УДК 539.9

Lavrenyuk M.V.[1], phD.

The stress-strain state of the plane containing rectangular multicoated inclusion.

Modified boundary element method (MBEM) has been proposed, to solve the problem of defining stress-strain state of plane with multicoated inclusion of rectangular shape. Modification allows simultaneous defining of all the components of stress-strain state on the boundary of inclusion and each layer by means of using two-dimensional approximation of displacements and stresses on every element of discretization. The approach exposed allows generalization for the cases of inclusion of an arbitrary shape, the pair of two interacting multicoated inclusions, etc. The numerical examples presented demonstrate high accuracy as well as numerical efficiency of the MBEM.

Key words: multicoated inclusion, BEM, layer.

[1] Taras Shevchenko National University of Kyiv, 03680, Kyiv, Glushkova st., 4e,
e-mail: mykolalav@ukr.net

М. В. Лавренюк[1], к. ф.-м. н.

Напружено-деформований стан платівки з прямокутним багатошаровим включенням

Запропоновано модифікований метод граничних елементів (ММГЕ) для розв'язання задачі визначення напружено-деформованого стану платівки з багатошаровим включенням прямокутної форми. Модифікація методу дозволяє визначати одночасно всі компоненти напружено-деформованого стану на границі включення та кожного прошарку за допомогою двовимірної апроксимації складових вектора переміщень та тензора напружень на кожному елементі дискретизації. Запропонований підхід можна узагальнити на випадок включення довільної форми, пари двох взаємодіючих багатошарових включень, тощо. Наведені результати чисельного розрахунку вказують на високу точність та ефективність ММГЕ.

Ключові слова: прошарок, включення, метод граничних елементів.

[1] Київський національний університет імені Тараса Шевченка,03680, м. Київ, пр-т. Глушкова 4е, e-mail: mykolalav@ukr.net

Статтю представив д.ф.-м.н., проф. Маципура В.Т.

Introduction

In recent years, the use of composite materials has increased in many areas of technology especially in aerospace, automotive, and civil engineering. Composites usually consist of two or more materials that are combined to form a new material that exhibits a number of desired properties. Such composites can be often presented as matrix containing multicoated fibers. If the surface load and the stress field do not vary in fiber direction, the problem can be reduced to two-dimensional elasticity problem, namely, an elastic plane containing multicoated inclusions. To solve such kind of problems conventional computational methods with boundary (BEM [1]) discretizations are usually [2], [3] utilized.

The two-dimensional approximation of displacements on the boundary of inclusions [4] allows simultaneous defining of all the components of stress-strain state on the boundary of inclusion and each layer. Let us consider the problem of elastic equilibrium of space containing rectangular multicoated single fiber. The author's contribution to related problems can be found for example in [5], [6]. Also, in authors PhD degree diploma the problem of defining stress-strain of inclusion with single coating in elastic plane was considered. In the last Section, the results of numerical investigation are presented and possible ways to improve computational efficiency of the method are discussed.

Statement of the problem

Lets consider infinite plane with multicoated inclusion in it (Fig. 1). On the infinite boundary of plate distributed forces p_i is applied and on the infinity body is also subjected to the action of mass load X.

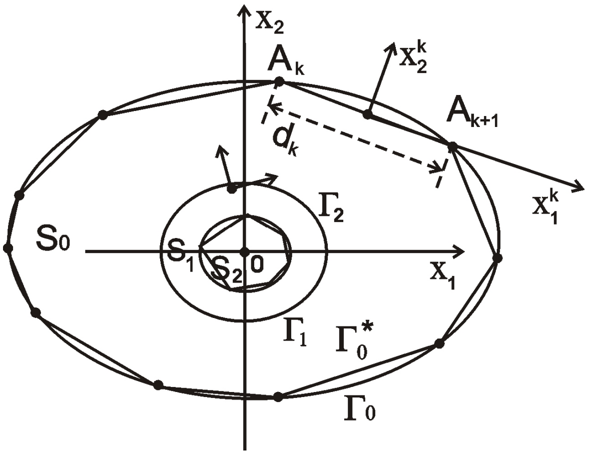


Figure 1. Scheme of discretized matrix with multicoated inclusion

The system of equilibrium equations can be written down as follows:

$$\begin{cases} \left(\lambda_0 u_{l,l}\delta_{ij} + \mu_0\left(u_{i,j}+u_{j,i}\right)\right)_{,j} + X_i = 0 \\ \left(\lambda_p u_{l,l}\delta_{ij} + \mu_p\left(u_{i,j}+u_{j,i}\right)\right)_{,j} + X_i = 0 \end{cases}, \quad (1)$$

