

УДК 519.9

Гончар М.С.<sup>1</sup>, д.ф.-м.н., проф.,  
Кучук-Яценко С.В.<sup>2</sup>, інженер

Gonchar N.S., doctor in physics and math., prof.  
Kuchuk-Iatsenko S.V., engineer

## Відсутність арбітражу в динамічних економічних системах

*В моделі економіки обміну знайдено умови строгої додатності розв'язків рівнянь економічної рівноваги. Встановлено нерівності знизу для всіх рівноважних цінових векторів. Сформульовано теорему про існування економічної динаміки. Представлено необхідні та достатні умови відсутності арбітражних можливостей для економічних агентів*

Ключові слова: економічні динамічні системи, арбітраж, економічна рівновага.

## Arbitrage absence in economy dynamical systems

*In the paper the necessary and sufficient conditions of strict positiveness of equilibrium price vectors are found for exchange economy model. For all solutions of the set of equations of equilibrium the inequalities from below are established. The theorem of existence of economy dynamics is stated. The sufficient conditions of absence of space and time arbitrage opportunities for economic agents are presented.*

Key words: economy dynamical systems, equilibrium, arbitrage

<sup>1</sup> Інститут теоретичної фізики ім. Миколи Боголюбова НАН України, 03143, м. Київ, вул. Метрологічна, 14Б, e-mail: [mhonchar@i.ua](mailto:mhonchar@i.ua)

<sup>1</sup> Bogolyubov Institute for Theoretical Physics of NAS of Ukraine, 03143, Kyiv, Metrologichna str., 14B, e-mail: [mhonchar@i.ua](mailto:mhonchar@i.ua)

<sup>2</sup> Київський національний університет імені Тараса Шевченка, 03680, м. Київ, пр.-т. Глушкова 4д, e-mail: [kuchuk.iatsenko@gmail.com](mailto:kuchuk.iatsenko@gmail.com)

<sup>2</sup> Taras Shevchenko National University of Kyiv, 03680, Kyiv, Glushkova st., 4d, e-mail: [kuchuk.iatsenko@gmail.com](mailto:kuchuk.iatsenko@gmail.com)

Статтю представив доктор фіз.-мат. наук професор Перестюк М.О..

### Introduction

In reality it is very important to find conditions of absence of space and time arbitrage opportunities in terms of macroeconomic parameters such as price of money and limitations on growth of prices of goods. The existing models give conditions of arbitrage absence in terms of martingale measures (see [1-3]). But in reality one does not know which mathematical model describes evolution of risk assets. Therefore, construction of models of general form and finding conditions under which arbitrage opportunities are absent is very actual. This paper is devoted to the construction of non-arbitrage economic dynamics in the case when strategies of consumer's behavior do not depend on price vector. This model is very important because the problem of finding equilibrium price vector in every period of economy operation is reduced to the solution of linear set of equations (see [5]).

The problem of arbitrage absence it is the problem of effective markets. Therefore, the main aim of the paper is to find general conditions on supply vector of goods under which spatial and time arbitrage opportunities are absent, knowing the demand vectors of consumer in every period of economy operation.

### Auxiliary results

In every period of operation a consumer is described by some demand vector that is given on some probability space. From another side every consumer has some goods that he wants to exchange on another set of goods defined by his demand vector. We suppose that the first component of this vector is the supply of money.

Constructing the economy system we assume that the demand on goods of previous period of economy operation defines the supply of goods in the next period of economy operation. The problem being

solved in this paper is to construct such economy dynamics in which spatial and time arbitrage opportunities are not possible. So in this stage of investigation we are not interested in determining the way in which consumers transform their demand vectors of the previous period into supply vectors of the next period. This investigation is the subject of the following papers.

In this section we consider an exchange model of economy with proportional consumption. The main problem is to clarify conditions under which the equilibrium price vector is strictly positive.

