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Модулі неперервності випадкових процесів з просторів Орліча випадкових величин, визначених на інтервалі

Доведено теорему про модулі неперервності випадкових процесів з просторів Орліча випадкових величин, визначених на інтервалі. Наведено приклади її застосування.

Ключові слова: простори Орліча, випадковий процес.

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Modules of continuity of random processes from Orlicz spaces of random variables, defined on the interval

A theorem about modules of continuity of random processes from the Orlicz spaces of random variables was proved. Examples of its application are presented.

Key words: Orlicz spaces, random process.

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1 Introduction

The article deals with modules of continuity of random processes from Orlicz spaces of random variables. There are given conditions under which the sample paths of these random processes satisfying Lipschitz condition. The results improve the estimates obtained in the monograph [1] in the general case, and in article [2]. The article specified space of random variables (considered the segment of axis of real numbers $[0, T]$).

2 Modules of continuity of random processes

Definition 1. A continuous even convex function $U = \{U(x),\, x \in \mathbb{R}\}$ is called a $\mathbb{C}$–function if U is monotone increasing for $x > 0$ and $U(0) = 0$.

By Definition 1, the function $U(x)$, $x \geq 0$, is invertible [1, p. 42-44].

Definition 2. A $\mathbb{C}$–function U satisfies Δ^2–condition if the functions U and U^2 are equivalent, that is there exist constants $x_0 \geq 0$ and $L > 1$ such that

$$U^2(x) \leq U(Lx). \tag{1}$$

for $x \geq x_0$.

Lemma 1. *Let U be a function satisfying Δ^2–condition. Then $\forall x \geq U(x_0)$:*

$$U^{(-1)}(x^2) \leq LU^{(-1)}(x).$$

Lemma 2. *[1, p. 55] Suppose that a $\mathbb{C}$–function U satisfies Δ^2–condition and constants x_0, L — from inequality (1). If $\xi_1, \ldots, \xi_n$ — arbitrary random variables from the Orlicz space $L_U(\Omega)$, then for $k \geq 1,\ x \geq z_0 = \max\{x_0, L\},\ n \geq U(z_0)$ and $m \geq n$ the inequality holds:*

$$\mathsf{P}\left\{\max_{j=1,\ldots,n} |\xi_j| > x \max_{j=1,\ldots,n} \|\xi_j\|_U L^k U^{(-1)}(m)\right\} \leq \leq \frac{1}{m^{2^k-1} U(x)}. \tag{2}$$

Consider a space $\mathbb{T} = [0, T]$, $T > 0$ with a metric

$$\rho(t, s) = |t - s|, \quad t, s \in \mathbb{T}.$$

Let $\{U(x),\, x \in \mathbb{R}\}$ be a $\mathbb{C}$–function, such that satisfies Δ^2–condition. Corresponding space $L_U(\Omega)$ is the Orlicz space with a norm $\|\cdot\|_U$.

Consider a stochastic process $X = (X(t),\, t \in \mathbb{T})$ with $L_U(\Omega)$–increments. Let

$$\rho_X(t, s) = \|X(t) - X(s)\|_U, \qquad t, s \in \mathbb{T}$$

be the pseudometric generated by the prosess X. Suppose that the process X is separable on $(\mathbb{T},\, \rho)$.

Let $N(u) = N_\rho(\mathbb{T}, u)$ be the metric massiveness of the space $(\mathbb{T},\, \rho)$. And let $\sigma(u)$, $u > 0$ be some continuous function, such that $\sigma(u) \to 0$ as $u \to 0$. Introduce a condition on the pseudometric ρ_X:

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$$\sup_{\rho(t,s)\le u} \|X(t)-X(s)\|_U \le \sigma(u), \quad \forall u>0. \quad (3)$$

Since the metric massiveness $N(u)$ denotes the smallest number of elements in an u–covering of the set $\mathbb{T}$ (in this case, the segment $[0,T]$), then $\frac{T}{2u} \le N(u) \le \frac{T}{2u}+1$. Or, for the inverse function $\sigma^{(-1)}(u)$ of the function $\sigma(u)$:

$$N(u) \le \frac{T}{2\sigma^{(-1)}(u)}+1.$$

Let us prove the following theorem.

