqrt{A^2-2H}) =: t_\pm^\alpha, t_\pm > 0$$

where $A = c_H\frac{\alpha\pi}{\sin\alpha\pi}$.

For $H < \frac{1}{2}$ we have that $A^2 - 2H > 0$ and $A \pm \sqrt{A^2-2H} > 0$.

Taking into account that $a > 0$, we obtain

for $t \in (0, t_-)$ the function $f(t,a)$ is increasing,

for $t \in (t_-, t_+)$ the function $f(t,a)$ is decreasing,

and for $t \in (t_+, +\infty)$ the function $f(t,a)$ is increasing.

Hence, $\max\limits_{t\in[0,1]} f(t,a) = \max(f(1,a), f(t_-,a))$.

Consider

$$f(1,a) = 1 - 2ac_H\frac{\alpha\pi}{\sin\alpha\pi}\frac{1}{\alpha+1} + a^2.$$

Find the derivative of $f(1,a)$ with respect to a:

$\frac{\partial f(1,a)}{\partial a} = 0$ when $a_0 = \frac{c_H}{\alpha+1}\frac{\alpha\pi}{\sin\alpha\pi}$, hence,

$$\min_{a>0} f(1,a) = 1 - \Big(\frac{c_H}{\alpha+1}\frac{\alpha\pi}{\sin\alpha\pi}\Big)^2.$$

Consider

$$f(t_-,a) = t_-\Big(t_-^{2\alpha} - \frac{2aA}{\alpha+1}t_-^\alpha + a^2\Big) =$$
$$= t_-\Big(\Big(\frac{a}{2H}\Big)^2(A-\sqrt{A^2-2H})^2-$$
$$-\frac{2aA}{\alpha+1}\frac{a}{2H}(A-\sqrt{A^2-2H}) + a^2\Big) =$$
$$= a^2t_-\Big(\frac{1}{(2H)^2}(A-\sqrt{A^2-2H})^2-$$
$$-\frac{A}{H(\alpha+1)}(A-\sqrt{A^2-2H}) + 1\Big).$$

For $a = a_0 = \frac{c_H}{\alpha+1}\frac{\alpha\pi}{\sin\alpha\pi}$ we can check using Mathematica that the following inequality holds:

$$f(t_-, a_0) < f(1, a_0),$$

hence

$$\min_{a>0}\max_{t\in[0,1]} f(t,a) = 1 - \Big(\frac{c_H}{\alpha+1}\frac{\alpha\pi}{\sin\alpha\pi}\Big)^2.$$

□

6 Comparison of approximation of fractional Brownian motion with Molchan and Mandelbrot–van Ness kernels.

Let us first prove an auxiliary lemma.

Lemma 5. *Let $\alpha > 0$ and $\beta > 0$. Then*

$$\frac{(\beta+1)B(\alpha+1,\beta-\alpha+1)}{\alpha+\beta+1} \geqslant B(\alpha+1,\beta+1).$$

Доведення. Simple calculations

$$\frac{(\beta+1)B(\alpha+1,\beta-\alpha+1)}{(\alpha+\beta+1)B(\alpha+1,\beta+1)} =$$

$$\frac{(\beta+1)\Gamma(\alpha+1)\Gamma(\beta-\alpha+1)\Gamma(\beta+\alpha+2)}{(\alpha+\beta+1)\Gamma(\beta+2)\Gamma(\beta+1)\Gamma(\alpha+1)} =$$

$$\frac{\Gamma(\beta-\alpha+1)\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)^2} \geqslant 1.$$

The last inequality follows from that $\ln\Gamma(x)$ is a convex function. □

Theorem 6.1. *Let $a(s) = as^\beta$ where $\beta > 0, s \geqslant 0, a > 0$. Let $B_t^H = \int_{-\infty}^t K_H(t,s)dW_t$ and $B_t^H = \int_0^t K(t,s)d\tilde{W}_s$ be the representations of fractional Brownian motion via Mandelbrot–van Ness kernel K_H and Molchan kernel K. Then $\max\limits_{0\leqslant t\leqslant 1} \mathrm{E}\Big(B_t^H - \int\limits_0^t a(s)dW_s\Big)^2 \geqslant \max\limits_{0\leqslant t\leqslant 1}\mathrm{E}\Big(B_t^H - \int\limits_0^t a(s)d\tilde{W}_s\Big)^2$.*

Доведення. Let $t > 0$.

Consider the Mandelbrot–van Ness kernel

$$h(t) := \mathrm{E}\Big(B_t^H - \int\limits_0^t a(s)dW_s\Big)^2 = t^{2H}+$$

$$+a^2\int\limits_0^t s^{2\beta}ds - 2ac_H\int\limits_0^t u^\beta(t-u)^\alpha du =$$

$$= t^{2H} + a^2\frac{t^{2\beta+1}}{2\beta+1} - 2ac_H B(\alpha+1,\, \beta+1)t^{\alpha+\beta+1}.$$

Consider Molchan kernel

$$\tilde{h}(t) = \mathrm{E}\Big(B_t^H - \int_0^t a(s)d\tilde{W}_s\Big)^2 = t^{2H} +$$
$$+a^2 \int_0^t s^{2\beta} ds - 2a \int_0^t s^{\beta} K(t,u)du =$$
$$= (\text{Lemma 4}) = t^{2H} + a^2 \frac{t^{2\beta+1}}{2\beta+1} -$$
$$-\frac{2ac_H(\beta+1)B(\alpha+1,\beta-\alpha+1)}{\alpha+\beta+1} t^{\alpha+\beta+1}.$$

We immediately get from Lemma 5 that $h(t) \geqslant \tilde{h}(t)$. Therefore, the approximation of the fBm with the power functions is better for the representation with Molchan kernel than for the representation with the Mandelbrot–van Ness kernel.

□

7 Conclusion

We find the best approximation of the fractional Brownian motion by Gaussian martingales in the integral norm for all $H \in (0,1)$. We calculate the best approximation of a fractional Brownian motion with Mandelbrot–van Ness and Molchan kernels and constant integrands. It is proved that the approximation with the Molchan kernel is better than with the Mandelbrot–van Ness kernel if the integrand is a power function.

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