Below we consider the set of equations

$$\sum_{i=1}^l C_{ki} \frac{\langle b_i, p \rangle}{\langle C_i, p \rangle} = \psi_k, \quad k = \overline{1, n}, \quad (1)$$

the solutions of which describe equilibrium in such economy systems, where  $C = \|C_{ki}\|_{k=1, i=1}^{n, l}$ ,  $B = \|b_{ki}\|_{k=1, i=1}^{n, l}$  are nonnegative matrices,  $C_i = \{C_{ki}\}_{k=1}^n$ , and  $b_i = \{b_{ki}\}_{k=1}^n$ ,  $i = \overline{1, l}$ , are nonnegative vectors constructed by these matrices correspondingly,  $\psi = \{\psi_k\}_{k=1}^n$  is a strictly positive vector.

The next Theorem is the special form of the Theorem 5.1.4 from [5].

**Theorem 1.** Let  $\psi = \{\psi_k\}_{k=1}^n$  be a strictly positive vector belonging to the interior of the cone generated by the set of vectors  $C_i = \{C_{ki}\}_{k=1}^n$ ,  $i = \overline{1, l}$ , being the column of the matrix  $C = \|C_{ki}\|_{k=1, i=1}^{n, l}$ , and also let the inequalities

$$\sum_{s=1}^l C_{ks} > 0, \quad k = \overline{1, n}, \quad \sum_{k=1}^n C_{ki} > 0, \quad i = \overline{1, l}, \quad (2)$$

hold.

Suppose that there exist a vector  $y = \{y_i\}_{i=1}^l$ ,  $y_i > 0$ ,  $i = \overline{1, l}$ , such that

$$\psi = \sum_{i=1}^l C_i y_i$$

and a matrix  $\tau_1 = \|\tau_{ki}^1\|_{k=1, i=1}^{l, n}$  satisfying conditions:

$$\sum_{i=1}^n \tau_{ki}^1 = 1, \quad k = \overline{1, l}.$$

If the matrix elements of the matrix  $\tau = \|\tau_{ki}\|_{k=1, i=1}^{l, n}$  constructed by the rule

$$\tau_{ki} = \frac{y_k \tau_{ki}^1}{\psi_i}, \quad k = \overline{1, l}, \quad i = \overline{1, n},$$

are such that the matrix  $\tau C$  is nonnegative and indecomposable, the matrix  $\tau C \tau$  is nonnegative and does not contain zero column, then for the matrix  $B = \|b_{ik}\|_{i=1, k=1}^{n, l}$  having the representation  $B = C B_1$ , where

$$B_1 = \|b_{ik}^1\|_{i=1, k=1}^{l, l}, \quad b_{ks}^1 = y_s \sum_{i=1}^n \tau_{ki} C_{is}, \quad k, s = \overline{1, l},$$

the set of equations (1) has a strictly positive solution. If the rank of the set of the vectors  $b_i - y_i C_i$ ,  $i = \overline{1, l}$ , equals  $n-1$ , then a strictly positive solution is unique up to constant factor.

The next Theorem gives the necessary and sufficient conditions of strict positiveness of solutions of the set of equations (1).

**Theorem 2.** Let the inequalities (2) hold. The set of equations (1) has a strictly positive solution if and only if there exists a strictly positive vector  $y = \{y_i\}_{i=1}^l$ ,  $y_i > 0$ ,  $i = \overline{1, l}$ , and a matrix

$$\tau_1 = \|\tau_{ki}^1\|_{k=1, i=1}^{l, n}, \quad \sum_{i=1}^n \tau_{ki}^1 = 1, \quad k = \overline{1, l},$$

such that:

1) for a strictly positive vector  $\psi$  the following representation holds:

$$\psi = \sum_{i=1}^l C_i y_i;$$

2) a matrix

$$\tau = \|\tau_{ki}\|_{k=1, i=1}^{l, n}, \quad \tau_{ki} = \frac{y_k \tau_{ki}^1}{\psi_i}, \quad k = \overline{1, l}, \quad i = \overline{1, n},$$

satisfy conditions: the matrix  $\tau C$  is non-negative and indecomposable, the matrix  $\tau C \tau$  is non-negative and does not contain zero column;

