УДК 519.21

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Поведінка об'єктивної ціни опціона на облігацію з відсотковою ставкою, що керується дробовим геометричним процесом Орнштейна–Уленбека

У роботі досліджується поведінка об'єктивної ціни Європейського опціона купівлі на облігацію з відсотковою ставкою, що керується модифікованим дробовим геометричним процесом Орнштейна–Уленбека, як функції індекса Хюрста.

Ключові слова: опціон, дробовий процес Орнштейна–Уленбека.

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Objective Call Option Price Behaviour of the Bond with Interest Rate Driven by Geometric Fractional Ornstein–Uhlenbeck Process

The paper examines behavior of the European call option price of the bond with interest rate driven by modified geometric fractional Ornstein–Uhlenbeck process as a function of Hurst index.

Key words: option, fractional Ornstein–Uhlenbeck process.

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Communicated by Prof. Kozachenko Yu. V.

1 Introduction

Fractional calculus is extensively used in different modelling, including financial modelling. The paper [1] studies Doob's transformation of fractional Brownian motion, which is identical in law with the Ornstein–Uhlenbeck diffusion. The book [2] represents stochastic calculus for fractional Brownian motion. It contains financial models and statistical problems that involve fractional Brownian motion. The paper [3] studies financial markets with stochastic volatilities driven by fractional Brownian motion with Hurst index $H > \frac{1}{2}$. Current paper examines behavior of the European call option price of the bond with interest rate driven by modified geometric fractional Ornstein–Uhlenbeck process as a function of Hurst index.

2 Price of the Bond with Interest Rate Driven by Geometric Ornstein–Uhlenbeck Process with Wiener Process

The Ornstein–Uhlenbeck process $X = \{X_t, t \geqslant 0\}$ is a Gaussian and Markov process that satisfies the following stochastic differential equation:

$$dX_t = -aX_t dt + \gamma dW_t,\ X\Big|_{t=0} = X_0,$$

where $a > 0$ and $\gamma > 0$ are constants and $W = \{W_t, t \geqslant 0\}$ is the Wiener process. The solution of this stochastic differential equation has the following form $X_t = X_0 e^{-at} + \gamma e^{-at} \int_0^t e^{as} dW_s$.

We suppose in what follows that $X_0 = 1$. The geometric Ornstein–Uhlenbeck process satisfies the following stochastic differential equation:

$$dX_t = X_t \Big\{ -ae^{-at}dt - a\gamma e^{-at}dt \int_0^t e^{as}dW_s + \gamma dW_t + \frac{1}{2}\gamma dt \Big\}, \quad (1)$$

and the solution of equation (3) has the following form

$$X_t = x_0 \exp\Big\{ e^{-at} + \gamma e^{-at} \int_0^t e^{as} dW_s \Big\} = x_0 \exp\Big\{ e^{-at} - a\gamma \int_0^t e^{-as} \int_0^s e^{au} dW_u ds + \gamma W_t \Big\}. \quad (2)$$

Consider the following model of financial market. Suppose we have the European call option with an expiration date T. The bond interest rate is driven by the modified geometric Ornstein–Uhlenbeck process with the Wiener process. It satisfies the following stochastic differential equation

$$dX_t^{(1)} = X_t^{(1)}\Big\{ -\mu e^{-\mu t}dt - \mu\gamma e^{-\mu t}\times \times \int_0^t e^{as}dW_s dt + \gamma e^{-\mu t+at}dW_t + \gamma e^{-2(\mu-a)t}dt\Big\}. \tag{3}$$

The solution of equation (3) has the following form

$$X_t^{(1)} = x_0 \exp\Big\{e^{-\mu t} + \gamma e^{-\mu t}\int_0^t e^{as}dW_s\Big\} = = x_0 \exp\Big\{e^{-\mu t} - \gamma\mu \int_0^t e^{-\mu s}\int_0^s e^{au}dW_u ds + + \gamma\int_0^t e^{-\mu s+as}dW_s\Big\}.$$

To find the price of this option we have to calculate the expectation value $E[X_T^{(1)} - K]^+$, where $K > 0$ is the strike price. Using indicator $I\{X_t^{(1)} > K\}$, we remove brackets in the expectation value

$$E[X_T^{(1)} - K]^+ = E[X_T^{(1)} - K]I\{X_T^{(1)} > K\} = = EX_T^{(1)}I\{X_T^{(1)} > K\} - KI\{X_T^{(1)} > K\}. \tag{4}$$