Theorem 1. *Let* $\varepsilon_0 = \sup_{t,s\in\mathbb{T}} \rho(t,s) = T$*;*

$$f(\sigma(u)) = \int_0^{\sigma(u)} U^{(-1)}\left(\frac{T}{2\sigma^{(-1)}(p)}+1\right)dp =$$

$$= [p=\sigma(t)] = \int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right)d\sigma(t),\ u>0;$$

$z_0 = \max\{x_0, L\}$*, where* x_0 *and* L *— constants from Definition 2; and let* $c = 3L(5+4L)$*,* $u_* = \sigma^{(-1)}(T)$*. The following statements are true:*
A) *There exist functions* $C(\sigma(u))$*,* $u\in(0,u_*)$ *and* $C_1(u)$*,* $u\in(0,T)$*, such that* $C(\sigma(u))>0$*,* $\forall u\in(0,u_*)$*,* $C_1(u)>0$ $\forall u\in(0,T)$*,* $C_1(u)\to 0$ *as* $u\to 0$*, and the inequality*

$$\mathsf{P}\left\{\sup_{\substack{t,s\in\mathbb{T}\\ 0<|t-s|\le u}} \frac{|X(t)-X(s)|}{C(\sigma(u))f(\sigma(|t-s|))} > x\right\} \le \frac{C_1(u)}{U(x)},$$

holds $\forall x\ge z_0$*,* $u\in(0,\sigma^{(-1)}(T))$*. If* u *is such that* $N(u)>U(z_0)$*, then* $C_1(u)\le 3+\sqrt{2}$ *and* $C(\sigma(u))\le c$*.*
B) *We have*

$$\limsup_{u\downarrow 0} \frac{\Delta(X;u)}{cz_0 f(\sigma(u))} \le 1$$

almost surely, where

$$\Delta(X;u) = \sup_{\substack{t,s\in\mathbb{T}\\ 0<|t-s|\le u}} |X(t)-X(s)|.$$

Proof. If $f(u_*)=\infty$ or $\sup_{u>0} N(u)<\infty$, then the claims of the theorem are clear. Assume that $f(u_*)<\infty$ and $N(u)\to\infty$, $u\to 0$. For simplicity of calculations, take u such that $N(u)>U(z_0)$.

Define a sequence $(\varepsilon_n, n\ge 0)$ by setting

$$\delta_n = 2\inf\{\varepsilon : N(\varepsilon) < 2N(\varepsilon_n)\},$$

$$\varepsilon_{n+1} = \min\left\{\frac{\varepsilon_n}{3}, \delta_n\right\},\ n\ge 0.$$

Then $\forall n\ge 0:$

$$\varepsilon_{n+1} \le \frac{\varepsilon_n}{3}, \qquad \varepsilon_n \le \frac{3}{2}(\varepsilon_n - \varepsilon_{n+1}); \qquad (*)$$

$$N(\varepsilon_{n+2}) \ge N\left(\frac{\varepsilon_{n+1}}{3}\right) \ge N\left(\frac{\delta_n}{3}\right) \ge 2N(\varepsilon_n);$$

$$N(\varepsilon_{n+1}) \ge 2\cdot 2N(\varepsilon_{n-3}) \ge \dots \ge 2^{(n-1)!!}\cdot N(\varepsilon_1) \ge$$
$$\ge 2^{(n-1)!!+1} \ge 2^{n/2}. \qquad (4)$$

Let us prove an auxiliary result.

Lemma 3. *Suppose that* $u\in(0,u_*)$ *is such that* $N(u)>U(z_0)$*. Take an integer* $k=k(u)\ge 0$*, such that the inequality* $\varepsilon_{k+1}<u\le\varepsilon_k$ *holds. Let* $c_l = \varepsilon_l L U^{(-1)}(N(\varepsilon_{l+1}))$*,* $l=k+1,k+2,\dots$*, and let* $b_k(u) = 4\sigma(u) L U^{(-1)}(N^2(\varepsilon_{k+1}))$*, where* x_0 *and* L *are the constants appearing in (1). Then*

$$2\sum_{l=k+1}^{\infty} c_l + b_k(u) \le cf(\sigma(u)), \qquad (5)$$

where $c = 3L(5+4L)$*,*

$$f(\sigma(u)) = \int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right)d\sigma(t).$$

