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Метрично реалізовні групи підстановок

Группа підстановок називаеться метрично реалізовною, якщо вона ізоморфна як група підстановок групі ізометрій деякого скінченного метричного простору. Розглянуто деякі властивості реалізовних груп. Введено конструкцію прямої суми метричних просторів і показано, що пряма сума, прямий добуток і вінцевий добуток двох метрично реалізовних груп буде метрично реалізовною групою. Охарактеризовано всі групи підстановок, що реалізуються як групи ізометрій скінченних ультраметричних просторів.

Ключові слова: група підстановок, метричний простір, група ізометрій, ультраметричний простір.

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Metric realizable permutation groups

A permutation group is said to be metric realizable if it is isomorphic as a permutation group to the isometry group of a metric space. Some properties of metric realizable groups are discussed. We introduce a new construction of direct sum of metric spaces and show that direct sums, direct products and wreath products of two metric realizable groups are metric realizable. All permutation group realizable as isometry group of finite ultrametric space are characterized.

Key Words: permutation group, metric space, isometry group, ultrametric space.

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1. A well-known problem that sometimes called generalized Koenig's problem is formulated as follows. For a given permutation group (G, X) find a discrete structure X (e.g., graph, metric space, ordered set, etc.) such that the automorphism group $Aut(X)$ is isomorphic to (G, X) as a permutation group (see [1]). Research in this direction naturally include studying operations on groups which preserve their property to be realized as automorphism groups of discrete structures. In [2] and [3] it was investigated constructions of graphs to show that wreath products and direct sums of permutation groups are realizable as their automorphism groups. In [5] and [6] constructions of metric spaces were studied to show the realizability of wreath products and direct products of permutation groups as isometry groups.

In this paper we consider only finite permutation groups, finite simple graphs (non directed, without loops) and finite metric spaces. We introduce a new construction of direct sum of metric spaces. Using this construction we will show that if two permutation group are isomorphic (as permutation groups) to the isometry groups of some metric space, then the direct sum of these groups also isomorphic to the isometry group of a finite metric space. We also characterize all permutation groups that are isomorphic to the isometry groups of finite ultrametric spaces.

2. Let (G, X) and (H, Y) be permutation groups. We say that the groups G and H are isomorphic as permutation groups ([4]) if there exist an isomorphism $f : G \to Y$ and a bijective map $\delta : X \to Y$ such that $\delta(x^g) = \delta(x)^{f(g)}$, where x^g is the image of the element $x \in X$ under $g \in G$.

Definition 1. A permutation group (G, X) is called *metric realizable* if there exists a metric space (Y, d_Y) such that the group (G, X) and the isometry group $(IsomY, Y)$ are isomorphic as permutation groups. In this case we say that the group (G, X) is *realized* on the metric space (Y, d_Y).

A permutation group (G, X) is said to be *graph realizable* if there exists a simple graph $\Gamma = (V, E)$ such that the group (G, X) and the automorphism group $(Aut\Gamma, V)$ are isomorphic as permutation groups.

Note that arbitrary simple graph $\Gamma = (V, E)$ can be considered as a metric space. The distance

$d_\Gamma(v_1, v_2)$ between vertices $v_1, v_2 \in V$ is defined as the length of the shortest path between v_1 and v_2 in Γ. In this case the automorphism group of the graph Γ equals to the isometry group of the metric space (Γ, d_Γ).

We can formulate some properties of metric realizable and graph realizable groups which are not difficult to verify.

1) A multiply transitive group (G, X) is metric realizable iff (G, X) is isomorphic as a permutation group to the symmetric group $Sym(X)$ naturally acting on the set X.

2) The class of graph realizable groups is a subclass of the class of metric realizable group. Moreover, there exist finite permutation groups which are not isomorphic to automorphism groups of any graphs, but are metric realizable. For example, a trivial group acting on 3 points can not be realized as the automorphism group of any simple graph. But this group is the isometry group of the metric space with the following matrix of distances:
$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}.$$

3) The class of graph realizable groups coincides with the class of metric realizable groups on metric spaces such that their metrics takes only three values.

4) There exist transitive groups that are not isometry groups of any finite metric spaces. For example, the regular cyclic group $(C_n, \{1, 2, \ldots, n\})$ of order $n \geqslant 3$ is not metric realizable.

5) If the permutation group (G, X) is metric realizable, then there exists a metric d on X such that the group (G, X) is realized on the space (X, d).

Every intransitive group can be decomposed in standard way into a semidirect product of its transitive components. Therefore, property (1) implies that it is sufficient to consider the problem of realization permutation groups as isometries of metric spaces for transitive groups only.