3) the conditions

$$\langle b_i, p^0 \rangle = \langle \bar{b}_i, p^0 \rangle, \quad i = \overline{1, l},$$

hold, where

$$\bar{B} = \|\bar{b}_{ik}\|_{i=1, k=1}^{n, l}, \quad \bar{B} = C B_1, \quad B_1 = \|b_{ik}^1\|_{i=1, k=1}^{l, l},$$

$$b_{ks}^1 = y_s \sum_{i=1}^n \tau_{ki} C_{is}, \quad k, s = \overline{1, l}, \quad b_i = \{b_{ki}\}_{k=1}^n,$$

$$\bar{b}_i = \{\bar{b}_{ki}\}_{k=1}^n,$$

$p^0$  is a strictly positive vector constructed in the Theorem 1 for the matrix  $\bar{B}$  instead of matrix  $B$ .  
*Necessity:* Let there exist a strictly positive solution  $p^0$  to the set of equations (1). Denote

$$y_i = \frac{\langle b_i, p^0 \rangle}{\langle C_i, p^0 \rangle}, \quad i = \overline{1, l},$$

and introduce the matrix

$$\tau_1 = \|\tau_{ki}^1\|_{k=1, i=1}^{l, n}, \quad \tau_{ki}^1 = \frac{p_i^0 \psi_i}{\langle \psi, p^0 \rangle}, \quad k = \overline{1, l}, \quad i = \overline{1, n}.$$

It is evident that  $\sum_{i=1}^n \tau_{ki}^1 = 1, k = \overline{1, l}$ .

A matrix  $\tau = \|\tau_{ki}\|_{k=1, i=1}^{l, n}$  with matrix elements

$$\tau_{ki} = \frac{y_k \tau_{ki}^1}{\psi_i} = \frac{y_k p_i^0}{\langle \psi, p^0 \rangle}, \quad k = \overline{1, l}, \quad i = \overline{1, n},$$

is such that the matrix

$$\tau C = \left\| \frac{y_k \langle C_s, p^0 \rangle}{\langle \psi, p^0 \rangle} \right\|_{k, s=1}^{l, n}$$

is nonnegative and indecomposable, a matrix

$$\tau C \tau = \left\| \frac{y_k p_i^0}{\langle \psi, p^0 \rangle} \right\|_{k=1, i=1}^{l, n}$$

is nonnegative and does not contain zero rows.

It is not difficult to show that the vector  $y$  is a strictly positive proper vector of the matrix  $\tau C$  with a proper value 1, that is,

$$\tau C y = y. \quad (3)$$

Let us construct the matrix

$$\bar{B} = \|\bar{b}_{ik}\|_{i=1, k=1}^{n, l}, \quad \bar{B} = C B_1, \quad B_1 = \|b_{ks}^1\|_{k=1, s=1}^{l, n},$$

where

$$b_{ks}^1 = y_s \sum_{i=1}^n \tau_{ki} C_{is} = \frac{y_s y_k \sum_{i=1}^n p_i^0 C_{is}}{\langle \psi, p^0 \rangle}, \quad k, s = \overline{1, l},$$

$$\bar{b}_{ki} = \sum_{j=1}^l C_{kj} b_{ji}^1, \quad k = \overline{1, n}, \quad i = \overline{1, l}.$$

There hold equalities

$$\sum_{i=1}^l \bar{b}_{ki} = \psi_k, \quad k = \overline{1, n}.$$

The conjugate problem

$$\sum_{k=1}^l \delta_k^0 \sum_{i=1}^n \tau_{ki} C_{is} = \delta_s^0, \quad s = \overline{1, l},$$

to the problem (3) has strictly positive solution

$$\delta^0 = \{\delta_s^0\}_{s=1}^l, \quad \delta_s^0 = \sum_{i=1}^n p_i^0 C_{is}.$$

In correspondence with the conditions of the Theorem 5.1.4 [5], let us construct the vector  $\bar{p}^0 = \{\bar{p}_i^0\}_{i=1}^n$ . We have

$$\bar{p}_i^0 = \sum_{k=1}^l \delta_k^0 \tau_{ki} = \sum_{k=1}^l \sum_{i=1}^n p_i^0 C_{ik} \frac{y_k p_i^0}{\langle \psi, p^0 \rangle} = p_i^0, \quad i = \overline{1, n}.$$

Since

$$\bar{b}_{ki} = \sum_{j=1}^l C_{kj} b_{ji}^1 = \frac{\psi_k y_i \sum_{m=1}^n p_m^0 C_{mi}}{\langle \psi, p^0 \rangle}, \quad k = \overline{1, n}, \quad i = \overline{1, l},$$

we have

$$\begin{aligned} \langle \bar{b}_i, p^0 \rangle &= \sum_{k=1}^n \bar{b}_{ki} p_k^0 = y_i \sum_{m=1}^n p_m^0 C_{mi} \\ &= \sum_{k=1}^n b_{ki} p_k^0 = \langle b_i, p^0 \rangle. \end{aligned}$$

The necessity is proved.