Proof of Lemma 3. Let us obtain a bound for the sum

$$\sum_{l=k+1}^{\infty} c_l = L\sum_{l=k+1}^{\infty} \varepsilon_l U^{(-1)}(N(\varepsilon_{l+1})).$$

Put

$$\sum_{l=k+1}^{\infty} \varepsilon_l U^{(-1)}(N(\varepsilon_{l+1})) = A_1 + A_2,$$

where

$$A_1 = \sum_{l\in D_1(k)} \varepsilon_l U^{(-1)}(N(\varepsilon_{l+1})),$$

$$A_2 = \sum_{l\in D_2(k)} \varepsilon_l U^{(-1)}(N(\varepsilon_{l+1})),$$

$$D_1(k) = \{l\ge k+1 : \varepsilon_{l+1} = \varepsilon_l/3\},$$
$$D_2(k) = \{l\ge k+1 : \varepsilon_{l+1} = \delta_l\}.$$

Inequalities (∗) imply that

$$A_1 = 3\sum_{l\in D_1(k)} \varepsilon_{l+1} U^{(-1)}(N(\varepsilon_{l+1})) \le$$

$$\le 3\sum_{l=k+1}^{\infty} \varepsilon_{l+1} U^{(-1)}(N(\varepsilon_{l+1})) \le$$

$$\le \frac{9}{2}\sum_{l=k+1}^{\infty} (\varepsilon_{l+1}-\varepsilon_{l+2})\, U^{(-1)}(N(\varepsilon_{l+1})) \le$$

$$\le \frac{9}{2}\sum_{l=k+1}^{\infty} \int\limits_{\varepsilon_{l+2}}^{\varepsilon_{l+1}} U^{(-1)}(N(s))ds =$$

$$= \frac{9}{2}\int\limits_{0}^{\varepsilon_{k+2}} U^{(-1)}(N(s))ds \le$$

$$\le \frac{9}{2}\int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}(N(s))ds \le$$

$$\le \frac{9}{2}\int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}\left(\frac{T}{2\sigma^{(-1)}(s)}+1\right) ds = [s=\sigma(t)] =$$

$$= \frac{9}{2}\int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t).$$

Since $U^{(-1)}(2x) \le 2U^{(-1)}(x)$ (property of the inverse function of the $\mathbb{C}$–function [1, p. 44]), $N(\delta_l) \le 2N(\varepsilon_l)$ and since (∗), we obtain:

$$A_2 = \sum_{l\in D_2(k)} \varepsilon_l U^{(-1)}(N(\delta_l)) \le$$

$$\le 2\sum_{l\in D_2(k)} \varepsilon_l U^{(-1)}(N(\varepsilon_l)) \le$$

$$\le 2\sum_{l=k+1}^{\infty} \varepsilon_l U^{(-1)}(N(\varepsilon_l)) \le$$

$$\le 3\sum_{l=k+1}^{\infty} (\varepsilon_l-\varepsilon_{l+1}) U^{(-1)}(N(\varepsilon_l)) \le$$

$$\le 3\sum_{l=k+1}^{\infty} \int\limits_{\varepsilon_{l+1}}^{\varepsilon_l} U^{(-1)}(N(s))ds =$$

$$= 3\int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}(N(s))ds \le$$

$$\le 3\int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}\left(\frac{T}{2\sigma^{(-1)}(s)}+1\right) ds = [s=\sigma(t)] =$$

$$\le 3\int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t).$$

Finally, from inequalities for A_1 and A_2, we obtain:

$$\sum_{l=k+1}^{\infty} \varepsilon_l U^{(-1)}(N(\varepsilon_{l+1})) \le$$

$$\le \left(\frac{9}{2}+3\right) \int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t).$$

And because $\varepsilon_{k+1} < u$, the following inequality holds:

$$2\sum_{l=k+1}^{\infty} c_l \le 15L \int\limits_{0}^{\varepsilon_{k+1}} U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t) \le$$

$$\le 15L \int\limits_{0}^{u} U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t).$$