For a finite metric space (X, d) denote by $Distd$ the set of all possible values of metric d.

Lema 1. *If the isometry group $Isom(X, d)$ of the space (X, d) acts transitively on X, then*
$$Distd \leqslant |X|. \qquad (1)$$

Proof. Let x, y be points of (X, d). Denote by D_x the set $\{d(u, x) \mid u \in X\}$. If f is an isometry of X and $f(x) = y$, then $D_x = D_y$. Hence, transitivity of $Isom(X, d)$ implies that for every $x \in X$ we have $Distd = D_x$. As $|D_x| \leqslant |X|$, the required inequality follows. □

For a two-point space the inequality (1) transforms into the equality. Assume that (X, d_X) is an n-point discrete space, i.e., for distinct points $u, v \in X$ we have $d_X(u, v) = 1$. Then $Distd_X = 2$. Hence, in this case we have the relation
$$\frac{Distd_X}{|X|} = \frac{2}{n}$$
which can be made arbitrarily small.

3. Let (X, d_X) and (Y, d_Y) be finite metric spaces with $X \cap Y = \emptyset$. Assume that b, c are positive numbers such that the inequalities
$$diamX \leqslant c, \qquad \min_{x,y \in X} > b \cdot diamY \qquad (2)$$
hold. Define a function $\rho_{c,b}$ on the union $X \cup Y$ by the rule:
$$\rho_{c,b}(x, y) = \begin{cases} d_X(x, y), & \text{if } x,\ y \in X \\ b \cdot d_Y(x, y), & \text{if } x,\ y \in Y \\ c, & \text{in other cases} \end{cases}.$$
The following proposition is clear.

Lema 2. *The function $\rho_{c,b}$ is a metric on the set $X \cup Y$.*

We call the metric space $(X \cup Y, \rho_{c,b})$ the *direct sum* of the metric spaces (X, d_X) and (Y, d_Y) and denote it by $X \boxplus_{c,b} Y$.

It is not difficult to verify

Proposition 1. *Let (X, d_X) and (Y, d_Y) be finite metric spaces, b, c be positive numbers such that inequalities (2) hold. Then the space $X \boxplus_{c,b} Y$ is ultrametric iff both metric spaces (X, d_X) and (Y, d_Y) are ultrametric.*

Following [4], we call a permutation group $(G \times H, X \cup Y)$ the direct sum of permutation groups (G, X) and (H, Y) if every element $(g, h) \in G \times H$ acts on arbitrary $t \in X \cup Y$ by the rule:

$$t^{(g,h)} = \begin{cases} t^g, & \text{if } t \in X \\ t^h, & \text{if } t \in Y \end{cases}. \qquad (3)$$

Denote the direct sum of permutation groups (G, X) and (H, Y) by $(G, X) \oplus (H, Y)$.

Theorem 1. *Let (X, d_X) and (Y, d_Y) be finite metric spaces, b, c be positive numbers such that inequalities (2) hold. Then the isometry group of the direct sum $X \boxplus_{c,b} Y$ of the metric spaces X and Y is isomorphic as a permutation group to the direct sum of isometry groups of these spaces:*

$$(Isom(X \boxplus_{c,b} Y), X \cup Y) \simeq \simeq (IsomX, X) \cup (IsomY, Y).$$

Proof. Let (g, h) be an element of the group $(IsomX, X) \oplus (IsomY, Y)$. We shall show that $\varphi = (g, h)$ is an isometry of $X \boxplus_{c,b} Y$. The group $(IsomX, X) \oplus (IsomY, Y)$ acts on $X \cup Y$. Hence (g, h) is a bijective map on $X \boxplus_{c,b} Y$. Moreover, for arbitrary points x, y from $X \boxplus_{c,b} Y$ we have

$$\rho_{c,b}(\varphi(x), \varphi(y)) = \begin{cases} d_X(x^g, y^g), & \text{if } x, y \in X \\ \rho_{c,b}(x^h, y^h), & \text{if } x, y \in Y \\ \rho_{c,b}(x^g, y^h), & \text{if } x \in X,\ y \in Y \\ \rho_{c,b}(x^h, y^g), & \text{otherwise} \end{cases} = \begin{cases} d_X(x^g, y^g), & \text{if } x, y \in X \\ b \cdot d_Y(x^h, y^h), & \text{if } x, y \in Y \\ c, & \text{if } x \in X,\ y \in Y \\ c, & \text{otherwise} \end{cases}.$$

As $g \in IsomX$, $y \in IsomY$, we obtain

$$\rho_{c,b}(\varphi(x), \varphi(y)) = \begin{cases} d_X(x, y), & \text{if } x, y \in X \\ b \cdot d_Y(x, y), & \text{if } x, y \in Y \\ c, & \text{otherwise} \end{cases} = \rho_{c,b}(x, y).$$

So φ preserves the metric $\rho_{c,b}$. Therefore φ is an isometry of $X \boxplus_{c,b} Y$.