We find both items of this equality. Ornstein–Uhlenbeck process is a Gaussian process, so

$$EX_T^{(1)}I\{X_T^{(1)} > K\} = \int_K^\infty e^x p(x)dx,$$

where $p(x)$ is a density function of normal distribution with parameters m and σ^2:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\Big\{-\frac{(x-m)^2}{2\sigma^2}\Big\},$$

here m is the expectation and σ^2 is the variance of modified Ornstein–Uhlenbeck process

$$m = E[X_T^{(1)}] = e^{-\mu T},$$

$$\sigma^2 = Var[X_T^{(1)}] = \frac{\gamma^2 e^{-2\mu T}}{2a}(e^{2aT} - 1). \tag{5}$$

We continue to calculate the first item in (4)

$$EX_T^{(1)}I\{X_T^{(1)} > K\} = \int_K^\infty e^x p(x)dx = = \frac{1}{\sigma\sqrt{2\pi}}\int_K^\infty \exp\Big\{-\Big(\frac{x-\sigma^2-m}{\sqrt{2}\sigma}\Big)^2 + m + \frac{\sigma^2}{2}\Big\}dx = = \frac{\exp\{m+\frac{\sigma^2}{2}\}}{\sqrt{2\pi}}\Big(1 - \Phi\Big(\frac{K-\sigma^2-m}{\sigma}\Big)\Big).$$

Consider the second item of (4). Evidently, $EI\{X_T^{(1)} > K\} = P\{X_T^{(1)} > K\}$ and

$$KEI\{X_T^{(1)} > K\} = K\int_K^\infty p(x)dx = \frac{K}{\sigma\sqrt{2\pi}}\times \times\int_K^\infty \exp\Big\{-\frac{(x-m)^2}{2\sigma^2}\Big\}dx = \frac{K}{\sqrt{2\pi}}\Big(1-\Phi\Big(\frac{K-m}{\sigma}\Big)\Big).$$

As a result we obtain the price for the option

$$E[X_T^{(1)} - K]^+ = \frac{\exp\{m+\frac{\sigma^2}{2}\}}{\sqrt{2\pi}}\times \times\Big(1-\Phi\Big(\frac{K-\sigma^2-m}{\sigma}\Big)\Big) - \frac{K}{\sqrt{2\pi}}\Big(1-\Phi\Big(\frac{K-m}{\sigma}\Big)\Big),$$

where the variance σ^2 is determined by (5).

3 Price of the Bond with Interest Rate Driven by Geometric Fractional Ornstein-Uhlenbeck Process

Now we consider a call option with the bond interest rate driven by geometric fractional Ornstein–Uhlenbeck process that satisfies the following stochastic differential equation

$$dX_t^{(2)} = X_t^{(2)}\Big\{ -\mu e^{-\mu t}dt - - \mu\gamma e^{-\mu t}\int_0^t e^{as}dB_s^H dt + \gamma e^{-\mu t+at}dB_t^H\Big\}.$$

The solution of the stochastic differential equation has the following form

$$X_t^{(2)} = x_0 \exp\{e^{-\mu t} + \gamma e^{-\mu t} \int_0^t e^{as} dB_s^H\} =$$

$$= x_0 \exp\{e^{-\mu t} - \gamma\mu \int_0^t e^{-\mu s} \int_0^s e^{au} dB_u^H ds +$$

$$+ \gamma \int_0^t e^{-\mu s + as} dB_s^H\}.$$

The expectation and the variance for the corresponding Ornstein-Uhlenbeck process $e^{-\mu t} + \gamma e^{-\mu t} \int_0^t e^{as} dB_s^H$ at the point T equal, respectively,

$$m = e^{-\mu T},$$

$$\sigma^2 = H(2H-1)\gamma^2 e^{-2\mu T} \int_0^T \int_0^T e^{as+au} |s-u|^{2H-2} du ds. \qquad (6)$$

The price for such option equals

$$E[X_T^{(2)} - K]^+ = \frac{\exp\{m + \frac{\sigma^2}{2}\}}{\sqrt{2\pi}} \times$$

$$\times \Big(1 - \Phi\Big(\frac{K - \sigma^2 - m}{\sigma}\Big)\Big) - \frac{K}{\sqrt{2\pi}} \Big(1 - \Phi\Big(\frac{K - m}{\sigma}\Big)\Big),$$

where the variance σ^2 is determined by (6).