$i,j=1,2$, $p=1,...,N$

The ideal contact conditions on the edge of neighboring layers are:

$$\sigma_{ij}^p n_j^p |_{\Gamma_p} = \sigma_{ij}^{p+1} n_j^{p+1} |_{\Gamma_p}, u_i^p |_{\Gamma_p} = u_i^{p+1} |_{\Gamma_p}, \quad (2)$$

$i,j=1,2$, $p=1,...,N$

Multiplying both parts of each equation (1) on the correspondent fundamental solutions $U_i^{k(0)}$ and $U_i^{k(p)}$, $p=1,...,N$ and integrating over the domains of plane, inclusion and layers, we have for matrix:

$$\int_{S_0\setminus S_1} \left(\mu_0 u_{l,l}\delta_{ij} + \mu_0\left(u_{i,j}+u_{j,i}\right)\right)_{,j} U_i^{k(0)} dS - \int_{S_0\setminus S_1} X_i U_i^{k(0)} dS = 0,$$

for every layer

$$\int_{S_p\setminus S_{p+1}} \left(\mu_p u_{l,l}\delta_{ij} + \mu_p\left(u_{i,j}+u_{j,i}\right)\right)_{,j} U_i^{k(p)} dS_p = 0,$$

and for inclusion domain

$$\int_{S_N} \left(\mu_N u_{l,l}\delta_{ij} + \mu_N\left(u_{i,j}+u_{j,i}\right)\right)_{,j} U_i^{k(N)} dS_N = 0.$$

At last we will have Somigliano identity for plane:

$$\chi(S)u_k^{(0)}(\xi) = \int_{\Gamma_1} \sigma_{ij}^{(0)} n_j^{(1)} U_i^{k(0)} d\Gamma_1 - \int_{\Gamma_1} g_i^{k(0)} u_i d\Gamma_1 + \int_{S_0\setminus S_1} X_i U_i^{k(0)} dS, \ \chi = \begin{cases} 1, \xi \in S_0 \setminus S_1 \\ 0, \xi \notin S_0 \setminus S_1 \end{cases} \quad (3)$$

for each layer:

$$\chi(S)u_k^{(p)}(\xi) = \int_{\Gamma_{p+1}} \sigma_{ij}^{(p)} n_j^{(p)} U_i^{k(p)} d\Gamma_{p+1} - \int_{\Gamma_p} g_i^{k(p)} u_i d\Gamma_p - \int_{\Gamma_{p+1}} g_i^{k(p)} u_i d\Gamma_{p+1} + \int_{\Gamma_p} \sigma_{ij}^{(p)} n_j^{(p)} U_i^{k(p)} d\Gamma, \ \chi = \begin{cases} 1, \xi \in S_p \setminus S_{p+1} \\ 0, \xi \notin S_p \setminus S_{p+1} \end{cases} \quad (4)$$

and for inclusion:

$$\chi(S)u_k^{(N)}(\xi) = \int_{\Gamma_N} \sigma_{ij} n_j U_i^{k(N)} d\Gamma_N - \int_{\Gamma_N} g_i^{k(N)} u_i d\Gamma_N, \ \chi = \begin{cases} 1, \xi \in S_N \\ 0, \xi \notin S_N \end{cases} \quad (5)$$