Let now φ be some isometry of $X \boxplus_{c,b} Y$, x be a point of $X \boxplus_{c,b} Y$. Using inequalities (2) we have that if $x \in X$ then $\varphi(x) \in X$, if $x \in Y$ then $\varphi(x) \in Y$. Hence the isometry φ acts on each point $x \in X \cup Y$ as some pair $(g, h) \in IsomX \times IsomY$ by rule (3). The proof of the theorem is complete. □

Note that there are graph realizable groups such that the direct sum of these groups in not a graph realizable group (see [3]).

The proof of the following proposition follows from Proposition 1 and Theorem 1.

Corollary 1. *If permutation groups (G, X), (H, Y) are metric realizable as isometry groups of some ultrametric spaces, then the group $(G, X) \oplus (H, Y)$ is metric realizable as isometry group of some ultrametric spaces too.*

Definition 2. We say that a binary operation $*$ on the class of finite permutation groups *preserves realizability* if for arbitrary metric realizable finite permutation groups (G, X) and (H, Y) the group $(G, X) * (H, Y)$ is metric realizable as well.

Lema 3. *[5] The direct product of finite metric realizable permutation groups is metric realizable.*

Theorem 2. *The operations of direct sum, direct product and wreath product preserve realizability.*

Proof. From Theorem 1 and Lemma 3 it follows that the operations of direct sum and direct product preserve realizability. Using Theorem 4 from [6] we obtain that the wreath product of finite metric realizable permutation groups is metric realizable. □

Let $B = \{T_1, T_2, \ldots\}$ be some countable alphabet. Define a notion of a formula on the alphabet B in three steps.

1) For every i, $i \geqslant 1$, the letter T_i from the alphabet B is *formula*.

2) If H_1 and H_2 are formulas on alphabet B then $(H_1 \wr H_2)$, $(H_1 \oplus H_2)$, $(H_1 \times H_2)$ are *formulas* on the alphabet B.

3) There are no other formulas.

Let now (G_1, X_1), (G_2, X_2), $\ldots$, (G_n, X_n) be permutation groups. For a formula $W(T_1, T_2, \ldots, T_n)$ denote by

$$W((G_1, X_1), (G_2, X_2), \ldots, (G_n, X_n))$$

its value on a sequence (G_1, X_1), (G_2, X_2), $\ldots$, (G_n, X_n).

Corollary 2. *Let $W(T_1, T_2, \ldots, T_n)$ be a formula. Then for arbitrary metric realizable permutation groups (G_1, X_1), (G_2, X_2), ..., (G_n, X_n) the permutation group*

$$W((G_1, X_1), (G_2, X_2), \ldots, (G_n, X_n))$$

is metric realizable too.

4. A permutation group is said to be *u-metric realizable* if it is isomorphic as a permutation group to the isometry group of an ultrametric space.

Theorem 3. *A finite permutation group (G, X) is u-metric realizable iff there exist positive integers k and $n_i, 1 \leqslant i \leqslant k$ such that the group (G, X) is isomorphic as a permutation group to the group $\oplus_{i=1}^{k}(\wr_{j=1}^{n_i} S_{m_j})$, where S_{m_j} is the symmetric group of degree m_j.*

Proof. It follows from [7] that for every finite ultrametric space (X, d) there exists a finite rooted tree $T = (V, E)$ such that (X, d) is isometric to the space (V, d_T). Then $(IsomX, X) \simeq (AutT, V)$. The automorphism group $(AutT, V)$ has a fixed point (root) and splits as a permutation group into the direct sums of transitive subgroups ([8]). A subgroup H of the automorphism group of a rooted tree is transitive iff H acts transitively on a spherically homogeneous rooted subtree of T. The automorphism group of a spherically homogeneous rooted tree is isomorphic as a permutation group to the wreath product $\wr_{j=1}^{n} S_{m_j}$ (for some n) of symmetric groups. Therefore, a finite permutation group (G, X) is u-metric realizable iff (G, X) is isomorphic to the direct sum of wreath products of symmetric groups. □

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