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Канонічна пуассонова структура на T*SE(3) в кватерніонних змінних

В даній роботі показано, що дужки Пуассона з кватерніонними змінними можуть бути виведені безпосередньо з канонічних дужок Пуассона на кодотичному розшаруванні до групи $SE(3)$ ($SO(3)$), що забезпечено стандартною симплектичною геометрією, відповідною формою Ліувіля. Отримані результати засновані на представленні кватерніонних змінних як явних функцій елементів матриці поворотів групи $SO(3)$.

Ключові слова: кватерніон, форма Ліувіля, SE(3), SO(3), пуассонові структури.

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Canonical Poisson structure on T*SE(3) in the quaternion variables

In this paper showed that Poisson bracket with quaternion variables can be deduce directly from the canonical Poisson brackets on the cotangent bundle to $SE(3)$ ($SO(3)$) group endowed with the canonical symplectic symplectic geometry corresponding Liouville one form. The obtained results based on quaternion variables representation as the explicit functions of rotation matrix elements of $SO(3)$ group.

Key Words: quaternion, Liouville one form, SE(3), SO(3), Poisson structure.

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1. Introduction

Quaternion applications in mechanics of rigid body there are greate number of publication are devoted. The most of them refer to the kinematic of the rigid body that is the description of rigid body orientation in space with quaternion parameters [1].

Thereinafter we specify those rare works that devote to dynamical aspects of quaternion application. In the V.V.Kozlov's work [2] descrition of the dynamics is given by quaternion variables based on the Lagrange approach that was modified by Poincaré for the nonholonomic basis. That approach also allow to put into consideration the momentum that conjugate to quaternion variables and give the hamiltonian form to the motion equations with redundant variables. Starting from deep relations between quaternion algebra and $SO(3)$, $SO(4)$ groupes A.V. Borisov and I.S. Mamaev [3-5] propose the expressions of Poisson brackets between quaternion's parameters and angular momentum of the rigid body. They consider them as the generatrix of some Lie-Poisson structure. So, the condition of normalized per unit quaternion that is required for rigid body description sprang into special value of the Casimir function in the present Lie-Poisson structure.

The expression of group matrix element by quaternions parameters is well known [1,4-6]. Additionally it clearly demonstrates the structure of the orthogonal matrix. Per se, inverse problem that is expressing the quaternion's parameters in terms of the elements of the corresponding rotation matrix is not so difficult. Though in the litterature we succeed in found (post factum) only one work where this task was posed and its solution presented [7]. But the main goal of this short (but very instructive) work was to present the comparative analysis of the necessary number of operation for quaternions and matrix computations. In terms of quantity was proven that quaternions computation is more effective. It is important to notice that as it was pointed in the book [4, c.104] the quaternions give one the balance of advantage also in the stability of numerical integration of rigid body motion equations.

From general relations of work [8] one can deduce the following expressions for Poisson brackets between the elements of rotation matrix and the angular momentum components in the inertial reference system. Here and further for determinacy we consider more wide group $SE(3)$, that describe not only rotational, but also translational degree of freedom of the rigid body.

$$\begin{cases} \{x_i,x_j\}=0, & \{p_i,p_j\}=0, & \{m_i,Q_{jk}\}=\varepsilon_{ijl}Q_{lk}, \\ \{x_i,p_j\}=\delta_{ij}, & \{p_i,Q_{jk}\}=0, & \{m_i,m_j\}=\varepsilon_{ijl}m_l, \\ \{x_i,Q_{jk}\}=0, & \{p_i,m_j\}=0, & \{x_i,m_j\}=0, \\ \{Q_{ij},Q_{kl}\}=0. \end{cases} \tag{1.1}$$

where x_i – coordinates of the body center of mass,

p_i – components of the momentum of translational motion, Q_{jk} – elements of rotation matrix that describe the body orientation with respect to the inertial reference system, m_j – components of the angular moment of the body with respect to the inertial reference system.

Notice that Poisson brackets in the system connected with a body is not difficult to get by simple canonical manipulation from (1.1), see also [4-5].

From the explicit expression of quaternion parameters as the functions of the elements of rotation matrix from (1.1) one can deduce the required Poisson brackets between the components of quaternion and the angular momentum components of the body. The construction ot these relations is the main goal of the paper.