*Sufficiency:* If sufficient conditions of the Theorem (2) are valid, then we are in the conditions of the Theorem (1). From the Theorem (1) and the conditions of the Theorem (2) we obtain that the set of equations

$$\sum_{i=1}^l C_{ki} \frac{\langle \bar{b}_i, p \rangle}{\langle \bar{C}_i, p \rangle} = \psi_k, \quad k = \overline{1, n},$$

has a strictly positive solution  $p^0$  satisfying conditions

$$\langle \bar{b}_i, p^0 \rangle = \langle b_i, p^0 \rangle, \quad i = \overline{1, l}.$$

This proves the Theorem.

**Corollary 1.** If  $p^0$  is a strictly positive solution to the set of equations (1), then for the vectors  $b_i$  the representations

$$b_i = \bar{b}_i + d_i, \quad i = \overline{1, l},$$

hold, where

$$\sum_{i=1}^l d_i = 0, \quad \langle p^0, d_i \rangle = 0, \\ \bar{b}_i = \frac{\psi y_i \langle C_i, p^0 \rangle}{\langle \psi, p^0 \rangle}, \quad i = \overline{1, l}.$$

**Corollary 2.** For given strictly positive vectors  $\psi, y, p^0$  and matrix  $C$  satisfying conditions (2) and the vector  $\psi$  having the representation  $\psi = \sum_{i=1}^l C_i y_i, C_i = \{c_{ki}\}_{k=1}^n, i = \overline{1, l}$ , the set of equations (1) has strictly positive solution  $p^0$  if and only if there exist vectors  $d_i, i = \overline{1, l}$ , such that

$$b_i = \bar{b}_i + d_i, \quad \langle p^0, d_i \rangle = 0, \quad i = \overline{1, l}, \quad \sum_{i=1}^l d_i = 0,$$

where

$$\bar{b}_i = \frac{\psi y_i \langle C_i, p^0 \rangle}{\langle \psi, p^0 \rangle}, \quad i = \overline{1, l}.$$

**Theorem 3.** Suppose that the conditions of the Theorem 1 are valid. If additionally matrix elements of the matrix  $\tau C \tau$  are strictly positive, the vector  $p^0 = \{p_j^0\}_{j=1}^n$  being the solution to the set of equations (1) satisfies the condition

$$\sum_{i=1}^n p_i^0 \psi_i = A, \quad A > 0,$$

then the following inequalities for components of vector  $p^0$  are valid

$$p_j^0 \geq \frac{A \min_{s,k} [\tau C \tau]_{sk}}{\max_s \sum_{j=1}^n [\tau C \tau]_{sj} \psi_j}, \quad j = \overline{1, n}. \quad (4)$$

*Proof:* Due to the Theorem 5.1.4 [5] the components of the vector  $p^0 = \{p_i^0\}_{i=1}^n$  of the solution to the problem (1) is given by the formula

$$p_i^0 = \sum_{k=1}^l \delta_k^0 \tau_{ki}, \quad i = \overline{1, n}, \quad (5)$$

where the vector  $\delta^0 = \{\delta_k^0\}_{k=1}^l$  is strictly positive left Frobenius vector of the problem

$$\sum_{k=1}^l \delta_k^0 \sum_{i=1}^n \tau_{ki} C_{is} = \delta_s^0. \quad (6)$$

From (5) and (6) we have equalities

$$p_j^0 = \sum_{s=1}^l \delta_s^0 [\tau C \tau]_{sj}, \quad j = \overline{1, n}, \quad (7)$$

$$A = \sum_{j=1}^n p_j^0 \psi_j = \sum_{s=1}^l \delta_s^0 \sum_{j=1}^n [\tau C \tau]_{sj} \psi_j \leq \\ \max_s \sum_{j=1}^n [\tau C \tau]_{sj} \psi_j \sum_{s=1}^l \delta_s^0.$$