4 Behavior of the Variance of Geometric Fractional Ornstein–Uhlenbeck Process as a Function of Hurst Index

We already know that the variance of geometric fractional Ornstein–Uhlenbeck process equals

$$\sigma^2 = H(2H-1)\gamma^2 e^{-2\mu T} \int_0^T \int_0^T e^{as+au} |s-u|^{2H-2} du ds.$$

After some transformation we obtain the following form for the variance

$$\sigma^2 = H(2H-1)\gamma^2 e^{-2\mu T} \frac{1}{a} \Big(e^{2aT} \int_0^T e^{-az} z^{2H-2} dz -$$

$$- \int_0^T e^{az} z^{2H-2} dz\Big).$$

As $H \to \frac{1}{2}$, easily can be shown that $z^{2H-1} \to 1$,

$$\lim_{H \to \frac{1}{2}} (2H-1) \int_0^T e^{-az} z^{2H-2} dz = 1,$$

$$\lim_{H \to \frac{1}{2}} (2H-1) \int_0^T e^{az} z^{2H-2} dz = 1.$$

Now we can observe that the asymptotic behavior of the variance as $H \to \frac{1}{2}$ is following:

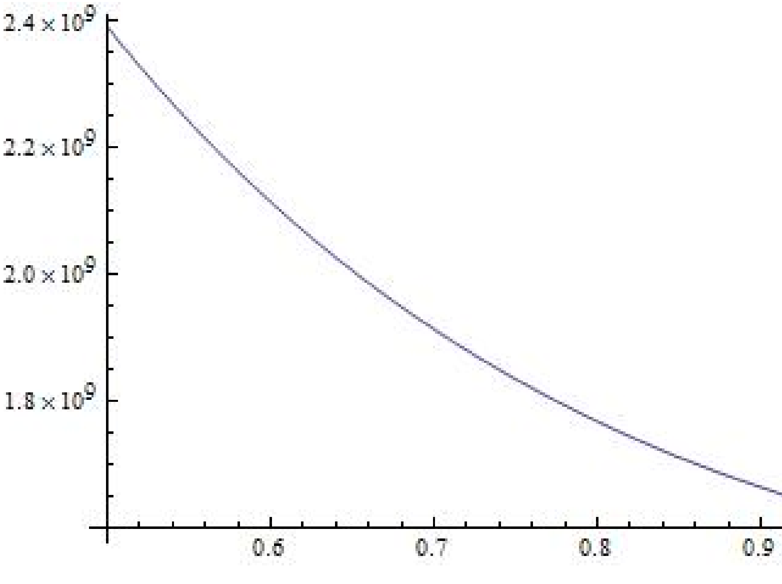
$$\lim_{H \to \frac{1}{2}} \sigma^2 = \frac{\gamma^2 e^{-2\mu T}}{2a} (e^{2aT} - 1).$$

As $H \to 1$ we obtain:

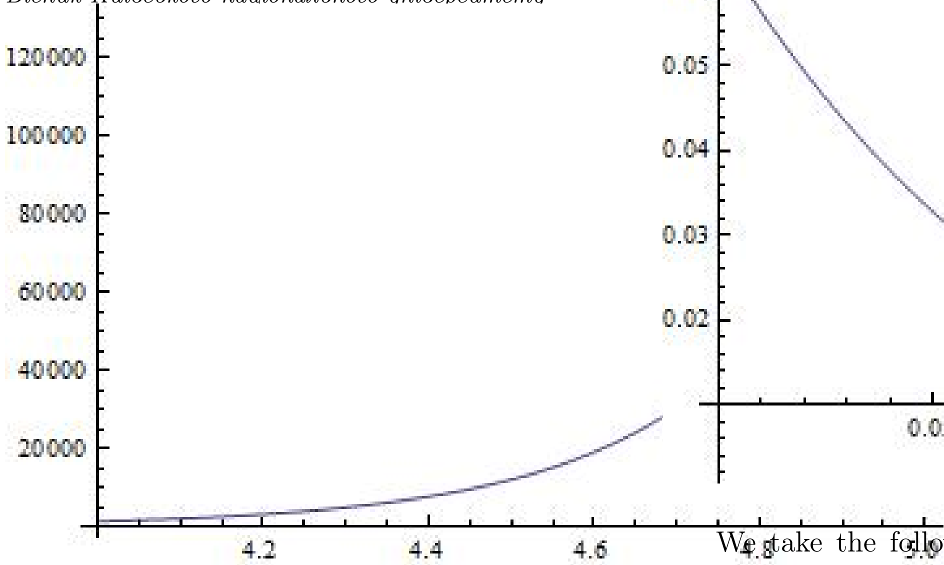
$$\lim_{H \to 1} \sigma^2 = \frac{\gamma^2 e^{-2\mu T}}{a} \Big(e^{2aT} \int_0^T e^{-az} dz - \int_0^T e^{az} dz\Big) =$$