Let us estimate $b_k(u)$. Lemma 1 imply that:

$$b_k(u) = 4\sigma(u) L U^{(-1)}(N^2(\varepsilon_{k+1})) \le$$

$$\le 4\sigma(u) L^2 U^{(-1)}(N(\varepsilon_{k+1})).$$

Since $\varepsilon_{k+1} = \min\{\varepsilon_k/3,\, \delta_k\}$, we have

$$\sigma(u)U^{(-1)}(N(\varepsilon_{k+1})) = \sigma(u)U^{(-1)}(N(\delta_k)) \le$$

$$\le \sigma(u)U^{(-1)}(2N(\varepsilon_k)) \le 2\sigma(u)U^{(-1)}(N(\varepsilon_k)) \le$$

$$\le 2\sigma(u)U^{(-1)}(N(u)) \le 2\int\limits_{0}^{\sigma(u)} U^{(-1)}(N(s))ds \le$$

$$\le 2\int\limits_{0}^{\sigma(u)} U^{(-1)}\left(\frac{T}{2\sigma^{(-1)}(s)}+1\right) ds = [s=\sigma(t)] =$$

$$= 2\int\limits_{0}^{u} U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t)$$

for $\varepsilon_{k+1} = \delta_k$, and

$$\sigma(u)U^{(-1)}(N(\varepsilon_{k+1})) = \sigma(u)U^{(-1)}\left(N\left(\frac{\varepsilon_k}{3}\right)\right) \leq$$

$$\leq \sigma(u)U^{(-1)}\left(N\left(\frac{u}{3}\right)\right) \leq \int_0^{\sigma(u)} U^{(-1)}\left(N\left(\frac{s}{3}\right)\right)ds =$$

$$= \left|\begin{matrix}\frac{s}{3}=p\\ ds=3dp\end{matrix}\right| = 3\int_0^{\sigma(u)/3} U^{(-1)}(N(p))dp \leq$$

$$\leq 3\int_0^{\sigma(u)} U^{(-1)}(N(p))dp \leq 3\int_0^{u} U^{(-1)}\left(\frac{T}{2t}+1\right)d\sigma(t)$$

for $\varepsilon_{k+1} = \varepsilon_k/3$. In any case, the following upper bound holds:

$$b_k(u) \leq 12L^2\int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right)d\sigma(t).$$

Finally, we have:

$$2\sum_{l=k+1}^{\infty} c_l + b_k(u) \leq 15L\int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right)d\sigma(t)+$$

$$+12L^2\int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right)d\sigma(t) =$$

$$=3L(5+4L)\int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right)d\sigma(t).$$

Lemma is proved. □

Let us continue the proof of Theorem 1. First of all we give the definition of α–procedure.

Assume that (S, ρ) is a completely bounded pseudometric space, $(\varepsilon_k, k \geq 1)$ is a sequence of positive numbers monotone decreasing to zero, and $\varepsilon_0 = \operatorname{diam} S$. If S_k is an ε_k–net in the set S with respect to the pseudometric ρ, then the set $S_\infty = \bigcup_{k=0}^{\infty} S_k$ is a countable everywhere dense subset of S.

Definition 3. An α_k–map $\alpha_k : S \to S_k$, $k \geq 0$, is defined as a map of S to S_k such that $\alpha_k(x) = x$ if $x \in S_k$, and $\alpha_k(x)$ is the point of S_k closest to x if $x \notin S_k$. If there is more than one closest point, then we may choose any of these points. The family of maps $\{\alpha_k, k \geq 0\}$ is called α–procedure for choosing points in S_∞.

Choose the same integer $k = k(u)$ as in Lemma 3: $\varepsilon_{k+1} < u \leq \varepsilon_k$. Suppose that $\{S_m, m \geq k\}$ is the set of centers of balls that form a minimal ε_m–covering of the space $(\mathbb{T}, \rho_X)$, and let $V_m = \bigcup_{m=k}^{\infty} S_m$. Since the process X is ρ_X–continuous in probability, the set V_k is a separant of the process X for any k.