It turns out that expressions computed in this way have the same form that the previously discussed Lie-Poisson brackets between generatrix of the Poisson structure, see [4, ф. (2.7), c.103].

Though, the essence of the getting relations is something else. Quaternion parameters in our case are not the generatrix of the Poisson structure, but the dynamic variables in the canonical Hamiltonian mechanics on $T^*SE(3)$ (it is not the Lie-Poisson structure). In our case the structural tensor of Lie algebra of the Poisson brackets is not nondegenerate, and, so, Casimir functions are absent then the condition of normalized quaternion are simply express the relation of quaternion dynamic variables on group. As an explanatory notes of this circumstance let us examine the example. Let φ is one of the Euler angel. Then the dynamic variable $\cos^2(\varphi)+\sin^2(\varphi)$ is identically equal to one, but it is not the Casimir function.

2. Quaternion's algebra

As it was already mention there is a vast amount of literature devoted to quaternions. Remind briefly the properties of quaternions [1,4-7,9].

The quaternions form the associative algebra with e_0 identity and e_i, $i=1,2,3,$ generatrix that satisfy the influential relations.

$$e_r e_s = -\delta_{rs} e_0 + \varepsilon_{rst} e_t \tag{2.1}$$

or that is equal

$$\begin{cases} [e_r, e_s] = e_r e_s - e_s e_r = 2\varepsilon_{rst} e_t, \\ e_r e_s + e_s e_r = -2\delta_{rs} e_0. \end{cases} \tag{2.2}$$

Thus the quaternions form 4-dimensional vector pace over field of real numbers

$$q = q^0 e_0 + q^1 e_1 + q^2 e_2 + q^3 e_3 \tag{2.3}$$

or expressed in terms of 4-dimensional column-vector $\mathrm{q} = (\mathrm{q}^0, q^1, q^2, q^3)$.

The component q^0 name is scalar part of quaternion q and components q^1, q^2, q^3 group in vector part. Thus quaternion can be presented as $q = (q^0, \mathbf{q})$.

If $q^0 = 0$ then it is acceptable to write $q = \mathbf{q}$ and that quaternions named pure quaternion [6, c.301]. Pure quaternions form the linear subspace of quaternion algebra, but it is not subalgebra, because of the associative product of two pure quaternions be the quaternion of the general type.

Multiplicative rule between a and b quaternions follow from (2.1)

$$\begin{aligned} ab &= (a^0 e_0 + a^r e_r)(b^0 e_0 + b^s e_s) = (a^0 e_0 + \mathbf{a})(b^0 e_0 + \mathbf{b}) \\ &= (a^0 b^0 - <\mathbf{a}, \mathbf{b}>) e_0 + a^0 \mathbf{b} + b^0 \mathbf{a} + \mathbf{a} \times \mathbf{b} \end{aligned} \tag{2.4}$$

Then for the pure quaternions the next expressions of scalar and vector product by associative multiplication are valid.

$$\begin{cases} <\mathbf{x}, \mathbf{y}> = -\tfrac{1}{2}(\mathbf{xy} + \mathbf{yx}), \\ \mathbf{x} \times \mathbf{y} = \tfrac{1}{2}(\mathbf{xy} - \mathbf{yx}). \end{cases} \tag{2.5}$$

The operation of quaternion conjugation specify as

$$e_0^{\dagger} = e_0, \; e_k^{\dagger} = -e_k \tag{2.6}$$

or

$$q^{\dagger} = (q^0, \mathbf{q})^{\dagger} = (q^0, -\mathbf{q}) \tag{2.7}$$

Thus the pure quaternions fully characterize of the next property

$$\mathbf{x}^{\dagger} = -\mathbf{x}. \tag{2.8}$$

It is follow from (2.7)

$$\begin{aligned} qq^{\dagger} = q^{\dagger} q &= (q^0 q^0 + (\mathbf{q}, \mathbf{q})) e_0 = \\ &= (q^0)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2. \end{aligned} \tag{2.9}$$

In addition, from the formulas (4,6a) follow

$$(ab)^{\dagger} = b^{\dagger} a^{\dagger}. \tag{2.10}$$

Let's define the quaternion norm

$$|q| = \sqrt{(qq^{\dagger})} \tag{2.11}$$

then

$$|ab| = |a||b|. \tag{2.12}$$

Moreover, from the formulas (2.9,2.11) follow the simple representation of the inverse quaternion

$$q^{-1} = \frac{q^{\dagger}}{|q|^2}. \qquad (2.13)$$

Formulas (2.9) and (2.13) show that all quaternions excepting the zero one have there inverse, so the quaternion algebra is a body.