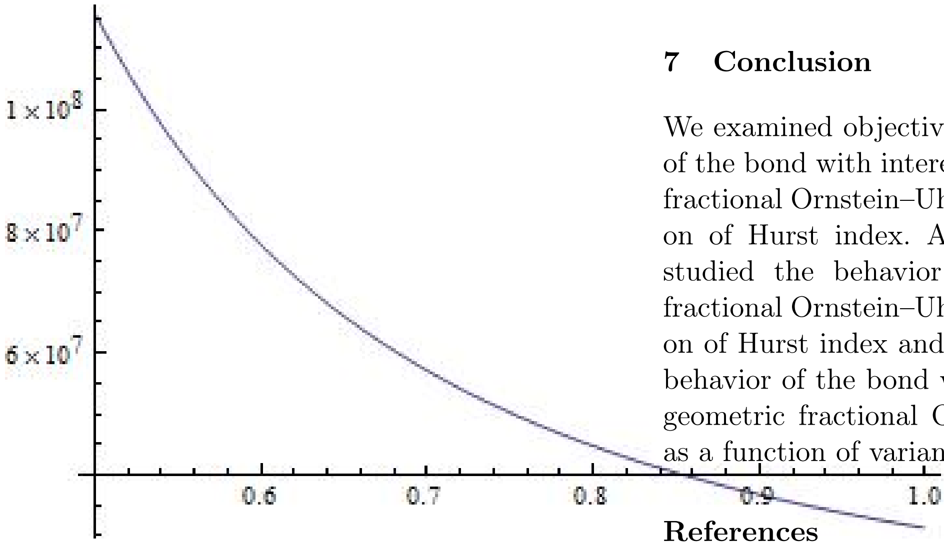
$$= \frac{\gamma^2 e^{-2\mu T}}{a^2} (1 - e^{aT})^2.$$

5 Monotonicity of the Variance of Geometric Fractional Ornstein–Uhlenbeck Process as a Function of Hurst Index. Monotonicity of Price of the Bond with Interest Rate Driven by Geometric Fractional Ornstein–Uhlenbeck Process as a Function of



The price of the bond with interest rate driven by geometric fractional Ornstein–Uhlenbeck process is an increasing function of σ^2.

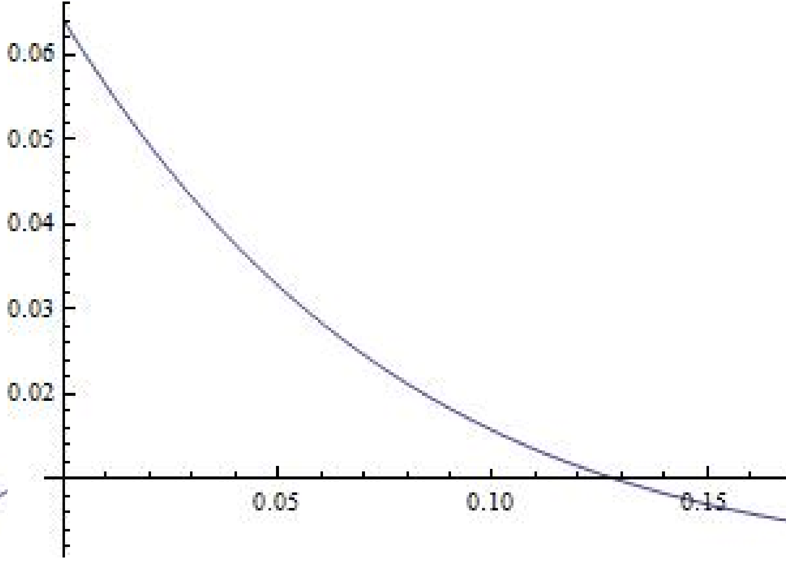




We take the following values of parameters: $\gamma = 1; \mu = 0,1; a = 2; T = 1; K = 5; m = 14$. A computational software program Mathematica is used.

6 Monotonicity of Price of the Bond with Interest Rate Driven by Geometric Fractional Ornstein–Uhlenbeck Process as a Function of μ

Since we consider the modified geometric fractional Ornstein–Uhlenbeck process it is interesting to examine the monotonicity of price of the bond as a function of μ. Parameter μ can be considered as a discount interest rate.



We take the following values of parameters: $\gamma = 1; a = 0,1; T = 1; K = 5; H = 3/2$.

7 Conclusion

We examined objective call option price behavior of the bond with interest rate driven by geometric fractional Ornstein–Uhlenbeck process as a function of Hurst index. As an auxiliary result were studied the behavior of variance of geometric fractional Ornstein–Uhlenbeck process as a function of Hurst index and examined call option price behavior of the bond with interest rate driven by geometric fractional Ornstein–Uhlenbeck process as a function of variance.

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Received: 16.11.2013