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Нормальна структура діагональної границі гіпероктаедральних груп з зануреннями дублювання

Розглянуто групу, що є діагональною границею гіпероктаедральних груп з зануреннями дублювання. Ця група реалізується також як скрізь щільна підгрупа групи ізометрій 2^∞-періодичного простору Хемінга. Охарактеризовано її нормальну структуру.

Ключові слова: гіпероктаедральна група, простір Хемінга, діагональна границя, вінцевий добуток, нормальний дільник.

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The normal structure of the diagonal limit of hyperoctahedral groups with doubling embeddings

The diagonal limit of hyperoctahedral groups with respect to doubling embeddings is considered. This group arises as an everywhere dense subgroup of the isometry group of the 2^∞-periodic Hamming space. We characterize its normal structure.

Key Words: hyperoctahedral group, Hamming space, diagonal limit, wreath product, normal subgroup.

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1. Consider the n-dimension Hamming space H_n, i.e. the space of all n-tuples $(a_1,\ldots,a_n)$, $a_i \in \{0,1\}$, $1 \leqslant i \leqslant n$, with the distance d_{H_n}. The distance d_{H_n} between points $\bar{x}=(x_1,\ldots,x_n)$, $\bar{y}=(y_1,\ldots,y_n)$ is defined by the rule:

$$d_{H_n}(\bar{x},\bar{y}) = |\{k : x_k \neq y_k, 1 \leqslant k \leqslant n\}|.$$

It easy to see that the isometry group $IsomH_n$ of the n-dimension Hamming space H_n can be realized as the group of orthogonal matrices over the ring of integers $\mathbb{Z}$. Moreover, it is isomorphic to the wreath product $W_n = Z_2 \wr S_n$, where Z_2 is the cyclic group of order 2 and S_n is the symmetric group of degree n. In other words, the isometry group $IsomH_n$ decomposes into the semidirect product of its subgroup S_n and its normal subgroup

$$K_n = \underbrace{Z_2 \times \ldots \times Z_2}_{n}.$$

In this case S_n acts on K_n by permutations of coordinates of vectors. We can write every element $u \in W_n$ as a so-called table $u=[\sigma; a_1,a_2,\ldots,a_n]$, where $\sigma \in S_n$, $a_i \in Z_2$, $1 \leqslant i \leqslant n$. The group operation in W_n is defined by the rule:

$$[\sigma; a_1,a_2,\ldots,a_n][\eta; b_1,b_2,\ldots,b_n] =$$

$$= [\sigma\eta; a_1+b_{1^\sigma},\ldots,a_n+b_{n^\sigma}], \quad (1)$$

where the symbol $+$ denotes addition in Z_2. The table

$$[\sigma^{-1}; a_{1^{\sigma^{-1}}},\ldots,a_{n^{\sigma^{-1}}}]. \quad (2)$$

is the inverse element to the element $[\sigma; a_1,\ldots,a_n]$. An element $u=[\sigma; a_1,\ldots,a_n]$ acts on a vector $\bar{t}=(t_1,\ldots,t_n) \in Z_2^n$ by the rule:

$$t^u = (t_{1^\sigma}+a_1,\ldots,t_{n^\sigma}+a_n). \quad (3)$$

The isometry group $(IsomH_n, H_n)$ of Hamming space H_n is isomorphic as a permutation group to the group (W_n, Z_2^n) defined by (1)-(3).

Let K_n^0 be the set of all sequences $(a_1,a_2,\ldots,a_n) \in K_n$ such that $\sum_{i=1}^n a_i = 0$. Denote by W_n' the commutator subgroup of the group W_n. We will need the following

Lema 1. *[1],[2] The commutator W_n' of the group W_n consists of all tables $[\sigma; a_1,a_2,\ldots,a_n]$ such that $\sigma \in A_n$ and $(a_1,a_2,\ldots,a_n) \in K_n^0$.*

2. The infinite sequence of positive integers $\mathbf{a}=(a_1,a_2,\ldots)$ is said to be *periodic* if there exists a natural number k (*a period* of the sequence $\mathbf{a}$) such that the equality $a_i = a_{i+k}$ holds for all $i \in \mathbb{N}$. A periodic sequence $\mathbf{a}$ is called 2^∞*-periodic* if there exists a nonnegative integer l such that the minimal period of $\mathbf{a}$ equals 2^l.