It is follow from (2.12) and (2.13) that unit quaternion or the quaternions with norma equal to identity form the group.

3. Right (left) action in the quaternion algebra

Its make sense to put into consideration the algebra representation of quaternions by 4×4-matrix of right action. Moreover, in some computation the using of rut matrix technic is offer advantages.

It is clear from (2.1) that for the right action of quaternion $b = (b^0, \mathbf{b})$ the matrix operation corresponds

$$R_b a = ab = \begin{bmatrix} b^0 & -b^1 & -b^2 & -b^3 \\ b^1 & b^0 & b^3 & -b^2 \\ b^2 & -b^3 & b^0 & b^1 \\ b^3 & b^2 & -b^1 & b^0 \end{bmatrix} \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix}. \qquad (3.1)$$

We can expand the matrix R_b into the present set of basis matrices

$$R_b = b^0 R_0 + b^1 R_1 + b^2 R_2 + b^3 R_3, \qquad (3.2)$$

where

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, R_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices R_0, R_1, R_2, R_3 satisfy the relations that are fully analogous to the relations (2.1) for quaternions generatrix. Therewith R_k is anti-symmetrical in $k = 1, 2, 3,$, and, then, the quaternion conjugation corresponds the operation of matrix transpose.

Quite similarly to the right action we can consider the matrix representation of the left action.

4. Representation of rotation by quaternions

Let's consider a linear subspace of the pure quaternions as a 3-dimensional Euclidian space with scalar product defined by (5).

Shown that interior automorphism of quaternion algebra that generated by unit quaternion q translate the space of pure quaternions by themselves.

The rule of conversion we write in form

$$\mathbf{x}' = q\mathbf{x}q^{-1} = q\mathbf{x}q^{\dagger} = Q[\mathbf{x}], \quad |q| = 1. \qquad (4.1)$$

Shown that resulting quaternion $\mathbf{x}'$ is also pure quaternion, really,

$$(q\mathbf{x}q^{\dagger})^{\dagger} = (q^{\dagger})^{\dagger}\mathbf{x}^{\dagger}q^{\dagger} = -q\mathbf{x}q^{\dagger}. \qquad (4.2)$$

Thus the operator $Q[\mathbf{x}]$ is the linear operator that acts in subspace of the pure quaternions.

Shown that operator $Q[\mathbf{x}]$ conserve the scalar product of the vectors by using expression of scalar product via associative multiplication in (2.5)

$$< Q[\mathbf{x}], Q[\mathbf{y}] >= -\frac{1}{2}(q\mathbf{x}q^{-1}q\mathbf{y}q^{-1} + q\mathbf{y}q^{-1}q\mathbf{x}q^{-1})$$

$$= -\frac{1}{2}q(\mathbf{xy} + \mathbf{yx})q^{-1} = q < \mathbf{x}, \mathbf{y} > q^{-1} =< \mathbf{x}, \mathbf{y} > .$$

Thus the following relations of invariance are true for the operator Q (it is proved analogous)

$$\begin{cases} < Q[\mathbf{x}], Q[\mathbf{y}] >=< \mathbf{x}, \mathbf{y} >, \\ Q[\mathbf{x}] \times Q[\mathbf{y}] = Q[\mathbf{x} \times \mathbf{y}], \\ < Q[\mathbf{z}], Q[\mathbf{x}] \times Q[\mathbf{y}] >=< \mathbf{z}, \mathbf{x} \times \mathbf{y} > . \end{cases} \qquad (4.3)$$

The first relation of (4.3) means that the operaror Q is orthogonal, and the third that it is unimodular, i.e. Q is the intrinsic rotation $(Q \in SO(3))$.