Lets discretize the contours of inclusion and layers using segments as elements of discretization (Fig. 1) and work in on these segments local coordinates system. Using boundary properties of single and double layer potentials we get the system of integral equations for defining displacements in zonal-homogeneous body. Let's carry out passage in integral representations, and direct (for every equation) the observation point step-by-step to the center of every segment of discretization. Thus we obtain the system of boundary integral equations for matrix:

$$\frac{1}{2}\delta_{kj}u_j^{(0)}(x) - u_k^{\infty} = \int_{S_0} X_i U_i^{k(0)} dS_0 + \int_{\Gamma_1} u_i^{(0)} g_i^{k(0)} d\Gamma_1 - \int_{\Gamma_1} U_i^{k(0)}(x,\xi)\sigma_{ij}^{(0)} n_j^{(0)} d\Gamma_1, \ x \in S_0 \setminus S_1, (6) \quad (6)$$

for every layer:

$$\frac{1}{2}\delta_{kj}u_j^{(p)}(x) = \int_{\Gamma_p} U_i^{k(p)}(x,\xi)\sigma_{ij}^{(p)} n_j^{(p)} d\Gamma_p - \int_{\Gamma_p} u_i^{(p)} g_i^{k(p)} d\Gamma_p, \ x \in S_p \setminus S_{p+1}, p=1,...,N \quad (7)$$

and for inclusion:

$$\frac{1}{2}\delta_{kj}u_j^{(N+1)}(x) = \int_{\Gamma_{N+1}} U_i^{k(N+1)}(x,\xi)\sigma_{ij}^{(N+1)} n_j^{(N+1)} d\Gamma_{N+1} - \int_{\Gamma_{N+1}} u_i^{(N+1)} g_i^{k(N+1)} d\Gamma_{N+1}, \ x \in S_N, \quad (8)$$

Here u_i^{∞} are displacements on the infinity defined as the solution of the problem of infinite plate subjected to loading on the infinity. Further unknown densities of potentials of double layer on the boundaries of inclusions can be represented as functions of two unknowns along the segments of discretization. Such representation allows after solving the system of boundary integral equations to define all the components of both displacements and stresses σ_{ij} on the boundary of inclusion and every layer. In the case of linear approximation the components of displacements on every segment of discretization can be written as follows:

$$u_1^{\prime i(p)}(x') = a_{1i}^p x_1' + a_{2i}^p x_2' + a_{3i}^p ,$$
$$u_2^{\prime i(p)}(x') = a_{4i}^p x_1' + a_{5i}^p x_2' + a_{6i}^p , \qquad (9)$$
$$p = 1,...,N, i = 1,...,M_p .$$

Using Hooke's law, we get correspondent expression for σ_{ij} on every segment of discretization:

$$\sigma_{11}^{i(p)}(x') = (\lambda_p + 2\mu_p) a_{1i}^p + \lambda_p a_{5i}^p$$
$$\sigma_{22}^{i(p)}(x') = (\lambda_p + 2\mu_p) a_{5i}^p + \lambda_p a_{1i}^p \qquad (10)$$
$$\sigma_{12}^{i(p)}(x') = \mu_p (a_{2i}^p + a_{4i}^p), p = 1,...,N, i = 1,...,M_p$$

It is obvious, that in considered case of linear approximation the components of tensor σ_{ij} (10) will be constant along every segment of integration. After substituting the expressions for displacements (9) and stresses (10) into boundary integral equations (6)-(8), and contact conditions (2) we have the system of linear algebraic equations, which must be made closed by adding to system the equations of continuity of displacements on every segment of discretization, that have the following form:

$$\frac{1}{2}(a_{1i}^p d_i^p + a_{3i}^p)\cos\alpha_i^p - \left(\frac{1}{2} a_{4i}^p d_i^p + a_{6i}^p\right)\sin\alpha_i^p =$$
$$= \left(-\frac{1}{2} a_{1i+1}^p d_{i+1}^p + a_{3i+1}^p\right)\cos\alpha_{i+1}^p -$$
$$-\left(-\frac{1}{2} a_{4i+1}^p d_{i+1}^p + a_{6i+1}^p\right)\sin\alpha_{i+1}^p ,$$
$$\left(\frac{1}{2} a_{1i}^p d_i^p + a_{3i}^p\right)\sin\alpha_i^p + \left(\frac{1}{2} a_{4i}^p d_i^p + a_{6i}^p\right)\cos\alpha_i^p =$$
$$= \left(-\frac{1}{2} a_{1i+1}^p d_{i+1}^p + a_{3i+1}^p\right)\sin\alpha_{i+1}^p +$$
$$+\left(-\frac{1}{2} a_{4i+1}^p d_{i+1}^p + a_{6i+1}^p\right)\cos\alpha_{i+1}^p ,$$
$$p = 0,...,N,\ i = 1,...,M_p$$