From here we obtain inequality

$$\sum_{s=1}^l \delta_s^0 \geq \frac{A}{\max_s \sum_{j=1}^n [\tau C \tau]_{sj} \psi_j}.$$

The equalities (7) give the inequalities

$$p_j^0 \geq \sum_{s=1}^l \delta_s^0 \min_{s,j} [\tau C \tau]_{sj} \geq \frac{A \min_{s,k} [\tau C \tau]_{sk}}{\max_s \sum_{j=1}^n [\tau C \tau]_{sj} \psi_j}.$$

The Theorem is proved.

### Statement of the problem

Let us assume that economy system functions during  $N$  periods,  $N < \infty$ . In the  $t$ -th period of economy operation demand vectors  $C_i^t(\omega) = \{C_{ki}^t(\omega)\}_{k=1}^n, i = \overline{1, l}$ , are given on a probability space  $\{\Omega, F, P\}$ . We also assume that consumption in the  $t$ -th period is characterized by some random vector  $y^t(\omega) = \{y_i^t(\omega)\}_{i=1}^l$  every component of which is strictly positive, that is  $y_i^t(\omega) > 0, i = \overline{1, l}$ .

This vector we call the vector of degrees of satisfaction of consumers' needs (see [4-5]). So, the supply vector in the  $t$ -th period is given by the formula

$$\psi^t(\omega) = \sum_{i=1}^l C_i^t(\omega) y_i^t(\omega).$$

Below we formulate a theorem giving sufficient conditions of the existence of economy dynamics and that is a consequence of previous results.

**Theorem 4.** Let the set of vectors  $C_i^t(\omega), i = \overline{1, l}$ , being the columns of the matrix

$C^t(\omega) = \|C_{ki}^t(\omega)\|_{k=1, i=1}^{n, l}$  satisfy the inequalities

$$\begin{aligned} \sum_{s=1}^l C_{ks}^t(\omega) &> 0, \quad k = \overline{1, n}, \quad t = \overline{1, N}, \\ \sum_{k=1}^n C_{ki}^t(\omega) &> 0, \quad i = \overline{1, l}, \quad t = \overline{1, N}, \end{aligned} \quad (8)$$

and let the vectors  $y^t(\omega) = \{y_i^t(\omega)\}_{i=1}^l$  be strictly positive for every  $\omega \in \Omega$ .

If for every matrix  $\tau_1^t = \tau_1^t(\omega)$  such that

$$\sum_{i=1}^n \tau_{ki}^{1,t}(\omega) = 1, \quad k = \overline{1, l}, \quad t = \overline{1, N},$$

and such that the matrix  $\tau^t = \tau^t(\omega) = \|\tau_{ki}^t(\omega)\|_{k=1, i=1}^{n, l}$ , elements of which are constructed by the rule

$$\tau_{ki}^t(\omega) = \frac{y_k^t(\omega) \tau_{ki}^{1,t}(\omega)}{\sum_{i=1}^l C_{ki}^t(\omega) y_i^t(\omega)},$$

is such that the matrix  $\tau^t C^t$  is nonnegative and indecomposable, the matrix  $\tau^t C^t \tau^t$  is nonnegative and does not contain zero columns, then for the matrix  $B^t(\omega) = \|b_{ik}^t(\omega)\|_{i=1, k=1}^{n, l}$  having representation  $B^t = C^t B_1^t$ , where

$$B_1^t \neq \|b_{ik}^{1,t}\|_{i, k=1}^l,$$

$$b_{ks}^{1,t} = y_s^t(\omega) \sum_{i=1}^n \tau_{ki}^t(\omega) C_{is}^t(\omega),$$

the set of equalities

$$\begin{aligned} \sum_{i=1}^l \frac{C_{ki}^t(\omega)}{\langle p^t, C_i^t(\omega) \rangle} \langle p^t, b_i^t \rangle &= \psi_k^t(\omega), \\ k &= \overline{1, n}, \quad t = \overline{1, N}, \end{aligned}$$

have strictly positive solution with probability 1 with respect to price vectors  $p^t = \{p_1^t, \dots, p_n^t\}$ .