Define a metric $d_{H_{2^\infty}}$ on the set H_{2^∞} of all 2^∞-periodic sequences by the rule:

$$d_{H_{2^\infty}}((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \\ = \frac{1}{k} d_{H_k}((x_1, \ldots, x_k), (y_1, \ldots, y_k)),$$

where k is a common period of sequences $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ from H_{2^∞}. The metric space $(H_{2^\infty}, d_{H_{2^\infty}})$ is said to be 2^∞-*periodic Hamming space*. This space was studied in [3] as a generalization of finite Hamming spaces to the infinite dimensional case.

The metric space H_{2^∞} admits another description using diagonal limits. A mapping

$$f_k : \frac{1}{k} H_k \to \frac{1}{2k} H_{2k}$$

is said to be *doubling* if it is determined as

$$f_k(x_1, \ldots, x_k) = (x_1, x_1, \ldots, x_k, x_k). \qquad (4)$$

The direct spectrum $\Phi = \langle \frac{1}{2^n} H_{2^n}, f_{2^n} \rangle$ of scaled Hamming spaces $\frac{1}{2^n} H_{2^n}$ is isometric to the space H_{2^∞}.

Recall that an embedding θ of a permutation group (G, X) into a permutation group (H, Y) is said to be strictly diagonal, if the restriction $\theta(G)$ on any orbit on the set Y is isomorphic to the group (G, X) (see [4]). The doubling defined by (4) induces a diagonal embedding φ_k of the isometry group $IsomH_{2^n}$ into the group $IsomH_{2^{n+1}}$. Then we obtain an embedding of $Z_2 \wr S_{2^n}$ into $Z_2 \wr S_{2^{n+1}}$ where K_{2^n} embeds into $K_{2^{n+1}}$ as we defined in (4) and S_{2^n} embeds into $S_{2^{n+1}}$ diagonally in sense of [4]. Denote by W_{2^∞} the diagonal limit of W_n. Note that

$$W_{2^\infty} = \bigcup_{i=1}^{\infty} W_n. \qquad (5)$$

The group W_{2^∞} is an everywhere dense subgroup of the isometry group $IsomH_{2^\infty}$ of the 2^∞-periodic Hamming space (see [5]).

The subgroup W_{2^∞} admits a natural description using wreath products of group.

Denote by $S(\mathbb{N})$ be the symmetric group on the set positive integers. Let r be a positive integer. A permutation $\pi \in S(\mathbb{N})$ is said to be r-periodic if π acts on a positive integer n by the rule:

$$n^\pi = t^\pi + qr,$$

where $n = q \cdot r + t$, $0 \leqslant t < r$. All r-periodic permutations form a subgroup $S(r)$ of the group $S(\mathbb{N})$. This subgroup is isomorphic to the symmetric group S_r. The set of all 2^∞-periodic permutations form a subgroup S_{2^∞} in $S(\mathbb{N})$. This subgroup decomposes as a union of the increasing chain of subgroups $S(2) \subset S(4) \subset \ldots$, i.e.

$$S_{2^\infty} = \bigcup_{i=1}^{\infty} S(2^i).$$

Similarly define the subgroup A_{2^∞}. Let $A(r)$ be the subgroup of the group $S(r)$ isomorphic to the alternating group A_r. Then A_{2^∞} is a union of the increasing chain of subgroups $A(2) \subset A(4) \subset \ldots$, in other words

$$A_{2^\infty} = \bigcup_{i=1}^{\infty} A(2^i).$$

Sequences of the form

$$[\pi, a_1, a_2, \ldots], \ \pi \in S_{2^\infty}, \ a_i \in Z_2, \ (i \geqslant 1)$$

are elements of the wreath product $W = Z_2 \wr S_{2^\infty}$ of the permutation group $(S_{2^\infty}, \mathbb{N})$ with the cyclic group of order 2. The operations of multiplication and taking inverses are defined analogously to (1) and (2). An element $u \in W$ acts on a sequence $\bar{t} = (t_1, t_2 \ldots) \in Z_2^\infty$ in a similar to (1) way.

Now consider the set $\widehat{W}$ of all elements $[\pi, a_1, a_2, \ldots] \in Z_2 \wr S_{2^\infty}$ such that $(a_1, a_2, \ldots)$ is a 2^∞-periodic sequence. The set $\widehat{W}$ is a subgroup of the wreath product $Z_2 \wr S_{2^\infty}$. We call the group $\widehat{W}$ the 2^∞-*wreath product* of groups S_{2^∞} and Z_2.