Here d_i^p — the length of i-th segment of p-th layer, M_p — the number of segments on p-th inclusion, and α_i^p — the angle at normal of i-th segment of p-th layer with respect to OX of global coordinates system. So we have the closed system of linear algebraic equations, which produce as solution vector unknown coefficients of approximation (9) or (10), and thus, all the components σ_{ij} and displacements on the contours of matrix and inclusions. Lets change the coordinate system to local for every discretization system, and change the densities of potentials on their expression using (9) and (10). Further using Somigliano formulas (3)-(5), we define displacements, and differentiating these formulas, also deformations. Then, on the base of Hooke's law we can define stresses in every internal point of the plane. And thus the elastic problem is solved.

Numerical results

There were considered the number of numerical examples to evaluate stress state of considered bodies, especially stresses concentration for composites with multicoated fibers. It was studied the influence of geometrical characteristic of inclusion ($r:a$ ratio), physical characteristics of material and geometrical characteristics of layers coating inclusion (thickness of layers h_i) on the stress concentration on the contour of inclusion.

In all solved problems the infinite plate is subjected to onelateral loading in vertical direction (along the axis OY). All results are obtained with respect to the value of distributed forces on the infinity.

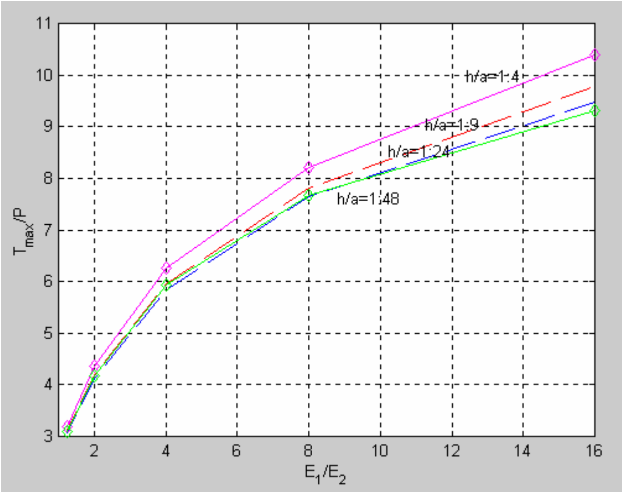
The form of rectangular inclusion was set with the help of Schwarz–Christoffel mapping. For example mapping function for quadrangle has the form:

$$\omega(\varsigma) = R\left(\frac{1}{\varsigma} - \frac{1}{6}\varsigma^3 + \frac{1}{56}\varsigma^7 - \frac{1}{176}\varsigma^{11} + ...\right) \qquad (11)$$

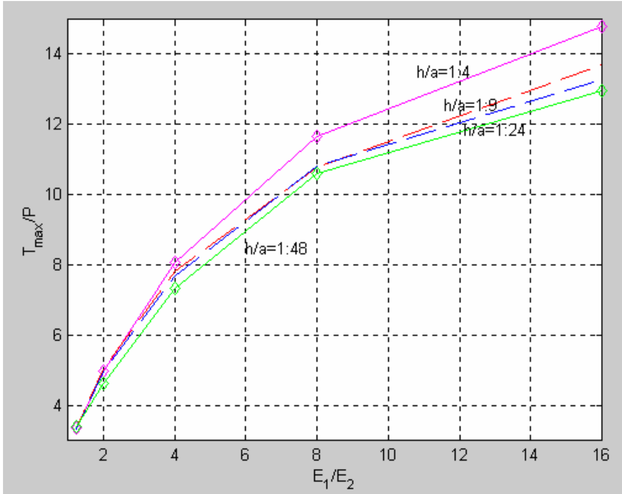
Varying the number of terms in (11) we will have the quadrangles with different radiuses of rounded corners, e.g. at 2 terms we have $r:a = 0.06$ and for 3 terms we have $r:a = 0.0245$, and so on.