Assume that $\alpha_l(t)$, $t \in V_k$, $l \geq k$, are the maps described in α–procedure (see Definition 3); that is, $\alpha_l(t)$ puts into correspondence a point $t \in V_k$ to a point belonging to S_l and satisfying $\rho_X(t, \alpha_l(t)) \leq \varepsilon_l$. Set

$$\xi_l = \max_{t\in S_{l+1}} |X(t) - X(\alpha_l(t))|,$$

$$\eta_l(v) = \max_{\substack{t,s\in S_{l+1}\\ \rho_X(t,s)\leq 4v}} |X(t) - X(s)|,\ l \geq k;\ v > 0.$$

The inequality

$$\sup_{\rho_X(t,s)\leq v} |X(t) - X(s)| \leq 2\sum_{l=m+1}^{\infty} \xi_l + \eta_m(v) \quad (6)$$

holds with probability one for any v satisfying $\varepsilon_{m+1} < v \leq \varepsilon_m$. Formula (6) and Lemma 3 imply that

$$\sup_{0<\rho_X(t,s)\leq u} \frac{|X(t) - X(s)|}{cf(\rho_X(t,s))} \leq$$

$$\leq \sup_{0<\sigma(u)\leq u}\left[\sup_{0<\rho_X(t,s)\leq\sigma(u)} \frac{|X(t) - X(s)|}{cf(\sigma(u))}\right] \leq$$

$$\leq \sup_{m\geq k}\ \sup_{\varepsilon_{m+1}<\sigma(u)\leq\varepsilon_m} \frac{2\sum_{l=m+1}^{\infty}\xi_l + \eta_m(\sigma(u))}{2\sum_{l=m+1}^{\infty} c_l + b_m(u)}.$$

Therefore we obtain the inequality

$$\mathsf{P}\left\{\sup_{0<\rho_X(t,s)\leq u} \frac{|X(t) - X(s)|}{cf(\rho_X(t,s))} > x\right\} \leq$$

$$\leq \sum_{l=k+1}^{\infty} \mathsf{P}\left\{\frac{\xi_l}{c_l} > x\right\} +$$

$$+\sum_{m=k}^{\infty} \mathsf{P}\left\{\sup_{\varepsilon_{m+1}<\sigma(u)\leq\varepsilon_m} \frac{\eta_m(\sigma(u))}{b_m(u)} > x\right\}$$

for any $x > 0$. By Lemma 2, for $m = N(\varepsilon_{l+1})$, $k = 1$ the following inequality holds:

$$\mathsf{P}\left\{\frac{\xi_l}{c_l} > x\right\} \leq \frac{1}{N(\varepsilon_{l+1})}\cdot\frac{1}{U(x)} \quad (7)$$

for $x \geq z_0$. Further, we have

$$\theta_m = \sup_{\varepsilon_{m+1}<\sigma(u)\leq\varepsilon_m} \frac{\eta_m(\sigma(u))}{b_m(u)} =$$

$$= \sup_{\substack{\sigma(u)\in \\ (\varepsilon_{m+1},\varepsilon_m]}} \max_{\substack{\rho_X(t,s)\in \\ (0,4\sigma(u)] \\ t,s\in S_{m+1}}} \frac{|X(t)-X(s)|}{4\sigma(u)LU^{(-1)}(N^2(\varepsilon_{m+1}))}$$

$$\leq \sup_{\substack{\sigma(u)\in \\ (\varepsilon_{m+1},\varepsilon_m]}} \left[\max_{\substack{0<\rho_X(t,s)\leq 4\sigma(u) \\ t,s\in S_{m+1}}} \frac{|X(t)-X(s)|}{\|X(t)-X(s)\|_U} \times \right.$$

$$\left. \times \frac{1}{LU^{(-1)}(N^2(\varepsilon_{m+1}))} \right] \leq$$

$$\leq \frac{1}{LU^{(-1)}(N^2(\varepsilon_{m+1}))} \max_{\substack{t,s\in S_{m+1} \\ t\neq s}} \frac{|X(t)-X(s)|}{\|X(t)-X(s)\|_U}$$

for any $m \geq k$. By the last inequality and Lemma 2, for $m = N^2(\varepsilon_{m+1})$, $k = 1$ we have:

$$\mathsf{P}\{\theta_m > x\} \leq \frac{1}{N^2(\varepsilon_{m+1})} \cdot \frac{1}{U(x)} \qquad (8)$$

for $x \geq z_0$. Then it follows from (7) and (8) that

$$\mathsf{P}\left\{ \sup_{0<\rho_X(t,s)\leq u} \frac{|X(t)-X(s)|}{cf(\rho_X(t,s))} > x \right\} \leq$$

$$\leq \frac{1}{U(x)} \sum_{l=k+1}^{\infty} \frac{1}{N(\varepsilon_{l+1})} +$$

$$+ \frac{1}{U(x)} \sum_{m=k}^{\infty} \frac{1}{N^2(\varepsilon_{m+1})} = \frac{C_1(u)}{U(x)}$$

for $x \geq z_0$, where

$$C_1(u) = \sum_{l=k+1}^{\infty} \frac{1}{N(\varepsilon_{l+1})} + \sum_{l=k}^{\infty} \frac{1}{N^2(\varepsilon_{l+1})}.$$

Inequality (4) implies that

$$C_1(u) \leq \sum_{l=k+1}^{\infty} \frac{1}{2^{l/2}} + \sum_{l=k}^{\infty} \frac{1}{2^l} = \frac{1}{2^{k/2}(\sqrt{2}-1)} + \frac{1}{2^{k-1}}$$

for all $k \geq 0$, and $C_1(u) \leq 3 + \sqrt{2}$. Part **A)** is proved.

To prove part **B)**, observe that the convergence of series $C_1(u)$, the Borel-Cantelli lemma, and inequalities (7) and (8) imply that $\xi_l \leq xc_l$ and $\theta_m \leq x$ almost surely for $x \geq z_0$ and for sufficiently large l, m. By (6) and (5), we conclude that

$$\sup_{\rho_X(t,s)\leq u} |X(t)-X(s)| \leq x \left(2\sum_{l=k}^{\infty} c_l + b_k(u) \right) \leq$$

$$\leq cxf(\sigma(u))$$

almost surely for sufficiently small u. □

3 Examples

Here are examples of applying of the proven theorem for particular functions $\sigma(u)$ and $U(x)$.

Example 1. *Let function $\sigma(u) = bu^\alpha$, $u > 0$, $b > 0$, $\alpha \geq 1$.*

The inverse function for the $\sigma(u)$ is $\sigma^{(-1)}(u) = \sqrt[\alpha]{u/b}$. In this case, the following inequality for the metric massiveness holds:

$$N(u) \leq \frac{T}{2\sqrt[\alpha]{\frac{u}{b}}} + 1 = \frac{T}{2}\sqrt[\alpha]{\frac{b}{u}} + 1.$$

Consider a process $X = (X(t), t \in \mathbb{T})$ with $L_U(\Omega)$–increments, which satisfies the condition (3), and $\rho_X(t,s)$ — the pseudometric generated by this process.

The function f is determined as:

$$f(\sigma(u)) = \int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t) =$$

$$= b\alpha \int_0^u t^{\alpha-1} U^{(-1)}\left(\frac{T}{2t}+1\right) dt.$$

Suppose that $N(u) > U(z_0)$, $\forall\, u \in (0,T)$.
By the Theorem 1, the following inequality holds true

$$\mathsf{P}\left\{ \sup_{\substack{t,s\in\mathbb{T} \\ 0<|t-s|\leq u}} \frac{|X(t)-X(s)|}{cb\alpha \int_0^{|t-s|} t^{\alpha-1}U^{(-1)}\left(\frac{T}{2t}+1\right) dt} > x \right\}$$

$$\leq \frac{3+\sqrt{2}}{U(x)}.$$

Moreover, we have

$$\limsup_{u\downarrow 0} \frac{\Delta(X;u)}{cz_0 b\alpha \int_0^{|t-s|} t^{\alpha-1}U^{(-1)}\left(\frac{T}{2t}+1\right) dt} \leq 1$$

almost surely, where

$$\Delta(X;u) = \sup_{\substack{t,s\in\mathbb{T}\\ 0<|t-s|\le u}} |X(t)-X(s)|.$$

Example 2. *Let the function $\sigma(u)$ is the same as in the Example 1. Consider a function $U(x) = e^x - 1$.*