In this context of the task the main interest invoke the explicit form of the matrix elements of Q-operator that can be obtained from (4.1) from the multiplication rule (2.4)

$$Q_{ik} = 2\left[\left((q^0)^2 - \frac{1}{2}\right)\delta_{ik} + q^i q^k - q^0 q^j \varepsilon_{jik}\right] \qquad (4.4)$$

that for the unit quaternion equal to the following expression of matrix operator Q:

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \qquad (4.5)$$

Shown that reflection $q \to Q$ is the reflection that cover whole group $SO(3)$.

In particular in the case $q_1 = 0,\ q_2 = 0,$ this matrix has the form

$$Q = \begin{bmatrix} q_0^2 - q_3^2 & -2q_0q_3 & 0 \\ 2q_0q_3 & q_0^2 - q_3^2 & 0 \\ 0 & 0 & q_0^2 + q_3^2 \end{bmatrix}$$

If appear that $q_0 = \cos\frac{1}{2}\theta,\ q_3 = \sin\frac{1}{2}\theta,$ then

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is the rotation matrix around the axis z on the angel θ (anticlockwise).

Similarly get the rotations around other two axes x and y.

R e m a r k. It is useful to notice that the arbitrary unit quaternion can be offered in the form

$$q = \cos(\varphi/2)\mathrm{e}_0 + \sin(\varphi/2)\mathbf{e}, \qquad (4.6)$$

where $\mathbf{e}$ – unit pure quaternion, that specify the rotation axis, and φ – the corresponding rotational angle around this axis. The expression (4.6) that concerned as a function φ is one-parameter subgroup of this group of unit quaternions.

Since $q \to Q$ is the homomorphism, so then the product of such rotations belong to direct image of this homomorphism. Well known that any rotation can be realized as the product of rotations around the axes of Cartesian coordinate system. Thus, really, matrix (4.6) represents the record of the arbitrary element of group $SO(3)$ via quaternion parameters.

So, formula (4.1) defines the homomorphism of the group of unit quaternions on the $SO(3)$ group. Therewith these groups are locally isomorphic and group of unit quaternions double cover $SO(3)$ [6]. Really, from (4.1) it is clear that quaternions q and $(-q)$ give us the same rotation Q.

5. Rotational representation of quaternions

In Section 4 it was proved that for each rotation of $SO(3)$ corresponds 2 and only 2 unit quaternions with opposite sign. Then the task about explicit form of this functional dependence is appeared.

Matrix Q corresponding to the quaternion q has the follows elements.

$$Q_{ik} = (2q_0^2 - 1)\delta_{ik} + 2q_iq_k - 2q_0q_j\varepsilon_{jik}. \qquad (5.1)$$

Let's evaluate the spur of the matrix with keep in mind that q is unit quaternion

$$\mathrm{Sp(Q)} = 4\mathrm{q}_0^2 - 1 \qquad (5.2)$$

i.e.

$$q_0^2 = \frac{1}{4}(\mathrm{Sp(Q)} + 1). \qquad (5.3)$$

Antisymmetric part of the matrix Q has a simple form, then we have

$$q_0q_i = -\frac{1}{4}\varepsilon_{ijk}Q_{jk}. \qquad (5.4)$$

Thus for $q_0 \neq 0$ we can express in the explicit form the components of quaternion that corresponding to the target rotation matrix by its elements

$$\begin{cases} q_0 = \frac{1}{2}(\mathrm{Sp(Q)} + 1)^{\frac{1}{2}}, \\ q_i = -\frac{1}{2}\frac{\varepsilon_{\mathrm{ijk}}\mathrm{Q_{jk}}}{(\mathrm{Sp(Q)}+1)^{\frac{1}{2}}}. \end{cases} \qquad (5.5)$$

We have two solutions of this system corresponding of two choose of root sign in the expression of q_0 (1-st line in (5.5)).

If we want to have the solution of these functional equations in the neighborhood of quaternion with zero scalar part $(\mathrm{Sp(Q)} = -1)$, i.e. in neighborhood of the pure quaternion then formulas (5.5) is not acceptable.