There were considered cases of inclusion in infinite matrix with 1 and 2 layers of different thickness, having different physical characteristics. The shape of inclusions was considered to be rectangular in above mentioned sense.

To provide the desired accuracy at high rates of $a:r$ and low thickness h of coating layers it is necessary to increase the number of discretized elements on the contour of inclusion, the less the thickness (or the greater $r:a$), the bigger this number. It is completely predicted result. In other words we must satisfy $d_{el} \le \min\{h_i\}$ condition. Lets roughly estimate the value d_{el} — the average size of discretization element, for $R_{incl} = 1$, N_{incl} being the number of segments on contour of inclusion. For circular inclusion, for instance, we have $d_{el} = 2\pi R_{incl}/N_{incl} \approx 3R_{incl}/N_{incl}$. So, if we want to



Graph 1. Maximal intensity for 2-terms rectangular single coated inclusion



Graph 2. Maximal intensity for 3-terms rectangular single coated inclusion

consider problem with $h = 0.01$ and $R_{incl} = 1$ we must guarantee that $N_{incl} \geq 300$.

Conclusions

An highly accurate numerical-analytical variation of boundary element method has been used to solve the problem of defining stress state of plane with multicoated inclusion of rectangular shape. Due to two dimensional approximation of unknown densities of potential on the boundary of considered discretized body it is possible to obtain simultaneously all the components of stress-strain state.

It was ascertained that there are three quite predictable factors influence the level of stresses concentration on the contour of coated inclusion, namely the thickness of layer (ratio $h_i : R_{incl}$), the ratio $r : a$ of rectangular inclusion and the difference of mechanical characteristics of matrix, layers and inclusion ($E_0 : E_i : E_{incl}$).

The problem can be generalized to problem of multicoated inclusion of an arbitrary shape in finite or infinite plane and pair of two interacting multicoated inclusions in plane. The proposed approach exhibits both high accuracy and numerical efficiency and thus can be effectively used for solving the problems of stress-strain state for the variety of composite materials, containing multicoated fibers.

References

1. *Brebbia K., Telles J., Wrobel L.* Boundary element techniques: Theory and applications in engineering. – Berlin: Springer-Verlag, 1984. – 464 p.
2. *Dong, C.Y.; Lo, S.H.; Cheung, Y.K.* Application of the boundary-domain integral equation in elastic inclusion problems // [J] Eng. Anal. Bound. Elem. 2002, 26, No.6, P. 471-477
3. *Vasanth S Kothnur, Subrata Mukherjee, Yu Xie Mukherjee*. Two-dimensional linear elasticity by the boundary node method // International Journal of Solids and Structures, 1999, 36, P. 1129-1147.
4. *Lavrenyuk V.I.* On the assessment of stress-strain state of matrix with inclusion using BEM // Visn., Ser. Fiz.-Mat. Nayky, Ky"iv Univ. im. Tarasa Shevchenka. – 1993. – N 2. – P. 27-35. (in Ukrainian)
5. *Lavrenyuk M.V.* Investigation of influence of contact conditions on stress-strain state of infinite elastic plate with two eliptical inclusions // Visn., Ser. Fiz.-Mat. Nayky, Ky"iv Univ. im. Tarasa Shevchenka. – 1998. – N 2. – P. 63-69. (in Ukrainian)
6. *Lavrenyuk M.V.* Analysis of stress-strain state of infinite elastic plate with rectangular inclusions at ideal and nonideal contact conditions // Visn., Ser. Fiz.-Mat. Nayky, Ky"iv Univ. im. Tarasa Shevchenka. – 1998. – N 4. – P. 17-26. (in Ukrainian)

Received on 27.03.13