It is very important to know under which conditions spatial and time arbitrage opportunities for economic agents are absent in the economy system. We suppose that in the  $t$ -th period of the economy operation the  $k$ -th component  $\bar{p}_k^t$  of random equilibrium price vector  $\bar{p}^t$  defines value of the  $k$ -th goods,  $k = \overline{1, n_t}$ . We assume that the first component  $\bar{p}_1^t$  of this vector defines equilibrium

price of money, and  $\psi_1^t(\bar{p}^t)$  is the supply of money in the  $t$ -th period of economy operation. We also assume that short sales of risk assets are only permitted within the limits of every period of the economy operation.

On a probability space  $\{\Omega, F, \bar{P}\}$ , where evolution of prices of goods is given in the Theorem 4, denote the evolution of prices of subset of  $n_0 \leq n_t, N_0 \leq t \leq N_1$ , goods by the rule

$$\begin{aligned} S_t &= \{\bar{p}_{i_1}^t, \dots, \bar{p}_{i_{n_0}}^t\}, \quad N_0 \leq t \leq N_1, \\ i_k &< i_{k+1}, \quad k = \overline{1, n_0 - 1}. \end{aligned} \quad (9)$$

We introduce evolution of risk-free asset by the law

$$B_t = \prod_{i=N_0}^t (1 + p_1^i), \quad t = \overline{N_0, N_1}, \quad (10)$$

where  $p_1^t$  is an equilibrium price of money in the  $t$ -th period of economy operation.

**Definition.** We call the economy dynamics defined in the Theorem 4 *non-arbitrage* if for any subset of goods numbering by indexes  $i_1, \dots, i_{n_0}$  and any numbers  $1 \leq N_0 < N_1 \leq N$  the evolution of prices of which is described by the law (9) and evolution of risk-free asset is given by the law (10) the set of self-financed arbitrage strategies without short selling between periods of economy operation is empty.

Now we give a statement of the general form of absence of arbitrage opportunities in dynamical systems.

**Theorem 5.** Let on a probability space  $\{\Omega, F, P\}$  with filtration  $F_n, n = \overline{1, N}$ , random evolution of risk-free asset be defined by the law  $B_n, n = \overline{1, N}$ , satisfying the condition: there exists non-random constant  $D < \infty$  such that  $1 \leq B_n \leq D, n = \overline{1, N}$ ,  $\omega \in \Omega$ , and let evolution of prices of  $n_0 \geq 1$  risk assets be given by the law  $S_n = \{S_n^i\}_{i=1}^{n_0}, n = \overline{1, N}$ .

If  $\left\{ \frac{S_n}{B_n}, F_n \right\}, n = \overline{1, N}$ , is a supermartingale, then

the set of self-financed arbitrage strategies without short selling is empty.

**Theorem 6.** The economic dynamics defined in the Theorem 4 is non-arbitrage one if with probability 1 the following inequalities

$$\bar{p}_k^{t+1} \leq \bar{p}_k^t \left( 1 + f^{t+1}(\bar{p}_1^{t+1}) \right), \quad k = \overline{2, n_t}, \quad t = \overline{1, N},$$

are valid, where  $\bar{p}^t = \{\bar{p}_k^t\}_{k=1}^{n_t}$  is an equilibrium price vector in the  $t$ -th period of economy operation,  $f^t(x), t = \overline{1, N}$ , is the set of nonnegative functions satisfying conditions  $0 \leq f^t(x) \leq x, t = \overline{1, N}$ .

Denote by

$$\begin{aligned} r_0^t &= r_0^t(\omega) = \min_i \psi_i^t(\omega), \\ R_0^t &= R_0^t(\omega) = \max_i \psi_i^t(\omega), \\ A^t &= A^t(\omega) = \sum_{i=1}^n p_i^t(\omega) \psi_i^t(\omega). \end{aligned}$$

There hold inequalities

$$p_i^t(\omega) \leq \frac{A^t(\omega)}{r_0^t(\omega)}, \quad i = \overline{1, n}.$$