The inverse function for the $U(x)$ is $U^{(-1)}(x) = \ln(x+1)$;

$$f(\sigma(u)) = b\alpha \int_0^u t^{\alpha-1} U^{(-1)}\left(\frac{T}{2t}+1\right) dt =$$

$$= b\alpha u^{1+\alpha} \cdot \frac{3+2\alpha+(1+\alpha)\ln(T/2u)}{(1+\alpha)^2}.$$

For simplicity of calculations, let $\psi(u) = 3 + 2\alpha + (1+\alpha)\ln(T/2u)$. In this case, by the Theorem 1, we have:

$$\mathsf{P}\left\{\sup_{\substack{t,s\in\mathbb{T}\\ 0<|t-s|\le u}} \frac{|X(t)-X(s)|(1+\alpha)^2}{|t-s|^{1+\alpha}\psi(|t-s|)} > cb\alpha x\right\} \le$$

$$\le \frac{3+\sqrt{2}}{e^x-1}$$

and

$$\limsup_{u\downarrow 0}\left[\sup_{\substack{t,s\in\mathbb{T}\\ |t-s|\in\\ (0,u]}} \frac{|X(t)-X(s)|(1+\alpha)^2}{|t-s|^{1+\alpha}\psi(|t-s|)}\right] \le cz_0 b\alpha$$

almost surely.

Example 3. *Consider a function $\sigma(u) = \left(\ln\left(1+\frac{1}{u}\right)\right)^{-1}$, $u>0$.*

The inverse function for the $\sigma(u)$ is $\sigma^{(-1)}(u) = \left(e^{1/u}-1\right)^{-1}$. So, we have the inequality for metric massiveness:

$$N(u) \le \frac{T\left(e^{1/u}-1\right)}{2} + 1.$$

Consider a process $X = (X(t),\, t\in\mathbb{T})$ with $L_U(\Omega)$–increments, which satisfies the condition (3), and $\rho_X(t,s)$ — the pseudometric generated by this process.

The function f is determined as:

$$f(\sigma(u)) = \int_0^u U^{(-1)}\left(\frac{T}{2t}+1\right) d\sigma(t) =$$

$$= \int_0^u \frac{1}{t(t+1)\ln^2\left(1+\frac{1}{t}\right)} U^{(-1)}\left(\frac{T}{2t}+1\right) dt.$$

For simplicity of calculations, let $\phi(t) = \left(t(t+1)\ln^2\left(1+\frac{1}{t}\right)\right)^{-1}$. Suppose that $N(u) > U(z_0)$, $\forall\, u \in (0,T)$. By the Theorem 1, for the process X next inequality holds:

$$\mathsf{P}\left\{\sup_{\substack{t,s\in\mathbb{T}\\ 0<|t-s|\le u}} \frac{|X(t)-X(s)|}{c\int_0^{|t-s|}\phi(t)\,U^{(-1)}\left(\frac{T}{2t}+1\right)dt} > x\right\} \le$$

$$\le \frac{3+\sqrt{2}}{U(x)}.$$

Moreover,

$$\limsup_{u\downarrow 0} \frac{\Delta(X;u)}{cz_0\int_0^{|t-s|}\phi(t)\,U^{(-1)}\left(\frac{T}{2t}+1\right)dt} \le 1$$

almost surely, where

$$\Delta(X;u) = \sup_{\substack{t,s\in\mathbb{T}\\ 0<|t-s|\le u}} |X(t)-X(s)|.$$

4 Conclusion

In the article are given conditions under which sample paths of random processes from Orlicz spaces of random variables, defined on the interval, satisfy the Lipschitz condition, and examples of application.

References

1. *Buldygin V. V., Kozachenko Yu. V.* Metric Characterization of Random Variables and Random Processes. — Providence: American Mathematical Society, – 2000.

2. *Kozachenko Yu. V.* Stochastic processes in Orlicz spaces. II. // Theory Prob. and Mathem. Statistics. – 1984. – №31. – p. 44-50.

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