It must be noticed that for the numarical computations the difficulties is possible even for small but not zero value of q^0. Therefore E. Salamin who investigate the task of comparative estimation of the efficiency of numarical computations with orthogonal matrix and quaternions [7], to put forward the set of experssions for finding the quaternion components that follow from the form of matrix Q.

$$\begin{cases} q_0^2 = \frac{1}{4}(1 + Q_{11} + Q_{22} + Q_{33}), \\ q_1^2 = \frac{1}{4}(1 + Q_{11} - Q_{22} - Q_{33}), \\ q_2^2 = \frac{1}{4}(1 - Q_{11} + Q_{22} - Q_{33}), \\ q_3^2 = \frac{1}{4}(1 - Q_{11} - Q_{22} + Q_{33}) \end{cases} \qquad (5.6)$$

and

$$\begin{cases} q_0q_1 = \frac{1}{4}(Q_{32} - Q_{23}), \\ q_0q_2 = \frac{1}{4}(Q_{13} - Q_{31}), \\ q_0q_3 = \frac{1}{4}(Q_{21} - Q_{12}), \\ q_1q_2 = \frac{1}{4}(Q_{12} + Q_{21}), \\ q_1q_3 = \frac{1}{4}(Q_{13} + Q_{31}), \\ q_2q_3 = \frac{1}{4}(Q_{23} + Q_{32}). \end{cases} \qquad (5.7)$$

For numarical computations the optimum is to choose the maximum component of quaternion by its absolute magnitude from the relations (5.6) (so this component sign will defined the sign of quaternion in general), then other components can be found from (5.7).

It is clear that for the unit quaternion at least we have one component nonvanishing.

Noticed that the present solution (5.5) hereinabove is corresponding to the 1-st line of (5.6) and for the first 3 lines in (5.7).

6. The Poisson brackets with quaternion components

If we want to get Poisson brackets with quaternion components it is required to deduce the expressions $\frac{\partial q_0}{\partial Q_{kl}}$ and $\frac{\partial q_i}{\partial Q_{kl}}$. For example,

$$\{m_i, q_0\} = \frac{\partial q_0}{\partial Q_{kl}}\{m_i, Q_{kl}\} \qquad (6.1)$$

where the Poisson bracket $\{m_i, Q_{kl}\}$ is known from (1.1).

From the relation (5.5) we have

$$\begin{cases} \frac{\partial q_0}{\partial Q_{kl}} = \frac{1}{8q_0}\delta_{kl}, \\ \frac{\partial q_j}{\partial Q_{kl}} = -\frac{1}{4q_0}\left(\varepsilon_{jkl} + \frac{1}{2}\frac{q_j}{q_0}\delta_{kl}\right). \end{cases} \qquad (6.2)$$

In consideration of $\{m_i, Q_{kl}\} = \varepsilon_{ikn}Q_{nl}$ and using (6.2) we receive

$$\{m_i, q_0\} = \frac{\partial q_0}{\partial Q_{kl}}\{m_i, Q_{kl}\} = \frac{1}{2}q_i \qquad (6.3)$$

and

$$\{m_i, q_j\} = \frac{1}{2}(\varepsilon_{ijk}q_k - q_0\delta_{ij}). \qquad (6.4)$$

Take into consideration $\{Q_{ij}, Q_{kl}\} = 0$, finally we get

$$\begin{cases} \{q_\mu, q_\nu\} = 0, \qquad \mu = 0,1,2,3 \\ \{m_i, q_0\} = \frac{1}{2}q_i, \\ \{m_i, q_j\} = \frac{1}{2}(\varepsilon_{ijk}q_k - q_0\delta_{ij}). \end{cases} \qquad (6.5)$$

As it was mentioned above in Section 1 such Poisson bracket were deduced by A.V. Borisov and I.S. Mamaev as the relation for the generatrix of some Lie-Poisson structure. Then the value $C(q) = (q^0)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2$ is the Casimir function for this Poisson structure and for deduce of motion equations of the rigid body it is required to pass on the symplectic sheet of this Poisson structure that corresponds to the $C(q) = 1$.

7. The invariance properties of the Poisson structure on T*(SE(3)) (T*(SO(3)))

Take note on the Poisson brackets (1.1) that includes the matrix elements of the rotations. Examine, for example, the relation

$$\{m_i, Q_{jk}\} = \varepsilon_{ijl}Q_{lk} \qquad (7.1)$$

and multiply it on the fixed matrix $B \in SO(3)$ by right hand.