**Theorem 7.** Let the set of vectors  $C_i^t(\omega)$ ,  $i = \overline{1, l}$ , being the columns of the matrix  $C^t(\omega) = \|C_{ki}^t(\omega)\|_{k=1, i=1}^{n, l}$  satisfy the inequalities (8) and let the vectors  $y^t(\omega) = \{y_i^t(\omega)\}_{i=1}^l$  be strictly positive for every  $\omega \in \Omega$ . Moreover, let the inequalities

$$\begin{aligned} \min_{s, j} [\tau^t C^t \tau^t]_{sj}(\omega) &> 0, \\ \max_{s, j} [\tau^t C^t \tau^t]_{sj}(\omega) &< \infty, \\ t &= \overline{1, N}, \quad \omega \in \Omega, \end{aligned}$$

are valid with probability 1.

If for every matrix  $\tau_1^t = \tau_1^t(\omega), t = \overline{1, N}$ , satisfying conditions:

$$\sum_{i=1}^n \tau_{ki}^{1, t}(\omega) = 1, \quad k = \overline{1, l}, \quad t = \overline{1, N},$$

and such that the matrix  $\tau^t = \tau^t(\omega) = \|\tau_{ki}^t(\omega)\|_{k=1, i=1}^{n, l}$ , elements of which are constructed by the rule

$$\tau_{ki}^t(\omega) = \frac{y_k^t(\omega) \tau_{ki}^{1, t}(\omega)}{\sum_{i=1}^l C_{ki}^t(\omega) y_i^t(\omega)},$$

is such that the matrix  $\tau^t C^t$  is nonnegative and indecomposable, the matrix  $\tau^t C^t \tau^t$  is nonnegative and does not contain zero columns, then for the matrix  $B^t(\omega) = \|b_{ik}^t(\omega)\|_{i=1, k=1}^{n, l}$  having representation

$$B^t = C^t B_1^t, \text{ where}$$

$$B_1^t = \|b_{ik}^{1, t}\|_{i, k=1}^l,$$

$$b_{ks}^{1, t} = y_s^t(\omega) \sum_{i=1}^n \tau_{ki}^t(\omega) C_{is}^t(\omega),$$

economy dynamics, defined in the Theorem 4, is non-arbitrage if with probability 1 the set of inequalities

$$\begin{aligned} \frac{A^{t+1}}{r_0^{t+1}} &\leq \frac{A^t \min_{s, j} [\tau^t C^t \tau^t]_{sj}(\omega)}{\max_s \sum_{j=1}^n [\tau^t C^t \tau^t]_{sj}(\omega) \psi_j^t(\omega)} \times \\ &\left( 1 + \delta \frac{A^{t+1} \min_{s, j} [\tau^{t+1} C^{t+1} \tau^{t+1}]_{sj}(\omega)}{\max_s \sum_{j=1}^n [\tau^{t+1} C^{t+1} \tau^{t+1}]_{sj}(\omega) \psi_j^{t+1}(\omega)} \right), \\ t &= \overline{1, N}, \end{aligned} \quad (11)$$

are valid for some  $0 < \delta < 1$ .

*Proof:* provided by inequalities (11) the following inequalities

$$p_i^{t+1} \leq p_i^t (1 + \delta p_1^{t+1}), \quad i = \overline{2, n},$$

are valid with probability 1 due to the Theorem 3. Using the Theorem 6 with  $f^t(x) = \delta x, 0 < \delta < 1$ , we obtain the proof of the Theorem.

## References

1. Harrison J.M., Pliska S.R. Martingales and stochastic integrals in the theory of continuous trading // Stochastic Processes and their Application. – 1981. – II '3. – P. 215-260
2. Dalang R.S., Morton A., Willinger W. Equivalent martingale measures and no-arbitrage in stochastic securities models // Stochastics and Stochastics Reports. – 1990. – 29, '2. P. 185- 201
3. Shiryaev A.N. The fundamentals of stochastic financial mathematics: 2 vol. – M. Phasis, 1998, - Vol. 2 – 544p.
4. Gonchar N.S. Mathematical Model of Stock Market. // Cond. Matt. Phys. – 2000. – 3, '3(23). – P.461-496
5. Gonchar N.S. Mathematical Foundations of information economics. K.: Bogolyubov Institute for Theoretical Physics, 2008. – 468p.

Стаття надійшла до редколегії 25.12.2012.