It is not difficult to verify

Proposition 1. *The groups W_{2^∞} and $\widehat{W}$ are isomorphic.*

3. The set H_{2^∞} of all 2^∞-periodic sequences with coordinate-wise addition form a group. Denote it by K_{2^∞}. This group is isomorphic to a subgroup of 2^∞-wreath product of groups S_{2^∞} and Z_2. Denote by C the subgroup of K_{2^∞} containing only sequences $(0, 0, \ldots)$ and $(1, 1, \ldots)$.

Theorem 1. *The normal structure of the 2^∞-wreath product of group S_{2^∞} and Z_2 has the form*

$$E \lhd C \lhd K_{2^\infty} \lhd W'_{2^\infty} \lhd W_{2^\infty},$$

where W'_{2^∞} is the commutator subgroup of the group W_{2^∞}.

Proof. The proof of the theorem is divided into four parts.

1. We shall show that the equality

$$W'_{2^\infty} = A_{2^\infty} \cdot K_{2^\infty}$$

holds. Indeed, hyperoctahedral group W_n is a semidirect product of K_n and S_n. It follows from Lemma 1 that $W'_n = A_n \cdot K_n^0$. Using (5) we get that the equalities

$$W'_{2^\infty} = \bigcup_{i=1}^{\infty} W'_n = \bigcup_{i=1}^{\infty} A_n \cdot K_n^0 = \\ = \bigcup_{i=1}^{\infty} A_n \cdot \bigcup_{i=1}^{\infty} K_n^0 \quad (6)$$

hold. But $\bigcup_{i=1}^{\infty} K_n^0 = K_{2^\infty}$. Hence, from (6) we have

$$W'_n = A_{2^\infty} \cdot K_{2^\infty}.$$

2. Let $v = [e; b_1, b_2, \ldots]$ be an element of W_{2^∞} such that $(b_1, b_2, \ldots) \neq (0, 0, \ldots)$, $(b_1, b_2, \ldots) \neq (1, 1, \ldots)$. We shall show that the normal closure L of the element v contains the subgroup K_{2^∞}. Note, that elements $[\eta; 0, 0, \ldots]$, $\eta \in S_{2^\infty}$ act on v as permutations of coordinates of the sequence $(b_1, b_2, \ldots)$. Assume that 2^l is the minimal period of $(b_1, b_2, \ldots)$. As $(b_1, b_2, \ldots) \neq (0, 0, \ldots)$ and $(b_1, b_2, \ldots) \neq (1, 1, \ldots)$, there exist b_i and b_j such that $b_i \neq b_j$, $1 \leqslant i < j \leqslant 2^l$. Denote by $(c_1, c_2, \ldots)$ the sequence obtained from the sequence $(b_1, b_2, \ldots)$ using the 2^l-periodic permutation, which permutes ith and jth coordinates inside the minimal period. Define an element $w = [\eta; 0, 0, \ldots]$ such that

$$w^{-1}vw = [\tau; c_1, c_2, \ldots).$$

Then the sequence $v + w^{-1}vw$ has only two units inside the minimal period. But elements of these type generate K_{2^∞}. Therefore, $K_{2^\infty} \subseteq L$.

3. Let $u = [\sigma; a_1, a_2, \ldots]$ be an element of W_{2^∞} such that $\sigma \neq e$. We shall show that the normal closure M of the element u contains the subgroup W'_{2^∞}. Indeed, define an element $v = [e; b_1, b_2, \ldots] \in W_{2^\infty}$ such that $(u, v) \neq [e; 0, 0, \ldots]$. Then $(u, v) = [e; c_1, c_2, \ldots]$. So, from the second part of the proof it follows that $K_{2^\infty} \subseteq M$.

Now define a table $w = [\nu; 0, 0, \ldots] \in W_{2^\infty}$ such that $(u, w) \neq [e; 0, 0, \ldots]$. Then $(u, w) = [\eta; q_1, q_2, \ldots]$, where $\eta \in A_{2^\infty}$ and $(q_1, q_2, \ldots) \in K_{2^\infty}$. As $K_{2^\infty} \subseteq M$, we have $A_{2^\infty} \subseteq M$. Hence $A_{2^\infty} \cdot K_{2^\infty} \subseteq M$ and from the first part of the proof we get $W'_{2^\infty} \subseteq M$.

4. The abelianization $W_{2^\infty}/W'_{2^\infty}$ of the group W_{2^∞} is a cyclic group of order 2. Hence W'_{2^∞} is a maximal subgroup of the group W_{2^∞}. From the second and third parts of the proof it follows that other nontrivial normal subgroup of W_{2^∞} are C and K_{2^∞} only. □

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