Thus the elements of the matrix is constant, so we can be bring them behind the sign of the Poisson brackets in the left side of (7.1), and then the relations are valid

$$\{m_i, (QB)_{jn}\} = \varepsilon_{ijl}(QB)_{ln} \qquad (7.2)$$

After doing the analogous conversion with all of these Poisson brackets, we see, that matrix elements of the rotation $P = QB$ satisfy to all of those Poisson brackets as well as the matrix elements of the initial rotation Q.

Let's shown that the Poisson brackets of the Section 6 can be transform to more compact form by using the multiplicative rule of the quaternions.

Multiply the quaternion $q = q_0e_0 + q_ke_k$ left hand on e_i

$$e_iq = q_0e_i + q_k(e_ie_k) = q_0e_i + q_k(-\delta_{ik}e_0 + \varepsilon_{ikj}e_j)$$
$$= -q_ie_0 + q_0\delta_{ij}e_j + q_k\varepsilon_{ikj}e_j,$$

$$e_iq = -q_ie_0 + q_0\delta_{ij}e_j + q_k\varepsilon_{ikj}e_j \qquad (7.3)$$

$$\begin{cases} (e_iq)_0 = -q_i, \\ (e_iq)_j = -\varepsilon_{ijk}q_k + q_0\delta_{ij}. \end{cases} \qquad (7.4)$$

Then

$$\begin{cases} \{q_0, m_i\} = \frac{1}{2}(e_iq)_0, \\ \{q_j, m_i\} = \frac{1}{2}(e_iq)_j \end{cases} \qquad (7.5)$$

$$\{q_\mu, m_i\} = \frac{1}{2}(e_iq)_\mu, \quad \mu = 0,1,2,3. \qquad (7.6)$$

Formally it is possible to multiply (by left hand) the expressions (7.6) on the fixed quaternions e_0, e_1, e_2, e_3 and sum by index μ. Moreover, using the constancy of e_μ, let's bring them behind the sign of the Poisson bracket.

In result we have

$$\{m_i, q\} = -\frac{1}{2}e_iq. \qquad (7.7)$$

As far as in the matrix case by multiplying the expression (7.7) (by right hand) on some fixed quaternion $b, |b| = 1$, we make sure that new quaternion dynamic variable $p = qb$ is also satified those Poisson brackets (7.7) for q. In fact, this is the simple algebraic conversion that is equal to the the action of right hand matrix shift (3.2) on the set of (7.7). Consequently, p satisfy to all Poisson brackets that are analogous to the Poisson brackets (7.5-7.6) of the Section 6.

8. Poisson bracket with quaternion components in the special case

Deduce of the relations (7.5-7.6) hereinabove, strictly speaking, will not be correct in the neighborhood of the pure quaternion $\mathrm{Sp(Q)} = -1 \rightarrow \mathrm{q}_0 = 0$, because of (5.5) in this point is not valid. But it does not mean that relation (7.5-7.6) in the neighborhood of the pure quaternion is not valid. Quaternion with $q_0 = 0$ is not looks as the special point for this relations. But the proof of this relations in general case required another approach.

First of all, it is possible to use the Salamin's solutions (5.6-5.7) by taken as a base those component of quaternion that is nonzero in the target neighborhood, for example q^1. But this approach, unfortunally, result in hungus and non-transparent computations.

Let's examine the properties of invariance that was presented in the Section 7 and the fact that reflection of $q \rightarrow Q$ is homomorphism on $SO(3)$, group as it was describe hereinabove.

Suppose that we want to compute Poisson brackets with quaternions in the neighborhood of a pure quaternion q with $q_0 = 0$.

Let's take such fixed quaternion $b : |b| = 1$ that the dynamical quaternion variable $p = qb$ with value $p^0 \neq 0$ in the target point (it is always possible for $q : |q| = 1$, as it was shown hereinabove). Quaternion p reflects into the matrix $P \in SO(3)$ for the mentioned homomorphism $(p \rightarrow P)$.

Quaternion variable p lie in the same relation to the matrix variable P as far as variable q relates to Q in computation of the Poisson brackets with quaternions presented hereinabove, it being known that $p^0 \neq 0$.

Therefore, the Poisson bracket (7.5-7.6) is valid for quaternion variable p. Apllling the inverse conversion $q = pb^{-1}$ to the expressions that was received result in the expressions (7.5-7.6), but for the quaternion variable q.

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